The Dottie Number

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The Dottie number was the nickname among my graduate school friends for the unique real root of $\cos(x) = x$. The story goes that Dottie, a professor of French, noticed that whenever she put a number in the calculator and hit the `cos` button over and over again, the number on the screen always went to the same value, about 0.739085 . . . . She asked her math-professor husband why the calculator did this no matter what number she started with. He looked. He tried it. He said he had no idea, at least not that day. The next day he realized not only what was happening, but that his wife had found a beautiful, simple example of a global attractor.

Dottie was computing the sequence defined by the recursion relation, $s_{n+1} = \cos(s_n)$. This sequence has a unique fixed point at the root of $\cos(x) = x$ where x is, of course, expressed in radians. Moreover, the domain of attraction for this fixed point was the entire real line. So any value used for s_0 will generate a sequence that converges to the same root.

In my own teaching, I have been inserting the Dottie number story into my courses. I use the story to teach students that when they get stuck on a problem, it is okay to stop and come back to the problem later, refreshed. I also follow up the story with a homework problem related to the Dottie number. For convenience, I will denote the Dottie number by d in this paper.

I have first semester Calculus students demonstrate that the Dottie number is the unique root of $\cos(x) = x$. They then apply the Intermediate Value Theorem to the function, $f(x) = x - \cos(x)$, on the interval $(-\pi/2, 3\pi/2)$ to show the existence of d. Using Rolle's Theorem, they give a proof by contradiction that d is the unique root on that interval. Then they have to show that there are no roots outside the interval.

In differential equations, I show students how to use Euler's method to find roots of a function. The differential equation $x' = f(x)$ has fixed points or equilibrium solutions at the roots of $f(x)$. Using Euler's method, a numerical approximation to $x' = f(x, t)$ for the initial condition $x(t_0) = x_0$ can be found from the series, $t_{n+1} = t_n + \Delta t$ and $x_{n+1} = x_n + \Delta t f(x_n, t_n)$ where Δt is a given parameter. I have students find a numerical approximation to the Dottie number by finding a solution to $x' = \cos(x) - x$ with $x(0) = 0$ and $\Delta x = 1$. I ask them to write a report about what they are doing and what they found.

In my Problems in Math course or an equivalent Advanced Calculus course, I talk about inverse power series. Looking for a good problem, I discovered that the Dottie number has a power series in odd powers of π. I got my students to prove that the Dottie number can be written in the form

$$\mathsf{d} = \sum_{n=0}^{\infty} a_n \pi^{2n+1}$$

where each coefficient, a_n is rational. For the function $f(x) = x - \cos(x)$, $f(\mathsf{d}) = 0$. Hence $f^{-1}(0) = \mathsf{d}$ where f^{-1} is defined in the interval $(-\pi/2, 3\pi/2)$. Using the Taylor series for f^{-1} about $\pi/2$, I have students then find the first two coefficients,

a_0 and a_1 in the power series for $f^{-1}(0)$, the Dottie Number. This method requires students to compute the nth derivative of f^{-1} at $\pi/2$ in terms of the first n derivatives of f at $\pi/2$. By the way, $a_0 = 1/4$ and $a_1 = -1/768$.

In my Chaos course I make sure to tell the story of the Dottie number right away—without the punch line—and ask them what is going on. They repeat the experiment themselves and make conjectures. We come back to this example every time we learn one new element of finding attracting fixed points and domains of attraction.

In my complex variables class, we show that $\cos(z) = z$ has infinitely many complex roots that come in complex conjugate pairs (except for the Dottie number). We do this by studying the complex form of $\cos(z) = (e^{iz} + e^{-iz})/2$. Later in the complex variables course, I introduce complex dynamics. When we get to Julia sets, we compute the Julia set numerically for $\cos(z)$ and see the domain of attraction for d. The other roots are repelling.

It is unlikely that the Dottie number will enter the annals of great constants alongside e, π, the Golden Ratio and many others. However, the Dottie number and its story might make good teaching elements for others out there. I also imagine there are many other interesting facets of the Dottie number yet to be discovered. I look forward to hearing about what you find.

$$\mathsf{d} = 0.73908\ 51332\ 15160\ 64165\ 53120\ 87673\ 87340\ 40134\ 11758\ 90075$$
$$74649\ 65680\ 63577\ 32846\ 54883\ 54759\ 45993\ 76106\ 93176\ 65318\ldots$$

Proof Without Words: Alternating Sums of Squares of Odd Numbers

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If n odd:

$$\sum_{k=1}^{n} (2k-1)^2 (-1)^{k-1} = 2n^2 - 1$$

E.g. $n = 5$: