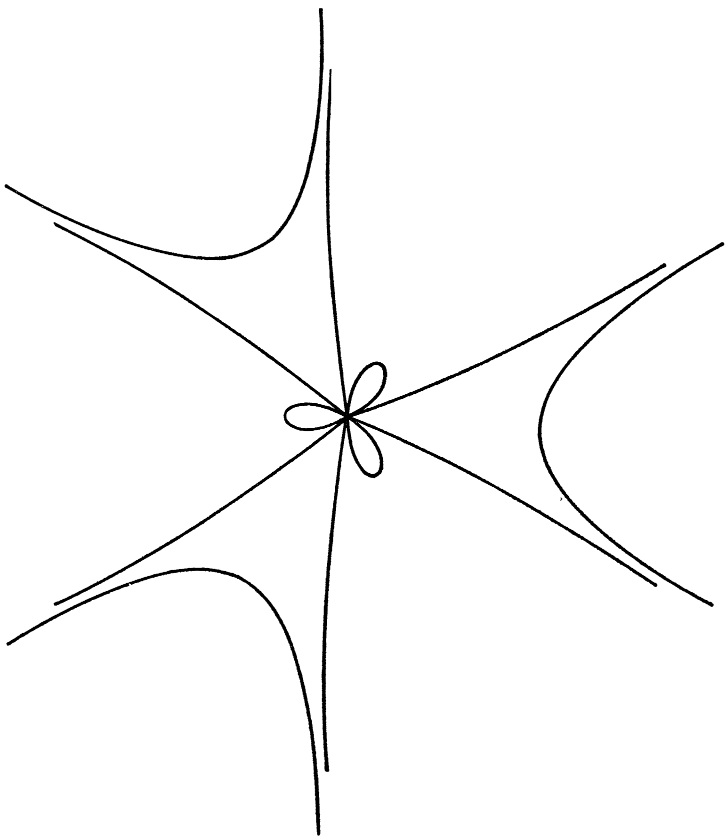

FIG. 2: $p=3$, $q=1$, $k=2a$.

LEONHARD EULER'S INTEGRAL: A HISTORICAL PROFILE OF THE GAMMA FUNCTION

IN MEMORIAM: MILTON ABRAMOWITZ

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Many people think that mathematical ideas are static. They think that the ideas originated at some time in the historical past and remain unchanged for all future times. There are good reasons for such a feeling. After all, the formula for the area of a circle was πr^2 in Euclid's day and at the present time is still πr^2. But to one who knows mathematics from the inside, the subject has rather the feeling of a living thing. It grows daily by the accretion of new information, it changes daily by regarding itself and the world from new vantage points, it maintains a regulatory balance by consigning to the oblivion of irrelevancy a fraction of its past accomplishments.

The purpose of this essay is to illustrate this process of growth. We select one mathematical object, the gamma function, and show how it grew in concept and in content from the time of Euler to the recent mathematical treatise of Bourbaki, and how, in this growth, it partook of the general development of mathematics over the past two and a quarter centuries. Of the so-called "higher mathematical functions," the gamma function is undoubtedly the most fundamental. It is simple enough for juniors in college to meet but deep enough to have called forth contributions from the finest mathematicians. And it is sufficiently compact to allow its profile to be sketched within the space of a brief essay.

The year 1729 saw the birth of the gamma function in a correspondence between a Swiss mathematician in St. Petersburg and a German mathematician in Moscow. The former: Leonhard Euler (1707–1783), then 22 years of age, but to become a prodigious mathematician, the greatest of the 18th century. The latter: Christian Goldbach (1690–1764), a savant, a man of many talents and in correspondence with the leading thinkers of the day. As a mathematician he was something of a dilettante, yet he was a man who bequeathed to the future a problem in the theory of numbers so easy to state and so difficult to prove that even to this day it remains on the mathematical horizon as a challenge.

The birth of the gamma function was due to the merging of several mathematical streams. The first was that of interpolation theory, a very practical subject largely the product of English mathematicians of the 17th century but which all mathematicians enjoyed dipping into from time to time. The second stream was that of the integral calculus and of the systematic building up of the formulas of indefinite integration, a process which had been going on steadily for many years. A certain ostensibly simple problem of interpolation arose and was bandied about unsuccessfully by Goldbach and by Daniel Bernoulli (1700–1784) and even earlier by James Stirling (1692–1770). The problem was posed to Euler. Euler announced his solution to Goldbach in two letters which were to be the beginning of an extensive correspondence which lasted the duration of Goldbach's life. The first letter dated October 13, 1729 dealt with the interpolation problem, while the second dated January 8, 1730 dealt with integration and tied the two together. Euler wrote Goldbach the merest outline, but within a year he published all the details in an article *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt.* This article can now be found reprinted in Volume I_{14} of Euler's *Opera Omnia.*

Since the interpolation problem is the easier one, let us begin with it. One of the simplest sequences of integers which leads to an interesting theory is 1, $1+2$, $1+2+3$, $1+2+3+4$, $\cdots$. These are the triangular numbers, so called because they represent the number of objects which can be placed in a triangular array of various sizes. Call the nth one T_n. There is a formula for T_n which is learned in school algebra: $T_n=\frac{1}{2}n(n+1)$.

What, precisely, does this formula accomplish? In the first place, it simplifies computation by reducing a large number of additions to three fixed operations: one of addition, one of multiplication, and one of division. Thus, instead of adding the first hundred integers to obtain T_{100}, we can compute $T_{100}=\frac{1}{2}(100)(100+1)=5050$. Secondly, even though it doesn't make literal sense to ask for, say, the sum of the first $5\frac{1}{2}$ integers, the formula for T_n produces an answer to this. For whatever it is worth, the formula yields $T_{5\frac{1}{2}}=\frac{1}{2}(5\frac{1}{2})(5\frac{1}{2}+1)=17\frac{7}{8}$. In this way, the formula extends the scope of the original problem to values of the variable other than those for which it was originally defined and solves the problem of interpolating between the known elementary values.

This type of question, one which asks for an extension of meaning, cropped up frequently in the 17th and 18th centuries. Consider, for instance, the algebra of exponents. The quantity a^m is defined initially as the product of m successive a's. This definition has meaning when m is a positive integer, but what would $a^{5\frac{1}{2}}$ be? The product of $5\frac{1}{2}$ successive a's? The mysterious definitions $a^0=1$, $a^{m/n}=\sqrt[n]{a^m}$, $a^{-m}=1/a^m$ which solve this enigma and which are employed so fruitfully in algebra were written down explicitly for the first time by Newton in 1676. They are justified by a utility which derives from the fact that the definition leads to continuous exponential functions and that the law of exponents $a^m \cdot a^n = a^{m+n}$ becomes meaningful for all exponents whether positive integers or not.

Other problems of this type proved harder. Thus, Leibnitz introduced the notation d^n for the nth iterate of the operation of differentiation. Moreover, he identified d^{-1} with $\int$ and d^{-n} with the iterated integral. Then he tried to breathe some sense into the symbol d^n when n is any real value whatever. What, indeed, is the $5\frac{1}{2}$th derivative of a function? This question had to wait almost two centuries for a satisfactory answer.

THE FACTORIALS

n:	1	2	3	4	5	6	7	8	$\cdots$
$n!$:	1	2	6	24	120	720	5040	40,320	$\cdots$

FIG. 1

INTELLIGENCE TEST

Question: What number should be inserted in the lower line half way between the upper 5 and 6?

Euler's Answer: 287.8852 $\cdots$. Hadamard's Answer: 280.3002 $\cdots$.

But to return to our sequence of triangular numbers. If we change the plus signs to multiplication signs we obtain a new sequence: $1, 1\cdot 2, 1\cdot 2\cdot 3, \cdots$. This is the sequence of factorials. The factorials are usually abbreviated $1!, 2!, 3!, \cdots$ and the first five are 1, 2, 6, 24, 120. They grow in size very rapidly. The number 100! if written out in full would have 158 digits. By contrast, $T_{100}=5050$ has a mere four digits. Factorials are omnipresent in mathematics; one can hardly open a page of mathematical analysis without finding it strewn with them. This being the case, is it possible to obtain an easy formula for computing the factorials? And is it possible to interpolate between the factorials? What should $5\frac{1}{2}!$ be? (See Fig. 1.) This is the interpolation problem which led to the gamma function, the interpolation problem of Stirling, of Bernoulli, and of Goldbach. As we know, these two problems are related, for when one has a formula there is the possibility of inserting intermediate values into it. And now comes the surprising thing. There is no, in fact there can be, no formula for the factorials which is of the simple type found for T_n. This is implicit in the very title Euler chose for his article. Translate the Latin and we have *On transcendental progressions whose general term cannot be expressed algebraically.* The solution to factorial interpolation lay deeper than "mere algebra." Infinite processes were required.

In order to appreciate a little better the problem confronting Euler it is useful to skip ahead a bit and formulate it in an up-to-date fashion: find a reasonably simple function which at the integers $1, 2, 3, \cdots$ takes on the factorial values $1, 2, 6, \cdots$. Now today, a function is a relationship between two sets of numbers wherein to a number of one set is assigned a number of the second set. What is stressed is the relationship and not the nature of the rules which serve to determine the relationship. To help students visualize the function concept in its full generality, mathematics instructors are accustomed to draw a curve full of twists and discontinuities. The more of these the more general the function is supposed to be. Given, then, the points $(1,1)$, $(2, 2)$, $(3, 6)$, $(4, 24)$, $\cdots$ and adopting the point of view wherein "function" is what we have just said, the problem of interpolation is one of finding a curve which passes through the given points. This is ridiculously easy to solve. It can be done in an unlimited number of ways. Merely take a pencil and draw some curve—any curve will do—which passes through the points. Such a curve automatically defines a function which solves the interpolation problem. In this way, too free an attitude as to what constitutes a function solves the problem trivially and would enrich mathematics but little. Euler's task was different. In the early 18th century, a function was more or less synonymous with a formula, and by a formula was meant an expression which could be derived from elementary manipulations with addition, subtraction, multiplication, division, powers, roots, exponentials, logarithms, differentiation, integration, infinite series, *i.e.*, one which came from the ordinary processes of mathematical analysis. Such a formula was called an *expressio analytica*, an analytical expression. Euler's task was to find, if he could, an analytical expression arising naturally from the corpus of mathematics which would yield factorials when a positive integer was inserted, but which would still be meaningful for other values of the variable.

It is difficult to chronicle the exact course of scientific discovery. This is particularly true in mathematics where one traditionally omits from articles and books all accounts of false starts, of the initial years of bungling, and where one may develop one's topic forward or backward or sideways in order to heighten the dramatic effect. As one distinguished mathematician put it, a mathematical result must appear straight from the heavens as a *deus ex machina* for students to verify and accept but not to comprehend. Apparently, Euler, experimenting with infinite products of numbers, chanced to notice that if n is a positive integer,

$$\text{(1)} \qquad \left[\left(\frac{2}{1}\right)^n \frac{1}{n+1}\right]\left[\left(\frac{3}{2}\right)^n \frac{2}{n+2}\right]\left[\left(\frac{4}{3}\right)^n \frac{3}{n+3}\right]\cdots = n!.$$

Leaving aside all delicate questions as to the convergence of the infinite product, the reader can verify this equation by cancelling out all the common factors which appear in the top and bottom of the left-hand side. Moreover, the left-hand side is defined (at least formally) for all kinds of n other than negative integers. Euler noticed also that when the value $n=\frac{1}{2}$ is inserted, the left-hand side yields (after a bit of manipulation) the famous infinite product of the Englishman John Wallis (1616–1703):

$$\text{(2)} \qquad \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\left(\frac{8\cdot 8}{7\cdot 9}\right)\cdots = \pi/2.$$

With this discovery Euler could have stopped. His problem was solved. Indeed, the whole theory of the gamma function can be based on the infinite product (1) which today is written more conventionally as

$$\text{(3)} \qquad \lim_{m\to\infty} \frac{m!(m+1)^n}{(n+1)(n+2)\cdots(n+m)}.$$

However, he went on. He observed that his product displayed the following curious phenomenon: for some values of n, namely integers, it yielded integers, whereas for another value, namely $n=\frac{1}{2}$, it yielded an expression involving π. Now π meant circles and their quadrature, and quadratures meant integrals, and he was familiar with integrals which exhibited the same phenomenon. It therefore occurred to him to look for a transformation which would allow him to express his product as an integral.

He took up the integral $\int_0^1 x^e(1-x)^n dx$. Special cases of it had already been discussed by Wallis, by Newton, and by Stirling. It was a troublesome integral to handle, for the indefinite integral is not always an elementary function of x. Assuming that n is an integer, but that e is an arbitrary value, Euler expanded $(1-x)^n$ by the binomial theorem, and without difficulty found that

$$\text{(4)} \qquad \int_0^1 x^e(1-x)^n dx = \frac{1\cdot 2\cdots n}{(e+1)(e+2)\cdots(e+n+1)}.$$

Euler's idea was now to isolate the $1\cdot 2\cdots n$ from the denominator so that he would have an expression for $n!$ as an integral. He proceeds in this way. (Here we follow Euler's own formulation and nomenclature, marking with an * those formulas which occur in the original paper. Euler wrote a plain $\int$ for $\int_0^1$.) He substituted f/g for e and found

$$(5) \qquad \int_0^1 x^{f/g}(1-x)^n dx = \frac{g^{n+1}}{f+(n+1)g}\cdot\frac{1\cdot 2\cdot\cdots\cdot n}{(f+g)(f+2\cdot g)\cdots(f+n\cdot g)}.$$

And so,

$$(6)^* \qquad \frac{1\cdot 2\cdots n}{(f+g)(f+2\cdot g)\cdots(f+n\cdot g)} = \frac{f+(n+1)g}{g^{n+1}}\int x^{f/g}dx(1-x)^n.$$

He observed that he could isolate the $1\cdot 2\cdots n$ if he set $f=1$ and $g=0$ in the left-hand member, but that if he did so, he would obtain on the right an indeterminate form which he writes quaintly as

$$(7)^* \qquad \int \frac{x^{1/0}dx(1-x)^n}{0^{n+1}}.$$

He now proceeded to find the value of the expression (7)*. He first made the substitution $x^{g/(f+g)}$ in place of x. This gave him

$$(8)^* \qquad \frac{g}{f+g}\, x^{-f/(g+f)}dx$$

in place of dx and hence, the right-hand member of (6)* becomes

$$(9)^* \qquad \frac{f+(n+1)g}{g^{n+1}}\int \frac{g}{f+g}\, dx(1-x^{g/(f+g)})^n.$$

Once again, Euler made a trial setting of $f=1$, $g=0$ having presumably reduced this integral first to

$$(10) \qquad \frac{f+(n+1)g}{(f+g)^{n+1}}\int_0^1 \left(\frac{1-x^{g/(f+g)}}{g/(f+g)}\right)^n dx,$$

and this yielded the indeterminate

$$(11)^* \qquad \int dx\frac{(1-x^0)^n}{0^n}.$$

He now considered the related expression $(1-x^z)/z$, for vanishing z. He differentiated the numerator and denominator, as he says, by a known (l'Hospital's) rule and obtained

$$(12)^* \qquad \frac{-x^z dz lx}{dz} \qquad (lx = \log x),$$

which for $z=0$ produced $-lx$. Thus,

$$(13)^* \qquad (1-x^0)/0 = -lx$$

and

$$(1 - x^0)^n/0^n = (-lx)^n. \tag{14*}$$

He therefore concluded that

$$n! = \int_0^1 (-\log x)^n dx. \tag{15}$$

This gave him what he wanted, an expression for $n!$ as an integral wherein values other than positive integers may be substituted. The reader is encouraged to formulate his own criticism of Euler's derivation.

Students in advanced calculus generally meet Euler's integral first in the form

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \qquad e = 2.71828 \cdots. \tag{16}$$

This modification of the integral (15) as well as the Greek Γ is due to Adrien Marie Legendre (1752–1833). Legendre calls the integral (4) with which Euler started his derivation the first Eulerian integral and (15) the second Eulerian integral. The first Eulerian integral is currently known as the Beta function and is now conventionally written

$$B(m, n) = \int_0^1 x^{m-1}(1 - x)^{n-1}dx. \tag{17}$$

With the tools available in advanced calculus, it is readily established (how easily the great achievements of the past seem to be comprehended and duplicated!) that the integral possesses meaning when $x>0$ and thus yields a certain function $\Gamma(x)$ defined for these values. Moreover,

$$\Gamma(n + 1) = n! \tag{18}$$

whenever n is a positive integer.* It is further established that for all $x>0$

$$x\Gamma(x) = \Gamma(x + 1). \tag{19}$$

This is the so-called recurrence relation for the gamma function and in the years following Euler it plays, as we shall see, an increasingly important role in its theory. These facts, plus perhaps the relationship between Euler's two types of integrals

$$B(m, n) = \Gamma(m)\Gamma(n)/\Gamma(m + n) \tag{20}$$

and the all important Stirling formula

* Legendre's notation shifts the argument. Gauss introduced a notation $\pi(x)$ free of this defect. Legendre's notation won out, but continues to plague many people. The notations Γ, π, and ! can all be found today.

$$\Gamma(x) \sim e^{-x}x^{x-1/2}\sqrt{(2\pi)}, \tag{21}$$

which gives us a relatively simple approximate expression for $\Gamma(x)$ when x is large, are about all that advanced calculus students learn of the gamma function. Chronologically speaking, this puts them at about the year 1750. The play has hardly begun.

Just as the simple desire to extend factorials to values in between the integers led to the discovery of the gamma function, the desire to extend it to negative values and to complex values led to its further development and to a more profound interpretation. Naive questioning, uninhibited play with symbols may have been at the very bottom of it. What is the value of $(-5\frac{1}{2})!$? What is the value of $\sqrt{(-1)}!$? In the early years of the 19th century, the action broadened and moved into the complex plane (the set of all numbers of the form $x+iy$, where $i=\sqrt{(-1)}$) and there it became part of the general development of the theory of functions of a complex variable that was to form one of the major chapters in mathematics. The move to the complex plane was initiated by Karl Friedrich Gauss (1777–1855), who began with Euler's product as his starting point. Many famous names are now involved and not just one stage of action but many stages. It would take too long to record and describe each forward step taken. We shall have to be content with a broader picture.

Three important facts were now known: Euler's integral, Euler's product, and the functional or recurrence relationship $x\Gamma(x)=\Gamma(x+1)$, $x>0$. This last is the generalization of the obvious arithmetic fact that for positive integers, $(n+1)n!=(n+1)!$ It is a particularly useful relationship inasmuch as it enables us by applying it over and over again to reduce the problem of evaluating a factorial of an arbitrary real number whole or otherwise to the problem of evaluating the factorial of an appropriate number lying between 0 and 1. Thus, if we write $n=4\frac{1}{2}$ in the above formula we obtain $(4\frac{1}{2}+1)!=5\frac{1}{2}(4\frac{1}{2})!$ If we could only find out what $(4\frac{1}{2})!$ is, then we would know that $(5\frac{1}{2})!$ is. This process of reduction to lower numbers can be kept up and yields

$$(5\tfrac{1}{2})! = (3/2)(5/2)(7/2)(9/2)(11/2)(1/2)! \tag{22}$$

and since we have $(\frac{1}{2})!=\frac{1}{2}\sqrt{\pi}$ from (1) and (2), we can now compute our answer. Such a device is obviously very important for anyone who must do calculations with the gamma function. Other information is forthcoming from the recurrence relationship. Though the formula $(n+1)n!=(n+1)!$ as a condensation of the arithmetic identity $(n+1)\cdot 1\cdot 2 \cdots n=1\cdot 2 \cdots n\cdot(n+1)$ makes sense only for $n=1$, 2, etc., blind insertions of other values produce interesting things. Thus, inserting $n=0$, we obtain $0!=1$. Inserting successively $n=-5\frac{1}{2}$, $n=-4\frac{1}{2}, \cdots$ and reducing upwards, we discover

$$(-5\tfrac{1}{2})! = (2/1)(-2/1)(-2/3)(-2/5)(-2/7)(-2/9)(1/2)! \tag{23}$$

Since we already know what $(\frac{1}{2})!$ is, we can compute $(-5\frac{1}{2})!$ In this way the recurrence relationship enables us to compute the values of factorials of negative numbers.

Turning now to Euler's integral, it can be shown that for values of the variable less than 0, the usual theorems of analysis do not suffice to assign a meaning to the integral, for it is divergent. On the other hand, it is meaningful and yields a value if one substitutes for x any complex number of the form $a+bi$ where $a>0$. With such substitutions the integral therefore yields a complex-valued function which is defined for all complex numbers in the right-half of the complex plane and which coincides with the ordinary gamma function for real values. Euler's product is even stronger. With the exception of $0, -1, -2, \cdots$ any complex number whatever can be inserted for the variable and the infinite product will converge, yielding a value. And so it appears that we have at our disposal a number of methods, conceptually and operationally different for extending the domain of definition of the gamma function. Do these different methods yield the same result? They do. But why?

The answer is to be found in the notion of an analytic function. This is the focal point of the theory of functions of a complex variable and an outgrowth of the older notion of an analytical expression. As we have hinted, earlier mathematics was vague about this notion, meaning by it a function which arose in a natural way in mathematical analysis. When later it was discovered by J. B. J. Fourier (1768–1830) that functions of wide generality and functions with unpleasant characteristics could be produced by the infinite superposition of ordinary sines and cosines, it became clear that the criterion of "arising in a natural way" would have to be dropped. The discovery simultaneously forced a broadening of the idea of a function and a narrowing of what was meant by an analytic function.

Analytic functions are not so arbitrary in their behavior. On the contrary, they possess strong internal ties. Defined very precisely as functions which possess a complex derivative or equivalently as functions which possess power series expansions $a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$ they exhibit the remarkable phenomenon of "action at a distance." This means that the behavior of an analytic function over any interval no matter how small is sufficient to determine completely its behavior everywhere else; its potential range of definition and its values are theoretically obtainable from this information. Analytic functions, moreover, obey the principle of the permanence of functional relationships; if an analytic function satisfies in some portions of its region of definition a certain functional relationship, then it must do so wherever it is defined. Conversely, such a relationship may be employed to extend its definition to unknown regions. Our understanding of the process of analytic continuation, as this phenomenon is known, is based upon the work of Bernhard Riemann (1826–1866) and Karl Weierstrass (1815–1897). The complex-valued function which results from the substitution of complex numbers into Euler's integral is an analytic function. The function which emerges from Euler's product is an analytic function. The recurrence relationship for the gamma function if satisfied in some region must be satisfied in any other region to which the function can be "continued" analytically and indeed may be employed to effect such extensions. All portions of the complex plane, with the exception of the values $0, -1, -2, \cdots$ are accessible to the complex gamma function which has become the unique, analytic extension to complex values of Euler's integral (see Fig. 3).

THE GAMMA FUNCTION

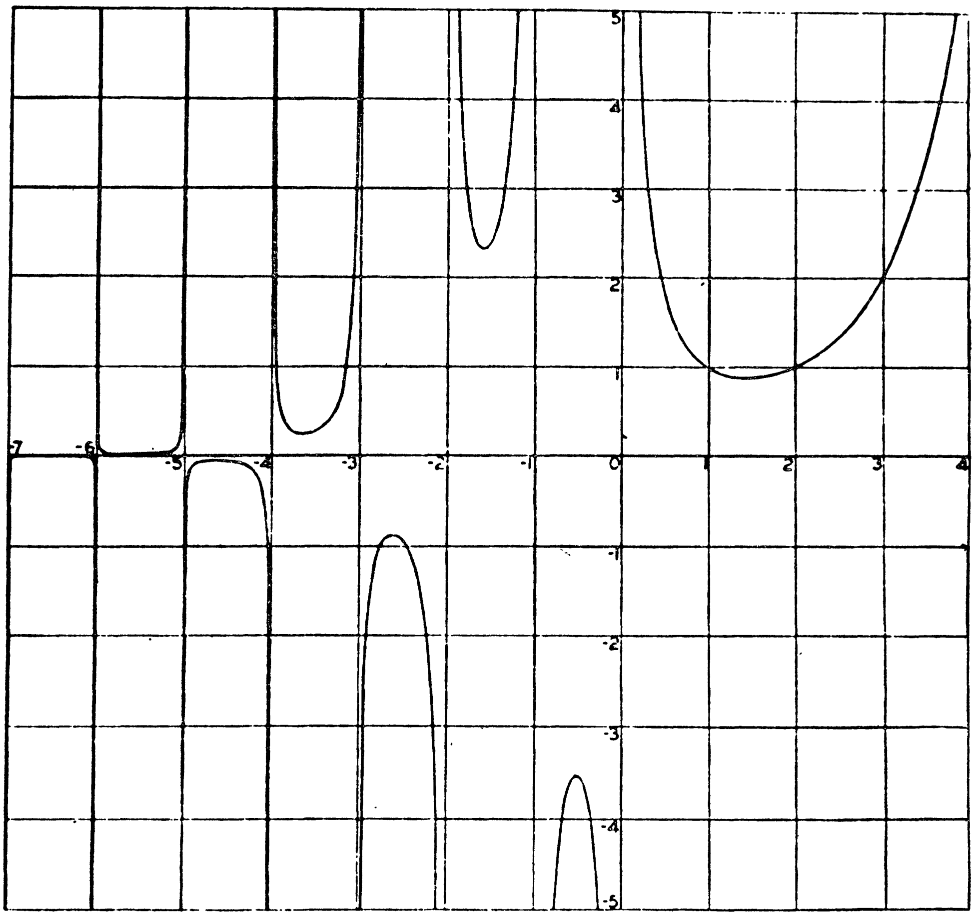

FIG. 2*

To understand why there should be excluded points observe that $\Gamma(x) = \Gamma(x+1)/x$, and as x approaches 0, we obtain $\Gamma(0) = 1/0$. This is $+\infty$ or $-\infty$ depending whether 0 is approached through positive or negative values. The

* From: H. T. Davis, Tables of the Higher Mathematical Functions, vol. I, Bloomington, Indiana, 1933.

functional equation (19) then, induces this behavior over and over again at each of the negative integers. The (real) gamma function is comprised of an infinite number of disconnected portions opening up and down alternately. The portions corresponding to negative values are each squeezed into an infinite strip one unit in width, but the major portion which corresponds to positive x and which contains the factorials is of infinite width (see Fig. 2). Thus, there are excluded points for the gamma function at which it exhibits from the ordinary (real variable) point of view a somewhat unpleasant and capricious behavior.

THE ABSOLUTE VALUE OF THE COMPLEX GAMMA FUNCTION, EXHIBITING THE POLES AT THE NEGATIVE INTEGERS

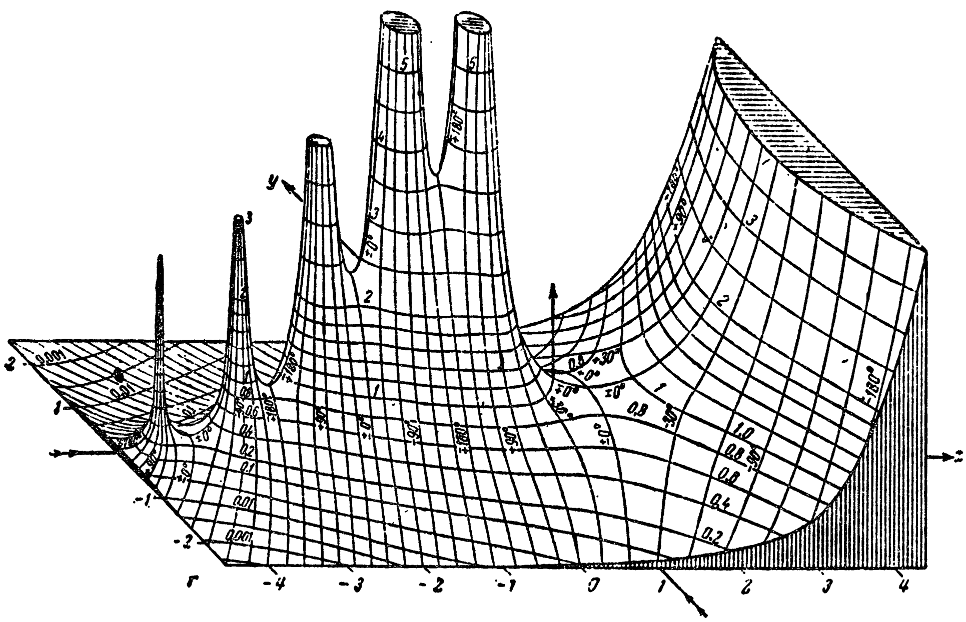

FIG. 3*

But from the complex point of view, these points of singular behavior (singular in the sense of Sherlock Holmes) merit special study and become an important part of the story. In pictures of the complex gamma function they show up as an infinite row of "stalagmites," each of infinite height (the ones in the figure are truncated out of necessity) which become more and more needlelike as they go out to infinity (see Fig. 3). They are known as poles. Poles are points where the function has an infinite behavior of especially simple type, a behavior which is akin to that of such simple functions as the hyperbola $y=1/x$ at $x=0$ or of $y=\tan x$ at $x=\pi/2$. The theory of analytic functions is especially interested

* From: E. Jahnke and F. Emde, Tafeln höherer Funktionen, 4th ed., Leipzig, 1948.

in singular behavior, and devotes much space to the study of the singularities. Analytic functions possess many types of singularity but those with only poles are known as meromorphic. There are also functions which are lucky enough to possess no singularities for finite arguments. Such functions form an elite and are known as entire functions. They are akin to polynomials while the meromorphic functions are akin to the ratio of polynomials. The gamma function is meromorphic. Its reciprocal, $1/\Gamma(x)$, has on the contrary no excluded points. There is no trouble anywhere. At the points $0, -1, -2, \cdots$ it merely becomes zero. And the zero value which occurs an infinity of times, is strongly reminiscent of the sine.

In the wake of the extension to the complex many remarkable identities emerge, and though some of them can and were obtained without reference to complex variables, they acquire a far deeper and richer meaning when regarded from the extended point of view. There is the reflection formula of Euler

$$\Gamma(z)\Gamma(1 - z) = \pi/\sin \pi z. \tag{24}$$

It is readily shown, using the recurrence relation of the gamma function, that the product $\Gamma(z)\Gamma(1-z)$ is a periodic function of period 2; but despite the fact that $\sin \pi z$ is one of the simplest periodic functions, who could have anticipated the relationship (24)? What, after all, does trigonometry have to do with the sequence 1, 2, 6, 24 which started the whole discussion? Here is a fine example of the delicate patterns which make the mathematics of the period so magical. From the complex point of view, a partial reason for the identity lies in the similarity between zeros of the sine and the poles of the gamma function.

There is the duplication formula

$$\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z + \tfrac{1}{2}) \tag{25}$$

discovered by Legendre and extended by Gauss in his researches on the hypergeometric function to the multiplication formula

$$\Gamma(nz) = (2\pi)^{1/2(1-n)}n^{nz-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{n}\right)\Gamma\left(z + \frac{2}{n}\right)\cdot \cdots \cdot\Gamma\left(\frac{z + n - 1}{n}\right). \tag{26}$$

There are pretty formulas for the derivatives of the gamma function such as

$$d^2 \log \Gamma(z)/dz^2 = \frac{1}{z^2} + \frac{1}{(z + 1)^2} + \frac{1}{(z + 2)^2} + \cdots. \tag{27}$$

This is an example of a type of infinite series out of which G. Mittag-Leffler (1846–1927) later created his theory of partial fraction developments of meromorphic functions. There is the intimate relationship between the gamma function and the zeta function which has been of fundamental importance in studying the distribution of the prime numbers,

$$\zeta(z) = \zeta(1-z)\Gamma(1-z)2^z\pi^{z-1}\sin\tfrac{1}{2}\pi z, \tag{28}$$

where

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots. \tag{29}$$

This formula has some interesting history related to it. It was first proved by Riemann in 1859 and was conventionally attributed to him. Yet in 1894 it was discovered that a modified version of the identity appears in some work of Euler which had been done in 1749. Euler did not claim to have proved the formula. However, he "verified" it for integers, for $\frac{1}{2}$, and for 3/2. The verification for $\frac{1}{2}$ is by direct substitution, but for all the other values, Euler works with divergent infinite series. This was more than 100 years in advance of a firm theory of such series, but with unerring intuition, he proceeded to sum them by what is now called the method of Abel summation. The case 3/2 is even more interesting. There, invoking both divergent series and numerical evaluation, he came out with numerical agreement to 5 decimal places! All this work convinced him of the truth of his identity. Rigorous modern proofs do not require the theory of divergent series, but the notions of analytic continuation are crucial.

In view of the essential unity of the gamma function over the whole complex plane it is theoretically and aesthetically important to have a formula which works for all complex numbers. One such formula was supplied in 1848 by F. W. Newman:

$$1/\Gamma(z) = ze^{\gamma z}\{(1+z)e^{-z}\}\{(1+z/2)e^{-z/2}\}\cdots, \quad \text{where } \gamma = .57721\,56649\cdots. \tag{30}$$

This formula is essentially a factorization of $1/\Gamma(z)$ and is much the same as a factorization of polynomials. It exhibits clearly where the function vanishes. Setting each factor equal to zero we find that $1/\Gamma(z)$ is zero for $z=0$, $z=-1$, $z=-2, \cdots$. In the hands of Weierstrass, it became the starting point of his particular discussion of the gamma function. Weierstrass was interested in how functions other than polynomials may be factored. A number of isolated factorizations were then known. Newman's formula (30) and the older factorization of the sine

$$\sin\pi z = \pi z(1-z^2)\left(1-\frac{z^2}{4}\right)\left(1-\frac{z^2}{9}\right)\cdots \tag{31}$$

are among them. The factorization of polynomials is largely an algebraic matter but the extension to functions such as the sine which have an infinity of roots required the systematic building up of a theory of infinite products. In 1876 Weierstrass succeeded in producing an extensive theory of factorizations which included as special cases these well-known infinite products, as well as certain doubly periodic functions.

In addition to showing the roots of $1/\Gamma(z)$, formula (30) does much more.

It shows immediately that the reciprocal of the gamma function is a much less difficult function to deal with than the gamma function itself. It is an entire function, that is, one of those distinguished functions which possesses no singularities whatever for finite arguments. Weierstrass was so struck by the advantages to be gained by starting with $1/\Gamma(z)$ that he introduced a special notation for it. He called $1/\Gamma(u+1)$ the *factorielle* of u and wrote $Fc(u)$.

The theory of functions of a complex variable unifies a hotch-potch of curves and a patchwork of methods. Within this theory, with its highly developed studies of infinite series of various types, was brought to fruition Stirling's unsuccessful attempts at solving the interpolation problem for the factorials. Stirling had done considerable work with infinite series of the form

$$A+Bz+Cz(z-1)+Dz(z-1)(z-2)+\cdots.$$

This series is particularly useful for fitting polynomials to values given at the integers $z=0, 1, 2, \cdots$. The method of finding the coefficients $A, B, C, \cdots$ was well known. But when an infinite amount of fitting is required, much more than simple formal work is needed, for we are then dealing with a bona fide infinite series whose convergence must be investigated. Starting from the series $1, 2, 6, 24, \cdots$, Stirling found interpolating polynomials via the above series. The resultant infinite series is divergent. The factorials grow too rapidly in size. Stirling realized this and put out the suggestion that if perhaps one started with the logarithms of the factorials instead of the factorials themselves the size might be cut down sufficiently for one to do something. There the matter rested until 1900 when Charles Hermite (1822–1901) wrote down the Stirling series for $\log \Gamma(1+z)$:

$$(32)\quad \log \Gamma(1+z) = \frac{z(z-1)}{1\cdot 2}\log 2 + \frac{z(z-1)(z-2)}{1\cdot 2\cdot 3}(\log 3 - 2\log 2) + \cdots$$

and showed that this identity is valid whenever z is a complex number of the form $a+ib$ with $a>0$. The identity itself could have been written down by Stirling, but the proof would have been another matter. An even simpler starting point is the function $\psi(z)=(d/dz)\log\Gamma(z)$, now known as the digamma or psi function. This leads to the Stirling series

$$(33)\quad \begin{aligned} &\frac{d}{dz}\log\Gamma(z) \\ &\qquad = -\gamma + (z-1) - \frac{(z-1)(z-2)}{2\cdot 2!} + \frac{(z-1)(z-2)(z-3)}{3\cdot 3!}\cdots, \end{aligned}$$

which in 1847 was proved convergent for $a>0$ by M. A. Stern, a teacher of Riemann. All these matters are today special cases of the extensive theory of the convergence of interpolation series.

Functions are the building blocks of mathematical analysis. In the 18th and 19th centuries mathematicians devoted much time and loving care to developing the properties and interrelationships between special functions. Powers, roots, algebraic functions, trigonometric functions, exponential functions, logarithmic functions, the gamma function, the beta function, the hypergeometric function, the elliptic functions, the theta function, the Bessel function, the Matheiu function, the Weber function, Struve function, the Airy function, Lamé functions, literally hundreds of special functions were singled out for scrutiny and their main features were drawn. This is an art which is not much cultivated these days. Times have changed and emphasis has shifted. Mathematicians on the whole prefer more abstract fare. Large classes of functions are studied instead of individual ones. Sociology has replaced biography. The field of special functions, as it is now known, is left largely to a small but ardent group of enthusiasts plus those whose work in physics or engineering confronts them directly with the necessity of dealing with such matters.

The early 1950's saw the publication of some very extensive computations of the gamma function in the complex plane. Led off in 1950 by a six-place table computed in England, it was followed in Russia by the publication of a very extensive six-place table. This in turn was followed in 1954 by the publication by the National Bureau of Standards in Washington of a twelve-place table. Other publications of the complex gamma function and related functions have appeared in this country, in England, and in Japan. In the past, the major computations of the gamma function had been confined to real values. Two fine tables, one by Gauss in 1813 and one by Legendre in 1825, seemed to answer the mathematical needs of a century. Modern technology had also caught up with the gamma function. The tables of the 1800's were computed laboriously by hand, and the recent ones by electronic digital computers.

But what touched off this spate of computational activity? Until the initial labors of H. T. Davis of Indiana University in the early 1930's, the complex values of the gamma function had hardly been touched. It was one of those curious turns of events wherein the complex gamma function appeared in the solution of various theoretical problems of atomic and nuclear theory. For instance, the radial wave functions for positive energy states in a Coulomb field leads to a differential equation whose solution involves the complex gamma function. The complex gamma function enters into formulas for the scattering of charged particles, for the nuclear forces between protons, in Fermi's approximate formula for the probability of β-radiation, and in many other places. The importance of these problems to physicists has had the side effect of computational mathematics finally catching up with two and a quarter centuries of theoretical development.

As analysis grew, both creating special functions and delineating wide classes of functions, various classifications were used in order to organize them for purposes of convenient study. The earlier mathematicians organized functions from without, operationally, asking what operations of arithmetic or calculus had to be performed in order to achieve them. Today, there is a much greater tendency to look at functions from within, organically, considering their construction as achieved and asking what geometrical characteristics they possess. In the earlier classification we have at the lowest and most accessible level, powers, roots, and all that could be concocted from them by ordinary algebraic manipulation. These came to be known as algebraic functions. The calculus, with its characteristic operation of taking limits, introduced logarithms and exponentials, the latter encompassing, as Euler showed, the sines and cosines of trigonometry which had been available from earlier periods of discovery. There is an impassable wall between the algebraic functions and the new limit-derived ones. This wall consists in the fact that try as one might to construct, say, a trigonometric function out of the finite material of algebra, one cannot succeed. In more technical language, the algebraic functions are closed with respect to the processes of algebra, and the trigonometric functions are forever beyond its pale. (By way of a simple analogy: the even integers are closed with respect to the operations of addition, subtraction, and multiplication; you cannot produce an odd integer from the set of even integers using these tools.) This led to the concept of transcendental functions. These are functions which are not algebraic. The transcendental functions count among their members, the trigonometric functions, the logarithms, the exponentials, the elliptic functions, in short, practically all the special functions which had been singled out for special study. But such an indescriminate dumping produced too large a class to handle. The transcendentals had to be split further for convenience. A major tool of analysis is the differential equation, expressing the relationship between a function and its rate of growth. It was found that some functions, say the trigonometric functions, although they are transcendental and do not therefore satisfy an algebraic equation, nonetheless satisfy a differential equation whose coefficients are algebraic. The solutions of algebraic differential equations are an extensive though not all-encompassing class of transcendental functions. They count among their members a good many of the special functions which arise in mathematical physics.

Where does the gamma function fit into this? It is not an algebraic function. This was recognized early. It is a transcendental function. But for a long while it was an open question whether the gamma function satisfied an algebraic differential equation. The question was settled negatively in 1887 by O. Hölder (1859–1937). It does not. It is of a higher order of transcendency. It is a so-called transcendentally transcendent function, unreachable by solving algebraic equations, and equally unreachable by solving algebraic differential equations. The subject has interested many people through the years and in 1925 Alexander Ostrowski, now Professor Emeritus of the University of Basel, Switzerland, gave an alternate proof of Hölder's theorem.

Problems of classification are extremely difficult to handle. Consider, for instance, the following: Can the equation x^7+8x+1 be solved with radicals? Is π transcendental? Can $\int dx/\sqrt{(x^3+1)}$ be found in terms of specified elementary functions? Can the differential equation $dy/dx=(1/x)+(1/y)$ be resolved with quadratures? The general problems of which these are representatives are even today far from solved and this despite famous theories such as Galois Theory, Lie theory, theory of Abelian integrals which have derived from such simple questions. Each individual problem may be a one-shot affair to be solved by individual methods involving incredible ingenuity.

HADAMARD'S FACTORIAL FUNCTION

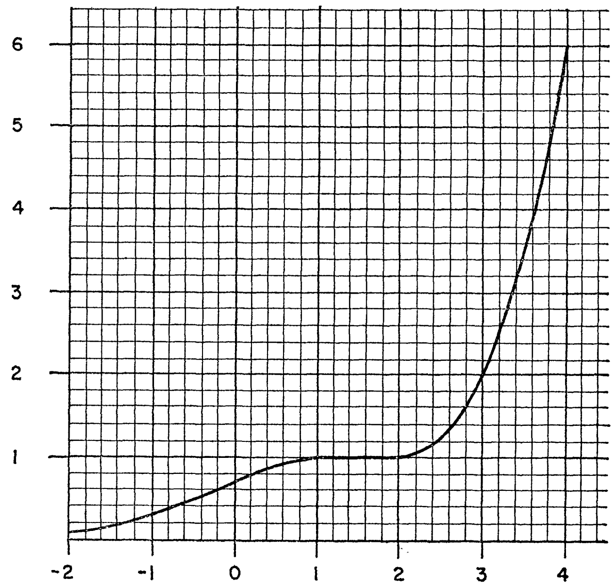

FIG. 4

There are infinitely many functions which produce factorials. The function

$$F(x) = (1/\Gamma(1-x))\,(d/dx)\log\{\Gamma((1-x)/2)/\Gamma(1-x/2)\}$$

is an entire analytic function which coincides with the gamma function at the positive integers. It satisfies the functional equation $F(x+1)=xF(x)+(1/\Gamma(1-x))$.

We return once again to our interpolation problem. We have shown how, strictly speaking, there are an unlimited number of solutions to this problem. To drive this point home, we might mention a curious solution given in 1894 by Jacques Hadamard (1865–). Hadamard found a relatively simple formula involving the gamma function which also produces factorial values at the positive integers. (See Figs. 1 and 4.) But Hadamard's function

$$y = \frac{1}{\Gamma(1-x)} \frac{d}{dx} \log\left[\Gamma\left(\frac{1-x}{2}\right) \Big/ \Gamma\left(1-\frac{x}{2}\right)\right], \tag{34}$$

in strong contrast to the gamma function itself, possesses no singularities anywhere in the finite complex plane. It is an entire analytic solution to the interpolation problem and hence, from the function theoretic point of view, is a simpler solution. In view of all this ambiguity, why then should Euler's solution be considered the solution par excellence?

From the point of view of integrals, the answer is clear. Euler's integral appears everywhere and is inextricably bound to a host of special functions. Its frequency and simplicity make it fundamental. When the chips are down, it is the very form of the integral and of its modifications which lend it utility and importance. For the interpolatory point of view, we can make no such claim. We must take a deeper look at the gamma function and show that of all the solutions of the interpolation problem, it, in some sense, is the simplest. This is partially a matter of mathematical aesthetics.

A PSEUDOGAMMA FUNCTION

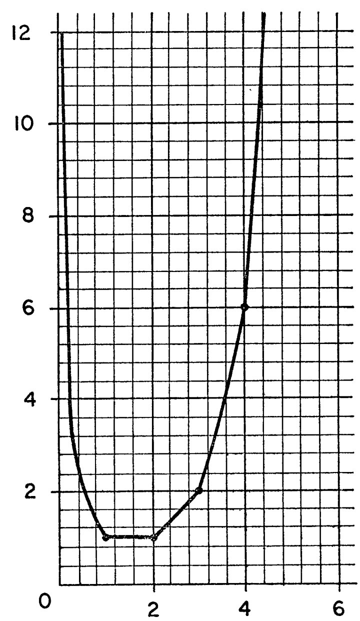

FIG. 5

The function illustrated produces factorials, satisfies the functional equation of the gamma function, and is convex.

We have already observed that Euler's integral satisfies the fundamental recurrence equation, $x\Gamma(x)=\Gamma(x+1)$, and that this equation enables us to compute all the real values of the gamma function from knowledge merely of its values in the interval from 0 to 1. Since the solution to the interpolation problem is not determined uniquely, it makes sense to add to the problem more conditions and to inquire whether the augmented problem then possesses a unique solution. If it does, we will hope that the solution coincides with Euler's. The recurrence relationship is a natural condition to add. If we do so, we find that the gamma function is again not the only function which satisfies this recurrence relation and produces factorials. One may easily construct a "pseudo" gamma function $\Gamma_S(x)$ by defining it between, say, 1 and 2 in any way at all (subject only to $\Gamma_S(1)=1$, $\Gamma_S(2)=1$), and allowing the recurrence relationship to extend its values everywhere else.

If, for instance, we let $\Gamma_S(x)$ be 1 everywhere between 1 and 2, the recurrence relation leads us to the function (see Fig. 5).

$$\begin{aligned} \Gamma_S(x) &= 1/x & 0 < x \leqq 1; \\ \Gamma_S(x) &= 1, & 1 \leqq x \leqq 2; \\ \Gamma_S(x) &= x - 1, & 2 \leqq x \leqq 3; \\ \Gamma_S(x) &= (x-1)(x-2), & 3 \leqq x \leqq 4; \cdots . \end{aligned} \tag{35}$$

We might end up with a fairly weird result, depending upon what we start with. Even if we require the final result to be an analytic function, there are ways of doing it. For instance, take any function which is both analytic and periodic with period 1. Call it $p(x)$. Make sure that $p(1)=1$. The function $1+\sin 2\pi x$ will do for $p(x)$. Now multiply the ordinary gamma function $\Gamma(x)$ by $p(x)$ and the result $\Gamma(x)p(x)$ will be a "pseudo" gamma function which is analytic, satisfies the recurrence relation, and produces factorials! Thus, we still do not have enough conditions. We must augment the problem again. But what to add?

By the middle of the 19th century it was recognized that Euler's gamma function was the only continuous function which satisfied simultaneously the recurrence relationship, the reflection formula and the multiplication formula. Weierstrass later showed that the gamma function was the only continuous solution of the recurrence relationship for which $\{\Gamma(x+n)\}/\{(n-1)n^x\}\to 1$ for all x. These conditions added to the interpolation problem will serve to produce a unique solution and one which coincides with Euler's. But they appear too heavy and too much like Monday morning quarterbacking. That is to say, the added conditions are hardly "natural" for they are tied in with the deeper analytical properties of the gamma function. The search went on.

Aesthetic conditions were not to be found in the older, analytic considerations, but in a newer, inner, organic approach to function theory which was developing at the turn of the century. Backed up by Cantor's set theory and an emerging theory of topology, the new function theory looked not so much at equations and identities as at the fundamental geometrical properties. The desired condition was found in notions of convexity. A curve is convex if the following is true of it: take any two points on the curve and join them by a straight line; then the portion of the curve between the points lies below the line. A convex curve does not wiggle; it cannot look like a camel's back. At the turn of the century, convexity was in the mathematical air. It was found to be intrinsic to many diverse phenomena. Over the period of a generation, it was sought out, it was generalized, it was abstracted, it was investigated for its own sake, it was applied. Called to attention by the work of H. Brunn in 1887 and of H. Minkowski in 1903 on convex bodies and given an independent interest in 1906 by the work of J. L. W. V. Jensen, the idea of convexity spread and established itself in mean value theory, in potential theory, in topology, and most recently in game theory and linear programming. At the turn of the century then, an application of convexity to the gamma function would have been natural and in order.

The individual curves which make up the gamma function are all convex. A glance at Figure 2 shows this to be true. If, as in the previous paragraph, a pseudogamma function satisfying the recurrence formula were produced by introducing the ripple $1+\sin 2\pi x$ as a factor, it would no longer be true. It must have occurred to many mathematicians to find out whether the gamma function is the only function which yields the factorial values, satisfies the recurrence relation, and is convex downward for $x>0$. Unfortunately, this is not true. Figure 5 shows a pseudogamma function which possesses just these properties. It remained until 1922 to discover a correct formulation. But it was not at too far a distance. The gamma function is not only convex, it is also logarithmically convex. That is to say, the graph of $\log \Gamma(x)$ is also convex down for $x>0$. This fact is implicit in formula (27). Logarithmic convexity is a stronger condition than ordinary convexity for logarithmic convexity implies, but is not implied by, ordinary convexity. Now Harald Bohr and J. Mollerup were able to show the surprising fact that the gamma function is the only function which satisfies the recurrence relationship and is logarithmically convex. The original proof was simplified several years later by Emil Artin, now professor at Princeton University, and the theorem together with Artin's method of proof now constitute the Bohr-Mollerup-Artin theorem. Its precise wording is this:

The Euler gamma function is the only function defined for $x>0$ which is positive, is 1 at $x=1$, satisfies the functional equation $x\Gamma(x)=\Gamma(x+1)$, and is logarithmically convex.

This theorem is at once so striking and so satisfying that the contemporary synod of abstractionists who write mathematical canon under the pen name of N. Bourbaki has adopted it as the starting point for its exposition of the gamma function. The proof: one page; the discovery: 193 years.

There is much that we know about the gamma function. Since Euler's day more than 400 major papers relating to it have been written. But a few things remain that we do not know and that we would like to know. Perhaps the hardest of the unsolved problems deal with questions of rationality and transcendentality. Consider, for instance, the number $\gamma=.57721\cdots$ which appears in formula (30). This is the Euler-Mascheroni constant. Many different expressions can be given for it. Thus,

$$\gamma = -\, d\Gamma(x)/dx\,\big|_{x=1}, \tag{36}$$

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \log n. \tag{37}$$

Though the numerical value of γ is known to hundreds of decimal places, it is not known at the time of writing whether γ is or is not a rational number. Another problem of this sort deals with the values of the gamma function itself. Though, curiously enough, the product $\Gamma(1/4)/\sqrt[4]{\pi}$ can be proved to be transcendental, it is not known whether $\Gamma(1/4)$ is even rational.

George Gamow, the distinguished physicist, quotes Laplace as saying that when the known areas of a subject expand, so also do its frontiers. Laplace evidently had in mind the picture of a circle expanding in an infinite plane. Gamow disputes this for physics and has in mind the picture of a circle expanding on a spherical surface. As the circle expands, its boundary first expands, but later contracts. This writer agrees with Gamow as far as mathematics is concerned. Yet the record is this: each generation has found something of interest to say about the gamma function. Perhaps the next generation will also.

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LINEAR DIFFERENTIAL OR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

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1. Introduction.* Solutions of a system of linear differential or difference equations with *real* constant coefficients a_{ij}, such as

$$(1) \qquad dx_i/dt = \sum_{j=1}^{n} a_{ij}x_j \quad \text{and} \quad x_i(t+h) = \sum_{j=1}^{n} a_{ij}x_j(t),$$

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