INTEGRABLE SYSTEMS AND GROUP ACTIONS

EVA MIRANDA

Abstract. The main purpose of this paper is to present in a unified approach to see different results concerning group actions and integrable systems in symplectic, Poisson and contact manifolds. Rigidity problems for integrable systems in these manifolds will be explored from this perspective.

Contents

Date: November 15, 2011.

1991 *Mathematics Subject Classification.* 53D17.

Key words and phrases. integrable system, momentum map, Poisson manifold, Contact manifold, symplectic manifold, group action.

Partially supported by the DGICYT/FEDER project MTM2009-07594: Estructuras Geometricas: Deformaciones, Singularidades y Geometria Integral.

1. Introduction

From the very beginning: symmetries, group actions and integrable systems have been close allies. The study of symmetries of the differential systems given by Hamilton's equation associated to an energy function H, led naturally to the existence of Poisson commuting functions and the method of integration by quadratures. When the number of commuting functions is maximal, the resolution by quadratures is possible and the set of commuting functions defines an integrable system on the phase space $T^*(\mathbb{R}^n)$. The idea can be naturally exported to any symplectic manifold and the reduction process can be seen as a reduction of the dynamics in the manifolds given by a Marsden-Weinstein reduction associated to a toric group.

One of the most striking and basic results in the theory of integrable systems on symplectic manifolds is the theorem of Liouville-Mineur-Arnold [53],[54],[2] which states that the foliation defined by a regular integrable system in the neighbourhood of a compact fibre is a fibration by tori and that the symplectic form can be given semilocally as a Darboux form. This theorem is achieved by constructing *action-angle* coordinates in a neighbhourhood of a regular fibre and proving a Darboux theorem in this coordinates. It is really intriguing to understand the difficulties in extending this theorem to a more general context. This is the case of including singularities into the picture or trying to construct global action-angle coordinates. The study of these problems concerning integrable systems is still a challenging issue for symplectic topologists, geometers and dynamicists.

Experts working in dynamical systems are interested in the properties of the Hamiltonian vector field and its flow. Topologists are interested in both constructing global examples of integrable systems and studying obstruction theories. Geometers are interested in understanding the geometrical structure (symplectic) of these objects and in proving classification theorems.

Symmetries are present in many physical problems and therefore they show up in integrable systems theory as well. The ace in the hole in the study of integrable systems is to look for symmetries. The very proof of Liouville-Mineur-Arnold uses this strategy. It finds a toric Hamiltonian action tangent to the fibres of the moment map. In this paper we will also consider additional symmetries showing up. We will try to study which properties hold for these additional symmetries. Those symmetries are encoded in actions of Lie groups.

Among all kind of symmetries the toric ones play a central rôle in this paper. Hamiltonian actions of tori in symplectic geometry have attracted the attention of many specialists. Along the way many results of symplectic uniqueness are obtained. A good example of this is Delzant's theorem [21] which enables to recover information of a compact $2n$-dimensional manifold by looking at the image of the moment map of a Hamiltonian torus action which is a convex polytope in $\mathbb{R}^n$. A lot of contributions in the area of Hamiltonian actions of Lie groups have been done ever since. Let us mention some of the references of the large list of results in that direction: the works of Lerman and Tolman to extend those result to symplectic orbifolds ([42]) and the works of Karshon and Tolman for complexity one Hamiltonian group actions ([38]) among many others. One of the current topics of interest are singularities of integrable systems. Besides the classical references on the symplectic geometry and topology of these singularities [27], [26], [52], [7], [79], and more recently the work in connection to (semi)-toric actions and singularities ([66], [67]).

The notion of integrability and its connection to group actions can be naturally studied in the contact context too. Toric contact manifolds have been largely studied by Banyaga and Molino [4, 5], Luzt [50] and Lerman [41]. In my thesis, I studied the integrable non-degenerate but not necessarily toric case. More recently, integrability in the contact context has been studied by Khesin and Tabaschnikov [22]. We will present classification results for integrable systems with non-degenerate singularities in contact manifolds and give complete proofs.

Last but not least, Poisson manifolds constitute a natural scenario to study Hamilton's equations. As a first and basic example of Poisson manifolds, we have the dual of a Lie algebra. Poisson manifolds are foliated by symplectic manifolds but this foliation is not necessarily regular. When considering a Hamiltonian system in a Poisson manifold we obtain families of Hamiltonian systems in the symplectic leaves of this foliation but there are some additional "transversal" structures given by extra symmetries (Casimir functions). We can therefore define integrable systems on these manifolds too and study similar topological/geometrical and dynamical properties.

Organization of this paper:

This paper is divided in three sections: The symplectic, contact and Poisson section. In this paper we wanted to give a global perspective and underline a common strategy in symplectic, contact and Poisson about the role of group actions in studying integrable systems: their (equivariant) normal and also some rigidity issues.

The results in the symplectic section concerning symplectic orthogonal decomposition and complete proof of symplectic equivalence with the linearized model are contained in the author's thesis. We include here an outline of this proof. A more detailed proof can be found in [55]. The Mather point of view in this proof is also new compared to Eliasson's point of view. We also re-state Eliasson's theorem using the foliation defined by the Hamiltonian vector fields. This formulation is necessary since because of the existence of hyperbolic singularities one cannot guarantee that the moment map is a function of the elements in Williamson's basis. (This inaccuracy has propagated somehow in the literature of integrable systems and had already been detected by myself [58] and other authors. A different approach and hypothesis are contained in the works of Matveev and Bolsinov [7] and [52]).

Most of the results contained in the Poisson case are joint results with other coauthors and the complete proof of the statements is contained somewhere else. The Poisson case is a summary of old results with a new perspective. We focus on the role of actions and symmetries in the proofs. As a bonus we also prove that, unlike the symplectic and contact case, integrable systems on Poisson manifolds are not rigid.

The contact case of integrable systems was just published in a short note before but contained no proofs. We offer here the detailed proof for the sake of completeness.

2. The Symplectic case

In June 29*th* of 1853 Joseph Liouville presented a communication entitled "Sur l'intégration des equations différentielles de la Dynamique" at the "Bureau des longitudes". In the resulting note [49] he relates the notion of integrability of the system to the existence of n integrals in involution with respect to the Poisson bracket attached to the symplectic form. These systems come to the scene with the classical denomination of "completely integrable systems". In another language, a particular choice of n-first integrals in involution determines the n components of a moment map $\mathbf{F} : M^{2n} \longrightarrow \mathbb{R}^n$. A lot of work has been done in the subject after Liouville. Let us outline some of the remarkable achievements from a geometrical and topological point of view.

2.1. Liouville-Mineur-Arnold Theorem: Torus actions meet integrable systems.

An integrable system on a symplectic manifold (M^{2n}, ω) is given by a set of generically independent functions $F = (f_1, \dots, f_n)$ satisfying $\{f_i, f_j\} = 0, \forall i, j$. The mapping $F : M^{2n} \longrightarrow R^n$ given by $F = (f_1, \dots, f_n)$ is called *moment map.*

The distribution generated by the Hamiltonian vector fields X_{f_i} is involutive and is tangent to the fibers of $F = (f_1, \dots, f_n)$.

Assume that the moment map is proper. Let L be a regular orbit of this distribution then this orbit is a Lagrangian submanifold. Moreover, it is a torus and the neighbouring orbits are also tori. Those tori are called Liouville tori. This is the topological contribution of a theorem which has been known in the literature as Arnold-Liouville theorem. The geometrical contribution of the above-mentioned theorem ensures the existence of symplectic normal forms in the neighbourhood of a compact regular orbit. To the author's knowledge, the works of Henri Mineur [53, 54] already gave the a complete description of the Hamiltonian system in a neighbourhood of a compact regular orbit. That is why we will refer to the classical Arnold-Liouville theorem as Liouville-Mineur-Arnold theorem. Let us state the theorem below,

Theorem 2.1. (Liouville-Mineur-Arnold Theorem)

Let (M^{2n}, ω) be a symplectic manifold and let $F : M^{2n} \longrightarrow \mathbb{R}^n$ be a proper moment map. Assume that the components f_i of F are pairwise in involution with respect to the Poisson bracket associated to ω and that $df_1 \wedge \dots \wedge df_n \neq 0$ on a dense set. Let $N = F^{-1}(c)$, $c \in \mathbb{R}^n$ be a connected levelset. Then there exists a neighbourhood $U(N)$ of N and a diffeomorphism $\phi : U(N) \longrightarrow D^n \times \mathbb{T}^n$ such that,

1. $\phi(N) = \{0\} \times \mathbb{T}^n$.
2. *A set of coordinates μ_i in D^n and a set of coordinates β_i in $\mathbb{T}^n$ for which,* $\phi^*(\sum_{i=1}^n d\mu_i \wedge d\beta_i) = \omega$.
3. *F depends only on $\phi^*(\mu_i) = p_i$ and it does not depend on $\phi^*(\beta_i) = \theta_i$.*

The new coordinates p_i obtained are called action coordinates. The coordinates θ_i are called angle coordinates. Mineur also showed that the action functions p_i can be defined via the period integrals. Let x be a point in a small neighbourhood of N, the period integrals are defined by the following formula:

$$p_i(x) = \int_{\Gamma_i(x)} \alpha \tag{2.1}$$

where α fulfils the condition $d\alpha = \omega$, and $\Gamma_i(x)$ is a closed curve which depends smoothly on x and which lies on the Liouville torus containing x. The homology classes of $\Gamma_1(x), ..., \Gamma_n(x)$ form a basis of the first homology group of the Liouville torus.

The existence of action-angle coordinates in a neighbourhood of a compact orbit provides a symplectic model for the Lagrangian foliation $\mathcal{F}$ determined by the symplectic gradients of the n component functions f_i of the moment map F. In fact, Liouville-Mineur-Arnold theorem entails a "uniqueness" result for the symplectic structures making $\mathcal{F}$ into a Lagrangian foliation. In other words, if ω_1 and ω_2 are two symplectic structures defined in a neighbourhood of N for which $\mathcal{F}$ is Lagrangian then there exists a symplectomorphism preserving the foliation, fixing N and carrying ω_1 to ω_2. This is due to the following observation: Let X_{f_i} be the symplectic gradients of the functions f_i for any $1 \leq i \leq n$, then the Lagrangian condition implies that in fact $\mathcal{F} =< X_{f_1}, \dots, X_{f_n} >$, further $\{f_j, f_k\}_i = 0$ where $\{.,.\}_i$ stands for the Poisson bracket attached to ω_i , $i = 1, 2$. Then by virtue of Liouville-Mineur-Arnold theorem there exists a foliation-preserving symplectomorphism ϕ_i taking ω_i to $\omega_0 = \sum_{i=1}^n dp_i \wedge d\theta_i$. In all, the diffeomorphism $\phi_2^{-1} \circ \phi_1$ does the job. It takes ω_1 to ω_2, it fixes N and it is foliation preserving.

So if the orbit is regular the existence of action-angle coordinates enables to classify the symplectic germs, up to foliation-preserving symplectomorphism, for which $\mathcal{F}$ is Lagrangian in a neighbourhood of a compact orbit. There is just one class of symplectic germs for which the foliation is Lagrangian.

One could look at the problem from a global perspective. There are topological obstructions to the existence of global action-angle coordinates as it was shown by Duistermaat in [25].

The problem of classification of symplectic germs for regular Lagrangian foliations can be taken further to consider the case of foliations not necessarily determined by a completely integrable system. Curras-Bosch and Molino have considered the following concomitant problem: They consider the problem of classification for germs of Lagrangian foliation defined in a neighbourhood of an torus equipped with an affine structure. The motivation for considering an affine structure on the torus is the Bott-Weinstein connection attached to the regular leaves of a Lagrangian foliation [**?**]. In the case the germ of Lagrangian foliation is determined by a completely integrable system this affine structure is trivial. In the above mentioned papers it is proved that there is no uniqueness result for the symplectic germ if the affine structure on the torus is non-trivial.

Consequences:

Action-angle coordinates give:

- An action of a torus $\mathbb{T}^n$ tangent to the Liouville tori which is Hamiltonian.
- A normal form for the set of first integrals in involution (action coordinates). In these coordinates, $F = (p_1, \dots, p_n)$.
- A normal form for the symplectic structure: the symplectic structure is Darboux $\omega = \sum_{i=1}^n dp_i \wedge d\theta_i$.

Remark 2.2. Indeed one of the author's favourite proof of the action-angle theorem is in the paper of Duistermaat [25]. The theorem uses strongly the existence of a torus action tangent to the fibration given by the moment map.

2.1.1. *Global action-angle coordinates: Some applications.* The global problem of existence of action-angle variables is related to Monodromy and the Chern class of the fibration given by the fibers of the moment map.

In the case of generalized global action-angle coordinates on compact manifolds, the semi-local torus action extend to a torus action $\mathbb{T}^n$ on the compact symplectic manifolds (M^{2n}, ω). We get global action-angle coordinates with singularities and a toric manifold. Symplectic geometry can be read from the Delzant polytope [21].

The existence of action-angle coordinates has many implications in dynamics: for instance the topological entropy of these systems is zero if there are no singularities in the way [71] and [45].

Other applications of global action-angle coordinates show up (not so unexpectedly) in the context of geometric quantization using real polarizations. We include here a short summary. A more extended version of these applications (with singularities in the way) can be found in the short note [56].

Let (M^{2n}, ω) be a symplectic manifold such that $[\omega]$ is integral. Under these circumstances (see for instance [75]), there exists a complex line bundle $\mathbb{L}$ with a connection ∇ over M such that $curv(\nabla) = \omega$. The symplectic manifold (M^{2n}, ω) is called prequantizable and the pair $(\mathbb{L}, \nabla)$ is called *a prequantum line bundle* of (M^{2n}, ω). In order to construct the geometric quantization of these objects, we need to restrict the space of sections to a subspace of sections which are flat in "priviledged" directions given by a polarization. In the case of real polarizations given by integrable systems, both problems are connected.

Consider the following:

Example 2.3. Consider $M = S^1 \times \mathbb{R}$ and $\omega = dt \wedge d\theta$. Take as $\mathbb{L}$ the trivial bundle with connection 1-form $\Theta = td\theta$. Now, let $\mathcal{P} =< \frac{\partial}{\partial \theta} >$ then flat sections satisfy, $\nabla_X \sigma = X(\sigma) - i < \theta, X > \sigma$. Thus flat sections $\sigma(t, \theta) = a(t).e^{it\theta}$ are defined along leaves are given by the condition $t = 2\pi k, k \in \mathbb{Z}$.

This example shows that flat sections are not globally defined but they exist along a subset of leaves of the polarization. These are called Bohr-Sommerfeld leaves. The characterization of Bohr-Sommerfeld leaves for regular fibrations under some conditions is a well-known result by Guillemin and Sternberg ([34]). In particular the set of Bohr-Sommerfeld leaves is discrete and is given by "action" coordinates.

Theorem 2.4 (Guillemin-Sternberg)**.** *If the polarization is a regular fibration with compact leaves over a simply connected base B, then the Bohr-Sommerfeld set is discrete and assuming that the zero-fiber is a Bohr-Sommerfeld leaf, the Bohr-Sommerfeld set is given by, $BS = \{p \in M, (f_1(p), \dots, f_n(p)) \in \mathbb{Z}^n\}$ where $f_1, \dots, f_n$ are global action coordinates on B.*

This result connects with Liouville-Mineur-Arnold theorem. When we consider a toric manifolds the base B may be identified with the image of the moment map by the toric action (Delzant polytope).

In view of the previous theorem, when the polarization is given by an integrable system with global action-angle coordinates it makes sense to "quantize" these systems counting integral Liouville tori.

This can be formalized following the idea of Kostant [43], in the case there are no global sections denote by $\mathcal{J}$ the sheaf of flat sections along the polarization, we can then define the quantization as $\mathcal{Q}(M) = \bigoplus_{k\geq 0} H^k(M, \mathcal{J})$. Then quantization is given by precisely the following theorem of Sniatycki [73]:

Theorem 2.5 (Sniatycki)**.** *If the leaf space B^n is a Hausdorff manifold and the natural projection $\pi : M^{2n} \to B^n$ is a fibration with compact fibres, then all the cohomology groups vanish except for degree half of the dimension of the manifold. Furthermore, $\mathcal{Q}(M^{2n}) = H^n(M^{2n}, \mathcal{J})$, and the dimension of $H^n(M^{2n}, \mathcal{J})$ is the number of Bohr-Sommerfeld leaves.*

2.2. **Rigidity.** In this section, we explore the consequences that normal forms have for structural stability problems.

For structural stability we understand the following: Given close geometrical "objects" in a certain topology, we want to see if these objects are equivalent.

As a general principle normal forms for geometrical structures give structural stability. Sometimes because of the type of singularities, this is not enough though (as we will see in the Poisson section).

When we have additional symmetries, it is still possible to prove rigidity using the averaging techniques. In this case, we would obtain the equivariant version.

As a first example of this, the equivariant version of Darboux theorem was stated by Weinstein in [76] and was proved by Chaperon in [17] for smooth compact group actions.

In the case the normal forms are given in a neighbhourhood of a fixed point for the action, we may linearize this action in such a way that the normal forms prevail. If the group is non-compact there is a hope to do it for analytic actions of semisimple groups/algebras in the Hamiltonian setting [60]. In the smooth case this is possible only for actions of semisimple actions of compact type [31].

We can also prove the following rigidity theorem for symplectic group actions on a compact symplectic manifold (see [62] and [65],

Theorem 2.6. *Let ρ_0 and ρ_1 be two C^2-close symplectic actions of a compact Lie group G on a compact symplectic manifold (M, ω). Then they can be made equivalent by conjugation via a symplectomorphism.*

In particular:

- Liouville-Mineur-Arnold theorem entails semi-local rigidity for integrable systems in a neighbourhood of a compact orbit.
- (equivariant) Darboux theorem gives (equivariant) local rigidity for symplectic forms.

One may think that a "key point" to obtain rigidity for integrable systems is the existence of compact action (of $\mathbb{T}^n$).

But, we also have normal forms and rigidity for some non-degenerate singular integrable systems. Indeed we also have infinitesimal rigidity for a certain type of singular integrable systems called non-degenerate.

2.3. **Singular counterparts to Arnold-Liouville.** What can be said about the corresponding classification problem for symplectic germs if the completely integrable systems has singularities?

This question is quite natural because singularities are present in many well-known examples of integrable systems. In fact, if the completely integrable system is defined on a compact manifold then the singularities cannot be avoided.

One of the main goals of [58] was to prove that the uniqueness result for symplectic germs for which the foliation determined by a completely integrable system is generically Lagrangian holds when L is a singular orbit.

In the singular case, the problem can be posed at three different levels:

(1) At the orbit level: In the neighbourhood of a compact singular orbit.
(2) At a semi-local level: In the neighbourhood of a compact singular leaf.
(3) At a global level.

The problem of topological classification of integrable Hamiltonian systems began with Fomenko [28] in some particular cases. Nguyen Tien Zung [79] studied the general case for the semi-local problem for non-degenerate singularities. It turns out that from a topological point of view we have a product-like description of the singularities in terms of the Williamson type. Nguyen Tien Zung also proved in [79] the existence of partial action-angle coordinates. The symplectic classification in the semi-local case for non-elliptic singularities has been studied in the hyperbolic case by Dufour, Molino and Toulet in [23]. The focus-focus case has been studied recently by San Vu Ngoc in [70]. In the hyperbolic and focus-focus case there are more invariants attached to the singularity. The symplectic germ in the hyperbolic case is determined by the jet of a function depending on a variable and in the focus-focus case is determined by the jet of a function in two variables. The singular global case has been studied by Nguyen Tien Zung in the paper [**?**] where the notion of Duistermaat-Chern class and monodromy (introduced by Duistermaat for regular foliations) is extended in order to include the singularities into the picture.

The condition of non-degeneracy is always present in the works cited above. There are also some contributions for degenerate singularities in the world of integrable systems. A recent contribution in that direction is contained in the paper [9] by Colin de Verdière. In that paper, among other things, the problem of classification of germs of singular Lagrangian manifolds is posed for more general singularities with a special emphasis on quasi-homogeneous singularities. For instance in this paper an explicit classification is obtained in the case of the cusp.

The singular analogue of Liouville-Mineur-Arnold theorem was considered by Eliasson in his thesis [26]. He constructed singular action coordinates for a special type of non-degenerate singularities. This is a major breakthrough which uses a clever combination of analysis and Moser's path method. However, there were some inaccuracies in some statements which lead to some confusion in the literature. We will try to clarify those here.

The singular achievements formerly specified often have a semiclassical version. Their semiclassical counterpart has been obtained by Colin de Verdière and San Vu Ngoc in [10, 9].

After this digression we will focus on the orbit-like case.

The singularity of the orbit can be described in terms of the singularity of the functions f_i.

Let us start with the case L is reduced to a point.

Observe that the Poisson bracket induces a Lie algebra structure in the set of functions. Since the functions f_i are in involution with respect to the Poisson bracket, the quadratic parts of the functions f_i commute defining in this way an abelian subalgebra of $Q(2n, \mathbb{R})$ (the set of quadratic forms on $2n$-variables). In the case the singularity of the functions f_i is of Morse type this subalgebra is indeed a Cartan subalgebra. We call these singularities of non-degenerate type.

The problem of classification of singularities for the quadratic parts of the functions f_i can be therefore converted into the problem of classification of Cartan subalgebras of $Q(2n, \mathbb{R})$. The singularities for the quadratic parts are well-understood thanks to a result of Williamson [78] where Cartan subalgebras of $Q(2n, \mathbb{R})$ are classified. Let us recall its precise statement,

Theorem 2.7. (Williamson)

For any Cartan subalgebra $\mathcal{C}$ of $Q(2n, \mathbb{R})$ there is a symplectic system of coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ in $\mathbb{R}^{2n}$ and a basis $f_1, \dots, f_n$ of $\mathcal{C}$ such that each f_i is one of the following:

$$
\begin{aligned}
&f_i = x_i^2 + y_i^2 \qquad \text{for } 1 \leq i \leq k_e\ , && \text{(elliptic)}\\
&f_i = x_i y_i \qquad \text{for } k_e + 1 \leq i \leq k_e + k_h\ , && \text{(hyperbolic)}\\
&\begin{cases} f_i = x_i y_{i+1} - x_{i+1} y_i, & \text{(focus-focus pair)}\\ f_{i+1} = x_i y_i + x_{i+1} y_{i+1} & \text{for } i = k_e + k_h + 2j - 1,\ 1 \leq j \leq k_f \end{cases}
\end{aligned}
\tag{2.2}
$$

The linear system given by the quadratic parts of the f_i is called the linear model for a singularity. Williamson's Theorem can be seen as a normal form theorem for the linear model.

We may attach a triple of natural numbers (k_e, k_h, k_f) to a non-degenerate singularity p of F, where k_e stand for the number of elliptic components in the linear model, k_h and k_f the number of hyperbolic and focus-focus components in the linear model respectively.

By virtue of Williamson theorem this triple is an invariant of the linear system. That is why this triple is often called the Williamson type of the singularity.

Now that the classification in the linear model has been carried out a natural question arises:

Can we linearize the completely integrable system symplectically in a neighbourhood of a point p?

We can reformulate the question as follows,

Problem 1

Consider a foliation $\mathcal{F}$ defined by a completely integrable system defined in a neighbourhood of a non-degenerate singular 0-dimensional orbit of $\mathcal{F}$. Assume that we are given two symplectic forms ω_1 and ω_2 for which the foliation $\mathcal{F}$ is Lagrangian. Does there exist a local diffeomorphism fixing p and taking ω_1 to ω_2?

This problem of symplectic linearization is closely related to another problem in the spirit of Morse lemma which was solved succesfully by Vey for analytic systems and by Vey and Colin de Verdire for smooth systems.

Problem 2

Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with a non-degenerate singularity at the origin and let ω be a volume form on $\mathbb{R}^n$ and let Q be its quadratic part at the origin. Does there exist a diffeomorphism $\phi : (R^n, 0) \longrightarrow (R^n, 0)$ such that $\phi^*(f) = Q$ and such that ω is taken to the volume form $\omega_0 = dx_1 \wedge \cdots \wedge dx_n$?

In [8] Colin de Verdière and Vey prove that there exists a smooth function χ such that $\phi^*(\omega) = \chi(Q) \cdot \omega_0$.

In that paper it is also proved that the function χ is characteristic of the pair (f, ω) if Q is definite, otherwise only the jet is characteristic for the pair.

As a corollary of this result we obtain normal forms for foliations defined by the levelsets of f because we can find a foliation-preserving diffeomorphism sending the volume form $\chi(Q) \cdot \omega_0$ to the volume form ω_0 as was observed in the paper cited above. Notice as well that this result provides an affirmative answer to Problem 1 in the case $n = 2$ because a volume form on a 2-dimensional manifold is a symplectic form and the Lagrangian condition for a curve is automatic in that dimension.

The affirmative answer to Problem 1 in any dimension was provided in the elliptic case by Eliasson in [27]. As a matter of fact the proof provided by Eliasson seems complete just in the case the singularity is completely elliptic (of Williamson type $(k_e, 0, 0)$).

What is not true in general is that the moment map has component functions which are, in turn, functions of the basis of the Cartan subalgebra. This fails essentially in the case the system has hyperbolic components. Errors stating this normal form result are unfortunately too common in the literature of integrable systems.

In this paper we will sketch a different proof of Eliasson's normal form in the context of linearization of associated foliation (not moment map) for general Williamson type. The proof that I am going to discuss here is essentially a slightly more sophisticated proof that the one in my thesis [58].

Observe that this normal form theorem can be seen as a symplectic linearization result which ensures that the initial completely integrable system can be taken to the linear system and that the symplectic form can be taken to the standard one. As a byproduct we obtain a multiple differentiable linearization result for n commuting vector fields with singularities of non-degenerate type.

The symplectic linearization in a neighbourhood of an orbit L with $\dim L > 0$ is due to Ito in the analytic case [40]. Partial results in that direction (with $\dim L = 1$ in a manifold of dimension 4) where obtained by Currás-Bosch and myself in [13] and independently by Colin de Verdire and San Vu Ngoc in [10]. The final result in any dimension was obtained by Nguyen Tien Zung and myself in [59]. In [59] it is also included a G-equivariant version of the symplectic linearization.

2.3.1. *The theorems in the non-degenerate singular case.* We begin by an elementary definition:

Definition 2.1. A point $x_0 \in M^{2n}$ is a singular point of the integrable Hamiltonian systems if the rank of $d_{x_0}F = (d_{x_0}f_1, \dots, d_{x_0}f_n)$ is less than n.

Let x_0 be as singular point of the foliation defined by F, we start by defining the rank and corank of a singular point.

Definition 2.2. Let $x_0 \in M^{2n}$ be a singular point of the integrable Hamiltonian system we say that the rank of x_0 is k if the rank of the moment map at x_0 is k, that is to say if $\operatorname{rank} d_{x_0}F = \operatorname{rank}(d_{x_0}f_1, \dots, d_{x_0}f_n) = k$.

We say that a singular point of rank k has corank $n - k$.

Recall that the foliation can be thought as a pair (χ, A) , where χ is an n-dimensional commutative Lie algebra of vector fields and A is a vector space of first integrals of the vector fields of χ. The foliation obtained from χ is called the *singular Lagrangian foliation*.

We follow Nguyen Tien Zung [79] for the definitions concerning the notion of *transversal linearization at a singular point*.

Let x_0 be a singular point and let χ_{x_0} be the subspace of $T_{x_0}M$ generated by $X_{x_0}, \forall X \in \chi$. Let $K_{x_0} = \cap_{f \in A} Ker d_{x_0} f$, and let B_{x_0} the set of $f \in A$ such that $d_{x_0}f = 0$. Then $\forall f \in B_{x_0}$ the 2-order jet of $f - f(x_0)$ gives a quadratic form on K_{x_0}, such that its kernel contains χ_{x_0}, so it gives a quadratic form $f^{\prime T}_{x_0}$ on K_{x_0}/χ_{x_0}, the set of quadratic forms obtained in this way which we denote by $A^{\prime T}_{x_0}$, is a commutative subalgebra under the Poisson bracket, which is often called the *transversal linearization of* F.

Notice that K_{x_0}/χ_{x_0} carries a natural symplectic structure $\overline{\omega}_{x_0}$, and it is symplectomorphic to a subspace $R_{x_0} \subset T_{x_0}M^{2n}$.

We are going to introduce the notion of nondegenerate point but first we need to recall the definition of Cartan subalgebra.

Definition 2.3. A Cartan subalgebra is a maximal self-centralizing abelian subalgebra.

Definition 2.4. A singular point of corank k is called non-degenerate if $A^{\prime T}_{x_0}$ is a Cartan subalgebra of the algebra of quadratic forms on R_{x_0}.

The linear model is given by Williamson's theorem stated above.

The rank 0 case

In this paragraph we state the analogue of Eliasson's theorem for general singularities, using foliations. Other statements using moment maps and additional hypotheses on the bifurcation diagrams have been provided by [52] and [7].

Assume that $\mathcal{F}$ is a linear foliation on M^{2n} with a rank 0 singularity at the origin p. Assume that the Williamson type of the singularity is (k_e, k_h, k_f). Recall that the foliation is then generated by the following vector fields,

$$\begin{aligned}
&X_i = -y_i\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial y_i} \quad \text{for} \quad 1 \leq i \leq k_e \ , \\
&X_i = y_i\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial y_i} \quad \text{for} \quad k_e + 1 \leq i \leq k_e + k_h \ , \\
&X_i = x_i\frac{\partial}{\partial x_{i+1}} - y_{i+1}\frac{\partial}{\partial y_i} - x_{i+1}\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_{i+1}} \quad \text{and} \\
&X_{i+1} = -x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i} - x_{i+1}\frac{\partial}{\partial x_{i+1}} + y_{i+1}\frac{\partial}{\partial y_{i+1}} \quad \text{for} \quad i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f
\end{aligned}$$

We can prove [58] that,

Theorem 2.8. *Let ω be a symplectic form defined in a neighbourhood of the origin for which $\mathcal{F}$ is Lagrangian, then there exists a local diffeomorphism $\phi : (U, p) \longrightarrow (\phi(U), p)$ such that ϕ preserves the foliation and $\phi^*(\sum_i dx_i \wedge dy_i) = \omega$, being x_i, y_i local coordinates on $(\phi(U), p)$.*

Remark 2.9. In the case the singularities are completely elliptic, this is equivalent to Eliasson's theorem.

Proof of theorem 2.8Infinitesimal rigidity and rigidity

In this paragraph we sketch a more evolved proof that the one contained in the author's thesis [58]. A detailed proof will be provided in [55].

The proof uses the classical scheme of *"infinitesimal rigidity" implies rigidity.*

In the spirit of Thom and Mather we can study (unfinitesimal) stability of an integrable system $\rightsquigarrow F = (f_1, \ldots, f_n)$ with constraints $\{f_i, f_j\} = 0$.

For non-degenerate singular integrable systems, we have proved in [61] that a singular Poincaré lemma holds.

Namely,

Theorem 2.10 (Miranda-Vu Ngoc, [61])**.** *Let $g_1, \ldots g_r$, be a set of germs of smooth functions on $(\mathbb{R}^{2n}, 0)$ with $r \leq n$ fulfilling the following commutation relations*

$$X_i(g_j) = X_j(g_i), \quad \forall i, j \in \{1, \ldots, r\}$$

where the X_i's are the vector fields defined above. Then there exists a germ of smooth function G and r germs of smooth functions f_i such that,

(1) $X_j(f_i) = 0$, $\forall i, j \in \{1, \ldots, r\}$.
(2) $g_i = f_i + X_i(G)$ $\forall i \in \{1, \ldots, r\}$.

We can restate this result in a different language.

Theorem 2.11 (Miranda-Vu Ngoc, [61])**.** *Integrable systems are infinitesimally stable in a neighbourhood of a non-degenerate fixed point.*

We can define a deformation complex using the action of $\mathbb{R}^n$ on $X = C^\infty(M)$ ($\mathbb{R}^n \times X \ni (\ell, g) \mapsto \{\mathbf{f}(\ell), g\} \in X$) (Chevalley-Eilenberg).

The cocyles of this deformation complex are defined as:

(cocycles) $Z^1(\mathbf{f}) \rightsquigarrow n$ functions $g_1 = \alpha(e_1), \ldots, g_n = \alpha(e_n)$ (mod. basic functions) such that

$$\forall i, j \qquad \{g_i, f_j\} = \{g_j, f_i\}. \tag{2.3}$$

A cocycle defines an infinitesimal deformation of the system since mod ε^2 we have

$$\{f_i + \varepsilon g_i, f_j + \varepsilon g_j\} \equiv 0.$$

Singular Poincaré lemma proves that every cocycle is a coboundary.

The condition $H^1(\mathbf{f}) = \mathbf{0}$ is equivalent to the existence of a singular Poincaré lemma and this can be used to prove an orthogonal symplectic decomposition.

The symplectic orthogonal decomposition allows to prove theorem 2.8 because it reduces the symplectic equivalence problem to 2 and 4-dimensional cells (for focus-focus singularities). The symplectic othogonal decomposition is attained via Moser's path method. A detailed proof will be contained in [55].

Indeed, Eliasson's normal form for non-degenerate singularities (orbits) of rank 0 give rigidity for integrable systems.

Normal forms for non-degenerate singularities in a neigbourhood of an orbit

Normal forms for higher rank have been obtained jointly with Nguyen Tien Zung.

These normal form results can be seen as a symplectic Morse-Bott theorem for integrable systems. The singular fibers can have non-compact symmetry group associated to it. This is why in order to obtain a complete proof of the symplectic equivalence problem, we need to consider the equivariant version of the statements.

So we may prove the following [58] and [79]:

Theorem 2.12. *Let $U(L)$ be a neighbourhood of a nondegenerate singular orbit of an integrable system with n degrees of freedom. Assume the corank of the orbit is $n - k = k_e + k_h + 2k_f$. Let $\mathcal{F}$ be the singular Lagrangian foliation defined by the integrable system. Then there exists a normal finite covering $\widetilde{U(L)}$ of $U(L)$ such that the foliation can be lifted to $\widetilde{\mathcal{F}}$ and a free Hamiltonian action of the torus $\mathbb{T}^k$ in the covering $\widetilde{U(L)}$ which preserves the moment map.*

Now we can introduce the linear model associated to the orbit L. Later, we will see that the invariants associated to the linear model are the Williamson type of the orbit and a twisting group Γ attached to it.

First we introduce the linear model in the covering,

Denote by $(p_1, ..., p_k)$ a linear coordinate system of a small ball D^k of dimension k, $(\theta_1, \ldots, \theta_k)$ is a standard periodic coordinate system of the torus $\mathbb{T}^k$, and $(x_1, y_1, ..., x_{n-k}, y_{n-k})$ a linear coordinate system of a small ball $D^{2(n-k)}$ of dimension $2(n-k)$. Now we consider the manifold

$$V = D^k \times \mathbb{T}^k \times D^{2(n-k)} \tag{2.4}$$

with the standard symplectic form $\sum dp_i \wedge d\theta_i + \sum dx_j \wedge dy_j$, and the following moment map:

$$F = (p_1, ..., p_k, f_1, ..., f_{n-k}) : V \to \mathbb{R}^n \tag{2.5}$$

where

$$
\begin{aligned}
&f_i = x_i^2 + y_i^2 \quad \text{for} \quad 1 \le i \le k_e \ , \\
&f_i = x_i y_i \quad \text{for} \quad k_e + 1 \le i \le k_e + k_h \ , \\
&f_i = x_i y_{i+1} - x_{i+1} y_i \quad \text{and} \\
&f_{i+1} = x_i y_i + x_{i+1} y_{i+1} \quad \text{for} \quad i = k_e + k_h + 2j - 1, \ 1 \le j \le k_f
\end{aligned} \tag{2.6}
$$

The *linearized foliation in the covering* is the foliation determined by the above moment map. This presentation of the foliation would be the one of $\mathcal{A}$, that is, the above components of the moment map are the first integrals of the system. We can also look for generators of χ to define the linearized foliation in the covering. After performing a linear change of coordinates in such a way that the hyperbolic functions can be written as $f_i = x_i^2 - y_i^2$, the following vector fields form a basis of χ,

$$
\begin{aligned}
&Y_i = \frac{\partial}{\partial \theta_i} \ \text{for} \ \ 1 \le i \le k \ , \\
&X_i = -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} \ \ \text{for} \ \ 1 \le i \le k_e \ , \\
&X_i = y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} \ \ \text{for} \ \ k_e + 1 \le i \le k_e + k_h \ , \\
&X_i = x_i \frac{\partial}{\partial x_{i+1}} - y_{i+1} \frac{\partial}{\partial y_i} - x_{i+1} \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_{i+1}} \ \ \text{and} \\
&X_{i+1} = -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}} + y_{i+1} \frac{\partial}{\partial y_{i+1}} \ \ \text{for} \ \ i = k_e + k_h + 2j - 1, \ 1 \le j \le k_f
\end{aligned}
$$

In order to prove equivalence in a neighbourhood of an orbit we need to first consider additional symmetries corresponding to the deck-transformations. We prove it for general symplectic actions preserving the system.

2.4. **Additional symmetries.** We assume also that the group acts symplectically and preserves the moment map which is underlying in the foliation.

We end up proving the equivariant version of the symplectic uniqueness result in a neighbourhood of a singular compact orbit. This result is contained in [58] and [59].

We are going to introduce the notion of linear action on the linear model associated to the orbit L for a given symplectic action preserving the system. Later, we will see that the invariants associated to the linear model are the Williamson type of the orbit and a twisting group Γ attached to it.

We recall the notion of linear model. Denote by $(p_1, ..., p_k)$ a linear coordinate system of a small ball D^k of dimension k, $(\theta_1 (mod 1), ..., \theta_k (mod 1))$ a standard periodic coordinate system of the torus $\mathbb{T}^k$, and $(x_1, y_1, ..., x_{n-k}, y_{n-k})$ a linear coordinate system of a small ball $D^{2(n-k)}$ of dimension $2(n-k)$. Consider the manifold

$$V = D^k \times \mathbb{T}^k \times D^{2(n-k)} \tag{2.7}$$

with the standard symplectic form $\omega_0 = \sum dp_i \wedge d\theta_i + \sum dx_j \wedge dy_j$, and the following moment map:

$$\mathbf{F} = (p_1, ..., p_k, f_{k+1}, ..., f_n) : V \to \mathbb{R}^n \tag{2.8}$$

where

$$
\begin{aligned}
&f_{i+k} = x_i^2 + y_i^2 \quad \text{for} \quad 1 \le i \le k_e \ , \\
&f_{i+k} = x_i y_i \quad \text{for} \quad k_e + 1 \le i \le k_e + k_h \ , \\
&f_{i+k} = x_i y_{i+1} - x_{i+1} y_i \quad \text{and} \\
&f_{i+k+1} = x_i y_i + x_{i+1} y_{i+1} \quad \text{for} \quad i = k_e + k_h + 2j - 1, \ 1 \le j \le k_f
\end{aligned} \tag{2.9}
$$

For the sake of simplicity we will denote by $\mathbf{p}$ the mapping whose components are the k regular first integrals p_i and $\mathbf{h}$ will stand for the mapping whose components are the singular first integrals f_i, $i \geq k$; following this convention we will write $\mathbf{F} = (\mathbf{p}, \mathbf{h})$. Let Γ be a group with a symplectic action $\rho(\Gamma)$ on V, which preserves the moment map $\mathbf{F}$. We will say that the action of Γ on V is *linear* if it satisfies the following property:

Γ acts on the product $V = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ componentwise; the action of Γ on D^k is trivial, its action on $\mathbb{T}^k$ is by translations (with respect to the coordinate system $(\theta_1, ..., \theta_k)$), and its action on $D^{2(n-k)}$ is linear with respect to the coordinate system $(x_1, y_1, ..., x_{n-k}, y_{n-k})$.

Suppose now that Γ is a finite group with a free symplectic action $\rho(\Gamma)$ on V, which preserves the moment map and which is linear. Then we can form the quotient symplectic manifold V/Γ, with an integrable system on it given by the induced moment map as above:

$$\mathbf{F} = (p_1, ..., p_k, f_{k+1}, ..., f_n) : V/\Gamma \to \mathbb{R}^n \tag{2.10}$$

The set $\{p_i = x_i = y_i = 0\} \subset V/\Gamma$ is a compact orbit of Williamson type (k_e, k_f, k_h) of the above system. We will call the above system on V/Γ, together with its associated singular Lagrangian foliation, the linear system (or linear model) of Williamson type (k_e, k_f, k_h) and twisting group Γ (or more precisely, twisting action $\rho(\Gamma)$). We will also say that it is a direct model if Γ is trivial, and a twisted model if Γ is nontrivial.

A symplectic action of a compact group G on V/Γ which preserves the moment map $(p_1, ..., p_k, f_{k+1}, ..., f_n)$ will be called linear if it comes from a linear symplectic action of G on V which commutes with the action of Γ. In our case, let $\mathcal{G}'$ denote the group of linear symplectic maps which preserve the moment map then this group is abelian and therefore this last condition is always satisfied.

In [58] and [59] we prove the following:

Theorem 2.13 (Miranda-Zung)**.** *Consider $\mathcal{F}$ the foliation defined by a completely integrable system and consider L, a compact orbit of Williamson type (k_e, k_h, k_f). Let ω be a symplectic for which the foliation $\mathcal{F}$ is Lagrangian. Then there exists a finite group Γ and a diffeomorphism taking the foliation to the linear foliation on V/Γ given by the linear model above , and taking ω to ω_0, which sends L to the torus $\{p_i = x_i = y_i = 0\}$. The smooth symplectomorphism ϕ can be chosen so that via ϕ, the system-preserving action of the compact group G near L becomes a linear system-preserving action of G on V/Γ. If the moment map $\mathbf{F}$ is real analytic and the action of G near L is analytic, then the symplectomorphism ϕ can also be chosen to be real analytic. If the system depends smoothly (resp., analytically) on a local parameter (i.e. we have a local family of systems), then ϕ can also be chosen to depend smoothly (resp., analytically) on that parameter.*

Remark 2.14. A nice consequence is the abelianity of the group of symplectomorphisms preserving the moment map. In particular, in the case the action of the group is effective then this group is Abelian, in all, since it is also compact it is a product of a torus with a finite group. In the end, in the case the group is connected we recover actions by tori in the spirit of the theorem of Delzant.

Remark 2.15. This theorem has interesting applications to geometric quantization of singular real polarizations (see [56] and [37].

3. The Contact case

Loosely speaking, the odd-dimensional counterpart of the theorems obtained would be considered in the contact case. That is we can consider foliations on a contact manifold as close as possible to the ones described by completely integrable systems on symplectic manifolds. The regular case started with Lutz ([50]) who studies the problem of classification for contact structures in a compact contact manifold under the constraint that they are invariant under the action of a torus. This problem is naturally linked with the analogous problem for symplectic manifolds exposed above. Recent contributions to that problem in the setting of contact orbifolds are due to Lerman [41] where a convexity result is also established. This problem has been considered by Molino and Banyaga in [4] and [5] also for singular foliations. The common property of the foliations considered by Lutz, Lerman, Molino and Banyaga is that their orbits are given by a torus action. Here we prove a similar result in the neighbourhood of a compact orbit but for foliations whose orbits are not necessarily given by a torus action but fulfil hypothesis of non-degeneracy. The foliation is determined as the enlarged foliation of a Legendrian foliation described by the horizontal parts of contact vector fields together with a Reeb vector field. We also assume that the Reeb vector field is the infinitesimal generator of an S^1-action. We study the problem of classification for Legendrian foliations under the assumption that the contact form has the same Reeb vector field.

Consider a contact manifold M^{2n+1} together with a contact form. We assume that the Reeb vector field associated to α coincides with the infinitesimal generator of an S^1 action. We assume further than there exists n-first integrals of the Reeb vector field which commute with respect to the Jacobi bracket. Then there are two foliations naturally attached to the situation. On the one hand, we can consider the foliation associated to the distribution generated by the contact vector fields. We call this foliation $\mathcal{F}'$. On the other hand we can consider a foliation $\mathcal{F}$ given by the horizontal parts of the contact vector fields. The functions determining the contact vector fields may have singularities. We will always assume that those singularities are of non-degenerate type.

Observe that $\mathcal{F}'$ is nothing but the enlarged foliation determined by the foliation $\mathcal{F}$ and the Reeb vector field.

Let α' be another contact form in a neighbourhood of a compact orbit $\mathcal{O}$ of $\mathcal{F}'$ for which $\mathcal{F}$ is Legendrian and such that the Reeb vector field with respect to α' coincides with the Reeb vector field associated to α. In this chapter we prove that then there exists a diffeomorphism from a neighbourhood of $\mathcal{O}$ to a model manifold taking the foliation $\mathcal{F}'$ to a linear foliation in the model manifold with a finite group attached to it and taking the initial contact form to the Darboux contact form. As it was done in the last chapter for Lagrangian foliations determined by a completely integrable system, we also prove the G-equivariant version of this fact for Legendrian foliations. That is, we prove that in the case there exists a compact

Lie group preserving the first integrals of the Legendrian foliation and preserving the contact form then the contactomorphism can be chosen to be G-equivariant.

The problem of determining normal forms for foliations related to Legendrian foliations has its own story. P. Libermann in [47] established a local equivalence theorem for α-regular foliations. Loosely speaking, those foliations are regular foliations containing the Reeb vector field and a Legendrian foliation. The problem of classifying contact forms is different from the problem of classification of contact structures. As a example of this, if M is a compact manifold then any two contact structures are equivalent as Gray's theorem asserts ([30]). Whereas one can find examples of two contact forms which are not equivalent (see for example [29]). The problem of classifying contact structures which are invariant under a Lie group was considered by Lutz in [50]. In particular he proves that two contact structures in a compact manifold M^{2n+1} which are invariant by a locally free action of $\mathbb{R}^{n+1}$ are equivalent in the sense that there exists an equivariant contactomorphism taking one to the other.

The foliations studied by Libermann and Lutz are regular. The singular counterpart to the result of Lutz was proved by Banyaga and Molino in [5] but for contact forms.

Namely, Banyaga and Molino study the problem of finding normal forms under the additional assumption of transversal ellipticity. The assumption of transversal ellipticity allows to relate the foliation $\mathcal{F}'$ of generic dimension $(n+1)$ with the foliation given by the orbits of a torus action.

In this section we extend these results for foliations which are related in the same sense to $(n+1)$-foliations but which are not necessarily identified with the orbits of a torus action. All our study of the problem is done in a neighbourhood of a compact orbit. Global results for contact manifolds admiting torus action have been obtained by Banyaga and Molino in [5] and recently by Lerman in [41]. Linearization results for contact vector fields in $\mathbb{R}^{2n}$ with an hyperbolic zero were considered by Guillemin and Schaeffer in [32].

We will prove that for any two contact forms for which $\mathcal{F}$ is Legendrian and having the same Reeb vector field, we can find a foliation preserving contactomorphism taking one to the other. It turns out that the Legendrian condition imposed on the foliation for the contact form α becomes a Lagrangian condition for the same foliation with the symplectic form $d\alpha$ defined in a convenient submanifold. The result appears then as an application of the symplectic equivalence results for the symplectic case spelled out before. We can also establish the G-equivariant version of contact equivalence. Applying this G-equivariant version to the particular case of the finite group attached to the finite covering, we obtain as a consequence the contact equivalence of any two contact forms fulfilling the above mentioned conditions.

In contrast to symplectic manifolds (M,ω) where the condition $i_X(\omega)=0$ implies $X=0$, in a contact manifold we can find non-trivial solutions X to the equation $i_X(\omega)=0$. A privileged solution of this equation has the particular name of Reeb vector field. It is a concept attached to the contact form rather than the contact structure.

Definition 3.1. Given a contact pair (M, α), the Reeb vector field Z is the unique vector field satisfying the following two conditions,

- $i_Z d\alpha = 0$.
- $\alpha(Z) = 1$.

The Reeb vector field is a particular case of what we call contact vector field.

Definition 3.2. Let f be a smooth function on the contact pair (M, α) the contact vector field associated to f is the unique vector field X_f fulfilling the following two conditions

- $i_{X_f} d\alpha_{|E} = -df_{|E}$.
- $\alpha(X_f) = f$.

Observe that the contact vector field associated to the function 1 is precisely the Reeb vector field.

As it is proved in [47], we can express any vector field X in $T(M)$ as a sum of two vector fields X_1 and X_2 where the vector field X_1 belongs to the subbundle E and its called the horizontal part of X and the vector field X_2 is the component in the direction of the Reeb vector field. The standard notation for the horizontal vector field associated to X is $\widehat{X}$.

We can now define the notion of Jacobi bracket of two functions, which is the contact counterpart to the Poisson bracket of two functions.

Definition 3.3. Let f, g be two smooth functions on a contact pair (M, α), we define the Jacobi bracket as,

$$[f, g] = \alpha([X_f, X_g]).$$

The following relations are proved in [47],

- $$X_{[f,g]} = [X_f, X_g] \tag{3.1}$$
- $$[f, g] = d\alpha(X_f, X_g) + f(Z(g)) - g(Z(f)) \tag{3.2}$$

Definition 3.4. A submanifold $N \subset M^{2n+1}$ is Legendrian if $dimN = n$ and $\alpha(X) = 0$ for any $X \in T(N)$.

3.1. **Toric and non-toric contact manifolds and integrability.** In this section we define the foliations that we will work with throughout this section and we will also define the linear model. Let (M^{2n+1}, α) be a contact pair and let Z be its Reeb vector field. We make the following assumptions,

- We assume Z coincides with the infinitesimal generator of an S^1 action. Let S be one of its orbits.
- We assume that there are n first integrals $f_1, \ldots, f_n$ of Z (that is $Z(f_i) = 0$) which fulfil the following additional hypotheses:
  (1) The first integrals are independent in an open dense set. That is, $df_1 \wedge \cdots \wedge df_n \neq 0$ in an open dense set.
  (2) The n-first integrals are in involution with respect to the Jacobi bracket associated to α. That is to say,

$$[f_i, f_j] = 0 \quad , \forall i, j.$$

  (3) The minimum rank of the differential $(df_1, \ldots, df_n)$ is k. Let p be a point in M^{2n+1} such that the rank is exactly k. Let $\mathcal{O}$ be the orbit of the contact vector fields through p. We will assume the following,
    (a) $\mathcal{O}$ is diffeomorphic to a torus of dimension $k+1$.
    (b) The first integrals $f_1, \ldots, f_k$ are non-singular along $\mathcal{O}$ and the first integrals $f_{k+1}, \ldots, f_n$ have a non-degenerate singularity in the Morse-Bott sense along $\mathcal{O}$.

Since $[f_i, f_j] = 0$ then due to formula 3.1, $[X_{f_i}, X_{f_j}] = 0$ and this implies that the distribution $< Z, X_{f_1}, \ldots, X_{f_n} >$ is involutive because the functions f_i are first integrals of the Reeb vector field. Thus, we can talk about the foliation generated by the contact vector fields of the functions $1, f_1, \ldots, f_n$. This foliation will be denoted by $\mathcal{F}'$.

On the other hand, consider the horizontal parts of the contact vector fields. They have the form $\widehat{X}_f = X_f - fZ$. Thus the distribution $< \widehat{X}_{f_1}, \ldots \widehat{X}_{f_n} >$ defines an involutive distribution. The foliation defined by this distribution will be denoted by $\mathcal{F}$. Observe that since $\alpha(X_f) = f$ and $\alpha(Z) = 1$ then the regular leaves of this foliation are Legendrian submanifolds with respect to α.

That is why this foliation will be called *the singular Legendrian foliation*.

In fact we will work with germ-like foliations. That is, we will assume that the foliation is defined in a neighbourhood of $\mathcal{O}$. Now let $p \in M$ be a singular point. We will say that the point has rank r if the dimension of the orbit through p is r.

Once the two foliations $\mathcal{F}$ and $\mathcal{F}'$ are defined we are ready to pose the following problem.

Problem

Study the contact forms α' defined in a neighbourhood of $\mathcal{O}$ for which $\mathcal{F}$ is Legendrian and such that the Reeb vector field with respect to α' coincides with the Reeb vector field with respect to α.

As far as this problem is concerned we will prove the following.

There exists a diffeomorphism ϕ defined in a neighbourhood of $\mathcal{O}$ such that $\phi^*(\alpha') = \alpha$ and ϕ preserves the foliations $\mathcal{F}$ and $\mathcal{F}'$.

In order to deal with this problem we will need to introduce coordinates in such a way that the foliations $\mathcal{F}$ and $\mathcal{F}'$ are really simple. This judicious choice of coordinates leads us to the linear model.

3.1.1. *Differentiable linearization.* In this subsection we want to prove that under the above assumptions there exist coordinates in a neighbourhood of $\mathcal{O}$ such that the foliation can be linearized.

We prove the following theorem,

Theorem 3.5. *There exist coordinates $(\theta_0, \dots, \theta_k, p_1, \dots, p_k, x_1, y_1, \dots, x_{n-k}, y_{n-k})$ in a finite covering of a tubular neighbourhood of $\mathcal{O}$ such that*

- *The Reeb vector field is $Z = \frac{\partial}{\partial\theta_0}$.*
- *There exists a triple of natural numbers (k_e, k_h, k_f) with $k_e + k_h + 2k_f = n-k$ and such that the first integrals f_i are of the following type, $f_i = p_i, \quad 1 \leq i \leq k$ and*

  $f_{i+k} = x_i^2 + y_i^2$ for $1 \leq i \leq k_e$,
  $f_{i+k} = x_i y_i$ for $k_e + 1 \leq i \leq k_e + k_h$,
  $f_{i+k} = x_i y_{i+1} - x_{i+1} y_i$ and
  $f_{i+k+1} = x_i y_i + x_{i+1} y_{i+1}$ for $i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f$
- *The foliation $\mathcal{F}$ is given by the orbits of the distribution $\mathcal{D} =< Y_1, \dots Y_n >$ where $Y_i = X_i - f_i Z$ being X_i the contact vector field of f_i with respect to the contact form $\alpha = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i$.*

Proof. First of all, since Z is the infinitesimal generator of an S^1-action, according to the Slice Theorem [?] a neighbourhood of $\mathcal{O}$ in M^{2n+1} is diffeomorphic to the bundle $S^1 \times_{S^1_x} W$ where S^1_x denotes the isotropy group at a point in the orbit. Thus we can choose coordinates

$$(\theta_0, \dots, \theta_k, p_1, \dots, p_k, x_1, y_1, \dots, x_{n-k}, y_{n-k})$$

in a finite covering of a neighbourhood of $\mathcal{O}$ such that the Reeb vector field has the form $Z = \frac{\partial}{\partial\theta_0}$. Now the 1-form α can be written as

$$\alpha = d\theta_0 + \overline{\alpha}.$$

Observe that since Z is the Reeb vector field in particular we obtain

$$i_Z d\overline{\alpha} = 0$$

Using Cartan's formula $L_Z(overline\alpha) = di_Z(\overline{\alpha}) + i_Z d\overline{\alpha}$ we deduce that $\overline{\alpha}$ does not depend on θ_0.

Further, the condition on the contact form $\alpha \wedge d\alpha^n \neq 0$ implies that $d\overline{\alpha}$ is a symplectic form in the submanifold $N_0 = \{p \in U(\mathcal{O}), \quad \theta_0 = 0\}$. Let f_i be the n first integrals. The equation

$$i_Y d\overline{\alpha} = -df_i$$

has a unique well-defined solution when restricted to the symplectic submanifold N_0. We denote by $X^s_{f_i}$ the n Hamiltonian vector fields of the functions f_i with respect to the symplectic structure $d\overline{\alpha}$ on N_0. We denote by $X^c_{f_i}$ the n contact vector fields of the functions f_i with respect to the contact structure α. With all these information at hand we can write

$$X^c_{f_i} = X^s_{f_i} + g_i Z \tag{3.3}$$

for certain smooth functions g_i.

We are going to focus our attention in the symplectic submanifold N_0 and in the Hamiltonian vector fields $X^s_{f_i}$ for a while.

First of all, we will check that $\{f_i, f_j\} = 0$ where $\{,\}$ stands for the Poisson bracket attached to $d\overline{\alpha}$. Thus, the vector fields $X^s_{f_i}$ define a completely integrable Hamiltonian system on N_0 and the foliation they define is a singular Lagrangian foliation.

We are going to check

$$\{f_i, f_j\} = [f_i, f_j]$$

Because of the definition of Poisson bracket,

$$\{f_i, f_j\} = d\overline{\alpha}(X^s_{f_i}, X^s_{f_j})$$

Since $d\alpha = d\overline{\alpha}$, we can write this last equality as, $d\alpha(X^s_{f_i}, X^s_{f_j})$

Taking into account this observation and due to 3.3 this equality can be written as,

$$\{f_i, f_j\} = d\alpha(X^c_{f_i} - g_i Z, X^c_{f_j} - g_i Z)$$

But Z is the Reeb vector field and the last expression reads

$$d\alpha(X^c_{f_i}, X^c_{f_j})$$

which is, by definition, the Jacobi bracket of the functions f_i and f_j. Thus $\{f_i, f_j\} = [f_i, f_j] = 0$

Denote by $\mathcal{O}_N$ a singular compact orbit of minimal rank of the singular Lagrangian foliation in N_0. According to the symplectic linearization theorem (theorem 2.13) for Lagrangian foliations. There exists a diffeomorphism in a neighbourhood of a singular compact orbit which takes the foliation to the linearized one and the symplectic structure $d\overline{\alpha}$ to the Darboux symplectic structure. Recall that the linearized foliation has a finite group attached to it. In particular, we can find a diffeomorphism in a covering of a tubular neighbourhood of $\mathcal{O}_N$, $\phi : (\widetilde{U(\mathcal{O}_N)}) \longrightarrow \phi((\widetilde{U(\mathcal{O}_N)}))$ such that in the new coordinates provided by the diffeomorphism the first integrals have the following simple form:

$$\begin{array}{l} f_i = p_i, \text{ for } 1 \leq i \leq k \ , f_{i+k} = x_i^2 + y_i^2 \text{ for } 1 \leq i \leq k_e \ , \\ f_{i+k} = x_i y_i \text{ for } k_e + 1 \leq i \leq k_e + k_h \ , \\ f_{i+k} = x_i y_{i+1} - x_{i+1} y_i \text{ and} \\ f_{i+k+1} = x_i y_i + x_{i+1} y_{i+1} \text{ for } i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f \end{array}$$

Now define,

$$\begin{array}{rccc} \varphi : & S^1 \times (\widetilde{U(\mathcal{O}_N)}) & \longrightarrow & \varphi(S^1 \times (\widetilde{U(\mathcal{O}_N)})) \\ & (\theta_0, z) & \longrightarrow & (\theta_0, \phi(z)) \end{array}$$

Observe that since

$$\phi^*(\sum_{i=1}^{k} dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i) = d\overline{\alpha}$$

Then by using the relative Poincaré lemma in a neighbourhood of compact orbit, we may assume $\phi^*(\sum \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum p_i d\theta_i + dH) = \overline{\alpha}$

this yields,

$$\varphi^*(d\theta_0 + \sum \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum p_i d\theta_i + dH) = d\theta_0 + \overline{\alpha}$$

Thus we may assume that in the new coordinates

$$\alpha = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i + dH$$

Now consider the path of contact forms

$$\alpha_t = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i + t dH$$

Observe that $\alpha_1 = \alpha$ and α_0 is the Darboux contact form. Consider now the diffeomorphism,

$\phi_1(\theta_0, \theta_1, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k}) = (\theta_0 - H, \theta_1, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k})$. So $\psi_1(\alpha_1) = \alpha_0$. Thus, $\psi_1^*(\alpha_1) = \alpha_0$ and in the new coordinates provided by ψ_1 we can assume that α is the Darboux contact form. That is to say, we can assume that $\alpha = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i$. In the new coordinates f_i have the same form.

Finally the foliation we are considering is generated by the horizontal parts of X_{f_i} which in the new coordinates are $Y_i = X_i - f_i Z$ being X_i the contact vector field of

f_i with respect to the contact form $\alpha = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i$. This ends the proof of the theorem.

□

This theorem establishes the existence of a linear foliation and a model manifold.

The model manifold is the manifold $M_0^{2n+1} = \mathbb{T}^{k+1} \times U^k \times V^{2(n-k)}$, where U^k and $V^{2(n-k)}$ are k-dimensional and $2(n-k)$ dimensional disks respectively. Now we introduce a contact form in this model manifold. We take coordinates $(\theta_0, \ldots, \theta_k)$ on $\mathbb{T}^{k+1}$, $(p_1, \ldots, p_k)$ on U^k and $(x_1, \ldots, x_{n-k}, y_1, \ldots y_{n-k})$ on $V^{2(n-k)}$ and we consider the following contact form

$$\alpha_0 = d\theta_0 + \sum_{i=1}^{k} p_i d\theta_i + \sum_{i=1}^{(n-k)} \frac{1}{2}(x_i dy_i - y_i dx_i).$$

The pair (M_0^{2n+1}, α_0) is called the contact model manifold. The Reeb vector field in the contact model manifold is the vector field $\frac{\partial}{\partial \theta_0}$.

Now consider functions of the following type, $f_i = p_i, \quad 1 \leq i \leq k$ and

$f_{i+k} = x_i^2 + y_i^2$ for $1 \leq i \leq k_e$,
$f_{i+k} = x_i y_i$ for $k_e + 1 \leq i \leq k_e + k_h$,
$f_{i+k} = x_i y_{i+1} - x_{i+1} y_i$ and
$f_{i+k+1} = x_i y_i + x_{i+1} y_{i+1}$ for $i = k_e + k_h + 2j - 1,\ 1 \leq j \leq k_f$

The linear foliation is the foliation given by the orbits of the distribution $\mathcal{D} = < Y_1, \ldots Y_n >$ where $Y_i = X_i - f_i Z$ being X_i the contact vector field of f_i in the contact model manifold.

In all, we have proved that there exists a finite covering of a neighbourhood $U(\mathcal{O})$ of the compact orbit considered such that the lifted foliation in the covering is differentiably equivalent to the linear foliation in the contact model manifold.

The linear model for the foliation $\mathcal{F}'$ is the foliation expressed in the coordinates provided by the theorem together with a finite group attached to the finite covering.

The different smooth submodels corresponding to the model manifold are labeled by a finite group which acts in a contact fashion and preserves the foliation in the model manifold. This is the only differentiable invariant. Therefore, our problem of contact equivalence will be studied in this model manifold and the equivalence will be established via the equivariant version equivalence which will be considered in the last section.

We now prove the following theorem,

Theorem 3.6. *Let α be a contact form on the model manifold M_0^{2n+1} for which $\mathcal{F}$ is a Legendrian foliation and such that the Reeb vector field is $\frac{\partial}{\partial \theta_0}$. Then there exists a diffeomorphism ϕ defined in a neighbourhood of the singular orbit $\mathcal{O} = (\theta_0, \ldots, \theta_k, 0, \ldots, 0)$ preserving $\mathcal{F}'$ and taking α to α_0.*

Proof. We are going to solve the problem by adjusting the contact form to a point where we can apply our symplectic linearization result.

Let us start by considering the contact 1-form α,

$$\alpha = Ad\theta_0 + \sum B_i dp_i + \sum C_i d\theta_i + \sum D_i dx_i + \sum E_i dy_i$$

Observe that the fact that the Reeb vector field is $\frac{\partial}{\partial\theta_0}$ imposes the following two conditions on α,

- $\alpha(\frac{\partial}{\partial\theta_0}) = 1$, that is to say $A = 1$.
  So far we can write $\alpha = d\theta_0 + \alpha'$, being $\alpha' = \sum B_i dp_i + \sum C_i d\theta_i + \sum D_i dx_i + \sum E_i dy_i$.
- $i_{\frac{\partial}{\partial\theta_0}} d\alpha = 0$,

Since $d\alpha = d\alpha'$ the condition becomes,

$$i_{\frac{\partial}{\partial\theta_0}} d\alpha' = 0$$

Now Cartan's formula yields,

$$0 = i_{\frac{\partial}{\partial\theta_0}} d\alpha' = L_{\frac{\partial}{\partial\theta_0}} \alpha' - di_{\frac{\partial}{\partial\theta_0}} \alpha'$$

Since the last term vanishes this chain of equalities give the condition,

$$L_{\frac{\partial}{\partial\theta_0}} \alpha' = 0$$

Therefore, the coefficient functions do not depend on θ_0. Let us see that the submanifold $\theta_0 = 0$ equipped with the form $d\alpha'$ is a symplectic submanifold of the model contact manifold. We denote this submanifold by N.

Since α is a contact form $d\alpha$ has to be symplectic in the vector bundle E defined by $E = \{(p, u) \in T(M), \quad \alpha_p(u) = 0\}$ and $d\alpha = d\alpha'$ then $d\alpha'$ defines a symplectic structure on N.

Observe that the vector fields $X_i = X_{f_i}$ are tangent to the submanifold N. Next step, we check that the vector fields X_i are Lagrangian for N, observe that $\alpha(X_i) = f_i$.

Now since, $d\alpha'(X_i, X_j) = X_i\alpha(X_j) - X_j\alpha(X_i) - \alpha([X_i, X_j])$

According to the computation above $X_i\alpha(X_j) = X_i(f_j)$ but f_i are first integrals for the foliation and therefore this term vanishes. Symmetrically, the second term vanishes. And since the Lie bracket of the vector fields are zero we obtain,

$$d\alpha'(X_i, X_j) = 0$$

Therefore, the foliation $\mathcal{F}$ is Lagrangian for $d\alpha'$ and we may apply the symplectic linearization result in a neighbourhood of $L = \mathbb{T}^k$ (theorem **??**) to find a local diffeomorphism $\varphi : U(L) \longrightarrow \varphi(U(L))$ in a neighbourhood of the leaf L, preserving the foliation $\mathcal{F}$ and satisfying $\varphi^*(\omega_0) = d\alpha'$, where $\omega_0 = \sum_i dp_i \wedge d\theta_i + \sum dx_i \wedge dy_i$. After shrinking the initial neighbourhood if necessary, the neighbourhood of $\mathbb{T}^{k+1}$ in the initial manifold M can be decomposed as a product, $\mathbb{S}^1 \times U(L)$. The $\mathbb{S}^1$ corresponds to an orbit of the Reeb vector field. We denote by z a point in $U(L)$. Now we define a diffeomorphism in the following way,

$$\begin{array}{rccc} \phi : & \mathbb{S}^1 \times U(L) & \longrightarrow & \phi(\mathbb{S}^1 \times U(L)) \\ & (\theta_0, z) & \longrightarrow & (\theta_0, \varphi(z)) \end{array}$$

Since φ preserves $\mathcal{F}$ it is clear that this diffeomorphism is foliation-preserving.

Now consider $\phi(\mathbb{S}^1 \times U(L))$ endowed with the Darboux contact form. That is with the contact form $\alpha_0 = d\theta_0 + \sum_{i=1}^{k} p_i d\theta_i + \sum_{i=1}^{(n-k)} \frac{1}{2}(x_i dy_i - y_i dx_i)$. It remains to check that the diffeomorphism above is indeed a contactomorphism.

First observe that since

$$\varphi^*(\omega_0) = d\alpha'$$

and $\omega_0 = d(\beta)$, being $\beta = (\sum_{i=1}^{k} p_i d\theta_i + \sum_{i=1}^{(n-k)} \frac{1}{2}(x_i dy_i - y_i dx_i))$ we can assert that $\varphi^*(\beta) = \alpha' + df$ for a smooth function f. Observe that since φ preserves the foliation the function f is a basic function for the foliation. Now consider the path $\alpha_t = \alpha_0 + t df$ being α_0 the contact form $\alpha_0 = d\theta_0 + \alpha'$. Consider

$\psi_1(\theta_0, \theta_1, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k}) = (\theta_0 - H, \theta_1, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k})$, we obtain $\psi_1^*(\alpha_1) = \alpha_0$.

Therefore ϕ is a contactomorphism and clearly it preserves the foliation.

□

Remark 3.7. The proof we have included here is the one provided in my thesis. There is another way to go which is to consider the reduction via the Hamiltonian S^1-action that the Reeb vector field determines in its symplectization. In the next section we use this idea to prove an equivariant statement of the theorem above.

3.2. **Additional symmetries.** In this subsection we consider a compact Lie group G acting on a contact model manifold in such a way that preserves the n first integrals of the Reeb vector field and preserves the contact form as well. We want to prove that there exists a diffeomorphism in a neighbourhood of $\mathcal{O}$ preserving the n first integrals , preserving the contact form and linearizing the action of the group. This result is a consequence of the equivariant symplectic linearization theorem.

The notion of linear action of a Lie group on the contact model manifold is analogous to the equivalent notion for the symplectic model manifold.

Let G be a group defining a smooth action $\rho : G \times M_0^{2n+1} \longrightarrow M_0^{2n+1}$ on M_0^{2n+1}. We assume that this action preserves the contact form α_0 of the contact model manifold. That is to say $\rho_g^*(\alpha_0) = \alpha_0$. Assume further that it preserves the n-first integrals $(f_1, \ldots, f_n)$, where $f_i = p_i, \quad 1 \leq i \leq k$. For the sake of simplicity we denote by F the collective mapping $F = (p_1, \ldots, p_k, f_{k+1}, \ldots, f_n)$. We will say that the action of G on M_0^{2n+1} is *linear* if it satisfies the following property:

G acts on the product $M_0^{2n+1} = D^k \times \mathbb{T}^{k+1} \times D^{2(n-k)}$ componentwise; the action of G on D^k is trivial, its action on $\mathbb{T}^{k+1}$ is by translations (with respect to the coordinate system $(\theta_0, \ldots, \theta_k)$), and its action on $D^{2(n-k)}$ is linear with respect to the coordinate system $(x_1, y_1, ..., x_{n-k}, y_{n-k})$.

Under the above notations and assumptions. Now we can state and prove the following theorem,

Theorem 3.8. *There exists a diffeomorphism ϕ defined in a tubular neighbourhood of $\mathcal{O}$ such that,*

- *it preserves the contact form* α_0 *i.e* $\phi^*(\alpha_0) = \alpha_0$.
- *it preserves* F.
- *it linearizes the action of* G. *That is to say* $\phi \circ \rho_g = \rho_g^{(1)} \circ \phi$.

Proof. Recall that $\alpha_0 = d\theta_0 + \overline{\alpha}_0$ being $\overline{\alpha}_0$ the 1-form $(\sum_{i=1}^k p_i d\theta_i + \sum_{i=1}^{(n-k)} \frac{1}{2}(x_i dy_i - y_i dx_i))$. Consider the symplectic manifold $S = M_0^{2n+1} \times (-\varepsilon, \varepsilon)$ endowed with the symplectic form $\omega_0 = dt \wedge d\theta_0 + d\overline{\alpha}_0$, where t stands for a coordinate function on $(-\varepsilon, \varepsilon)$. An action of G on M_0^{2n+1} can be extended in a natural way to an action of G on S as follows,

$$\begin{array}{rccc} \widehat{\rho}: & G \times M_0^{2n+1} \times (-\varepsilon,\varepsilon)+ & \longrightarrow & M_0^{2n+1} \times (-\varepsilon,\varepsilon) \\ & (g, z, t) & \longrightarrow & (\rho_g(z), t) \end{array}$$

On S we consider the moment mapping $\widehat{F} = (F, t)$. We can apply the equivariant linearization theorem to obtain a symplectomorphism $\widehat{\varphi}$ preserving $\widehat{F}$ and linearizing the action $\widehat{\rho}$. From the definition of the action $\widehat{\rho}$ and the definition of $\widehat{F}$, this symplectomorphism clearly descends to a diffeomorphism φ on M_0^{2n+1} which linearizes the action ρ and which satisfies $\varphi^*(d\alpha_0) = d\alpha_0$.

Therefore by applying relative Poincaré lemma,

$$\varphi^*(\alpha_0) = \alpha_0 + dh$$

Finally the diffeomorphism,

$$\phi(\theta_0, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k}) = (\theta_0 - h, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k})$$

takes the form $\alpha_0 + dh$ to α_0 and provides new coordinates for which the action is linear.

□

In the previous section we have attained the contact linearization in the covering. Now applying the theorem of equivariant linearization to the group of deck transformations we obtain as a corollary the following theorem,

Theorem 3.9. *Let* $\mathcal{F}$ *be a foliation fulfilling the hypotheses specified in section 3.1, let* $\mathcal{F}'$ *be the enlarged foliation with the Reeb vector field* Z *and let* α *be a contact form for which* $\mathcal{F}$ *is Legendrian and such that* Z *is the Reeb vector field then there exists a diffeomorphism defined in a neighbourhood of* $\mathcal{O}$ *taking* $\mathcal{F}'$ *to the linear foliation, the orbit* $\mathcal{O}$ *to the torus* $\{x_i = 0, y_i = 0, p_i = 0\}$ *and taking the contact form to the Darboux contact form* α_0.

3.2.1. *Another approach to the equivariant case: The rigidity problem.* In the same sense, that the rigidity problem was approached in the symplectic case, we can also prove that close contact structures are equivalent at the local, semilocal and global case in the

The local case a linearization result for compact contact group actions was already established by Marc Chaperon [17].

In the global case, we can use the path method in the contact setting due to [30] and reproduce the same ideas of the proof of the symplectic case.

This statement is implicit in [65],

Theorem 3.10 (Miranda-Monnier-Zung)**.** *Let ρ_0 and ρ_1 be two C^2-close contact actions of a compact Lie group G on a compact contact manifold (M, α). Then they can be made equivalent by conjugation via a contactomorphism.*

4. The Poisson case

In this section we are going to provide some normal form results for integrable systems in the Poisson setting.

We start this section by providing some natural examples of Hamiltonian and integrable systems in Poisson manifolds.

4.1. **Examples.**

4.1.1. *Example 1: Newton systems.* This example was found with Alain Albouy. I thank him for explaining me about Projective Dynamics.

Consider a system of the form:

$$\ddot{q} = f(q)$$

These kind of systems are called Newton system. It is a Hamiltonian system in moment and position coordinates.

Appell discovered that such systems can be projectivised.

Appell's transformation (central projection) allows to change the "screen" of projection (change of affine coordinates). Two such systems and their solutions are equivalent. This is the principle of "Projective Dynamics" (Appell, Killing, Albouy [1]...).

The study of projective dynamics allows to:

- Solve some problems by separation of variables.
- Simplify the solution of those systems by finding an appropriate screen. For instance the Neumann problem on the ellipsoid becomes a Newton System (Knoerrer).

  By means of Appell's transformation we transform both the dynamics and the initial symplectic structure. We are including singularities into the picture (coming from projectivisation). It motivates to look at the integrable system from the Poisson point of view.

An example of Newton system is the two fixed-center problem (Euler, 1760). A particle in the plane moves under the gravitational attraction of two fixed points A and B with masses m_A and m_B.

This system reads:

$$\ddot{q} = -m_A \frac{q_A}{\|q_A\|^3} - m_B \frac{q_B}{\|q_B\|^3}$$

($q_A = q - A$, $q_B = q - B$) Two first integrals are:

$$H = \frac{1}{2}\|\dot{q}\|^2 - \frac{m_A}{\|q_A\|} - m_B \frac{m_B}{\|q_B\|}$$

$$G = \langle q_A \wedge \dot{q}, q_B \wedge \dot{q}\rangle - \frac{m_A}{\|q_A\|}\langle q_A, u\rangle - m_B \frac{m_B}{\|q_B\|}\langle q_B, u\rangle$$

where $u = q_A - q_B$

They satisfy $\{H, G\} = 0$.

We can now perform central projection for the two-centers problem:

We start from the cotangent bundle in $T^*(\mathbb{R}^2)$. Consider the "position" homogeneous coordinates $[q_0 : q_1 : q_2]$. (the initial affine chart is $q_0 = 1$).

We now perform central projection to the screen $q_2 = 1$. After this, we change the momenta accordingly and we obtain an integrable system on the new screen.

The new "symplectic "structure (it is not symplectic we added "singularities" in the procedure) reads:

$$dv_1 \wedge dq_1 + \frac{q_1}{q_2}(dq_1 \wedge dv_2 + dq_2 \wedge dv_1) + \frac{(v_2 q_1 - v_1 q_2)}{q_2^2} dq_1 \wedge dq_2 + (\frac{q_1^2}{q_2^2} - 1) dv_2 \wedge dq_2$$

It makes sense to dualize the 2-form to get the hyperplane $q_2 = 0$ into the picture. By doing so, we can associate a bivector field which is a Poisson structure on a dense set together with an integrable system on it.

4.1.2. *Example 2: Gelfand-Cetlin system.* The Gelfand-Cetlin system has been classically (Guillemin-Sternberg) considered as an integrable system on a coadjoint orbit $\mathcal{O}$ of $\mathfrak{u}(n)^*$. A good reference for this system is [34].

The dual of a Lie algebra constitute a simple example of linear Poisson structure with Poisson brackets defined via the structure constants. The Gelfand-Cetlin system can be seen as a system on the dual of a Lie algebra, as follows. We dualize the increasing sequence of Lie algebra inclusions:

$$\mathfrak{u}(1) \subset \cdots \subset \mathfrak{u}(n-1) \subset \mathfrak{u}(n)$$

where $\mathfrak{u}(k)$ is considered as the left-upper diagonal block of $\mathfrak{u}(k+1)$ for $k = 1, \ldots, n-1$), we get a sequence of surjective Poisson maps:

$$\mathfrak{u}(n)^* \longmapsto \mathfrak{u}(n-1)^* \longmapsto \cdots \longmapsto \mathfrak{u}(1)^*$$

The family of functions on $\mathfrak{u}(n)^*$ obtained by pulling-back generators of the Casimir algebras of all the $\mathfrak{u}(k)^*$ for $k = 1, \ldots, n$ yields an integrable system on $\mathfrak{u}(n)^*$. The complete integrability of this system is a consequence of the method of Thim).

For particular generators, its restriction to an open subset of $\mathcal{O}$ gives the Gelfand-Cetlin system (a commendable reference for this paper is [34]).

This system is defined not only when restricted to the coadjoint orbit with the Kirillov-Kostant-Souriau symplectic structure but on $\mathfrak{u}(n)^*$.

4.1.3. *Example 3: Magnetic Flows on Homogeneous Spaces and coadjoint orbits.* The geodesic flow can be interpreted as the inertial motion of a particle on a Riemannian manifold (Q, g_{ij}) with kinetic energy given by:

$$H(x,p) = \frac{1}{2}\sum_{i,j} g^{ij} p_i p_j$$

The symplectic structure associated to g comes from the cotangent symplectic structure with momenta defined via $p_j = g_{ij}\dot{x}_i$. The motion of a particle under additional magnetic field given by a closed 2-form on Q:

$$\Omega = \sum_{i,j} F_{ij}(x) dx_i \wedge dx_j$$

is described by Hamilton's equations associated to the symplectic structure $\omega + \rho^*(\Omega)$ where ρ is the standard projection from T^*Q to Q.

Let G be a compact Lie group and H a closed subgroup. Let $a \in \mathfrak{h}$ be H-adjoint invariant.

In particular, $H \in G_a$. Let $\mathcal{O}(a)$ stand for the adjoint orbit. We then have a submersion of homogeneous spaces $\sigma : G/H \longrightarrow G/G_a \cong \mathcal{O}(a)$ and $\omega = \sigma^*(\Omega_{KKS})$ (with Ω_{KKS} the Kostant-Kirillov-Souriau symplectic form) gives a magnetic field on G/H. We then have [6]:

Theorem 4.1 (Bolsinov, Jovanovich)**.** *The magnetic geodesic flows of normal metric ds_0^2 in G/H with respect to the magnetic form ω is completely integrable in the non-commutative sense.*

By using a Theorem of Mischenko-Fomenko (non-commutative integrability $\rightsquigarrow$ commutative integrability) we obtain integrability of the magnetic geodesic flows on G/H. Again, this can be viewed in the Poisson manifold $T^*(G)/H$.

4.2. **The local structure of a Poisson manifold.** We start by defining what is a Poisson structure.

When working with Poisson structure we need to work with bivector fields instead of using forms.

Definition 4.1. A Poisson structure on a smooth manifold M is given by a smooth antisymmetric bivector field Π satisfying

$$[\Pi, \Pi] = 0$$

This defines a Poisson bracket on $\mathcal{C}^\infty(M)$,

$$\{f, g\} := \Pi(df, dg)$$

Hamiltonian vector fields are defined by the formula $X_f := \Pi(df, \dot{})$ and the manifold M is endowed with a smooth foliation (in the Sussmann sense) whose leaves are symplectic manifolds. This symplectic foliation integrates the distribution of Hamiltonian vector fields.

There is no Darboux theorem for Poisson manifolds. The best "normal form" that we can get is the following result due to Weinstein [77]

Theorem 4.2 (Weinstein). *Let (M^n, Π) be a smooth Poisson manifold and let p be a point of M of rank $2k$, then there is a smooth local coordinate system $(x_1, y_1, \dots, x_{2k}, y_{2k}, z_1, \dots, z_{n-2k})$ near p, in which the Poisson structure Π can be written as*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where f_{ij} vanish at the origin.

The Poisson manifold is locally a product of a symplectic manifold with a Poisson manifold with vanishing Poisson structure at the point.

$$(M^n, \Pi, p) \approx (N^{2k}, \omega, p_1) \times (M_0^{n-2k}, \Pi_0, p_2)$$

The symplectic foliation on the manifold is locally a product of the induced symplectic foliation on M_0 with the symplectic leaf through x.

4.3. **Integrable systems, normal forms and action-angle coordinates for Poisson manifolds.** The results contained in this section are based on joint work with Camille Laurent-Gengoux and Pol Vanhaecke and are mostly contained in our joint paper [33].

Let us start by defining what is an integrable system on a Poisson manifold.

Definition 4.2. Let (M, Π) be a Poisson manifold of (maximal) rank $2r$ and of dimension n. An s-tuplet of functions $\mathbf{F} = (\mathbf{f_1}, \dots, \mathbf{f_s})$ on M is said to define a Liouville integrable system on (M, Π) if

(1) $f_1, \dots, f_s$ are independent
(2) $f_1, \dots, f_s$ are pairwise in involution
(3) $r + s = n$

Viewed as a map, $\mathbf{F} : \mathbf{M} \to \mathbf{R}^\mathbf{s}$ is called the *momentum map* of $(M, \Pi, \mathbf{F})$.

There are several problems analogous to the symplectic case that we could consider in the Poisson setting.

We can also consider these problems for "easy" Poisson manifolds (but not as easy as symplectic). These will be the b-Poisson case. [35], [36]. This will be considered in a future paper.

4.3.1. *A Darboux-Carathéodory theorem in the Poisson context.* We start by stating the local normal theorem that we have for Poisson structures contained in [33]:

Theorem 4.3 (Laurent, Miranda, Vanhaecke [33]). *Let m be a point of a Poisson manifold (M, Π) of dimension n. Let $p_1, \dots, p_r$ be r functions in involution, defined on a neighborhood of m, which vanish at m and whose Hamiltonian vector fields are linearly independent at m. There exist, on a neighborhood U of m, functions $q_1, \dots, q_r, z_1, \dots, z_{n-2r}$, such that*

(1) *The n functions $(p_1, q_1, \ldots, p_r, q_r, z_1, \ldots, z_{n-2r})$ form a system of coordinates on U, centered at m;*
(2) *The Poisson structure Π is given on U by*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i,j=1}^{n-2r} g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{4.1}$$

where each function $g_{ij}(z)$ is a smooth function on U and is independent of $p_1, \ldots, p_r, q_1, \ldots, q_r$.

Observe that in this theorem we have an adaptation of Weinstein's splitting theorem to some of the first integrals of the integrable system but not all of them.

As we explained in [33], not every integrable system on a Poisson manifold can be splitted (in a compatible way with Weinstein's theorem).

The following example is contained in [33]:

On $\mathbf{R}^4$, with coordinates f_1, f_2, g_1, g_2, consider:

$$\Pi = \frac{\partial}{\partial g_1} \wedge \frac{\partial}{\partial f_1} + g_2^2 \wedge \frac{\partial}{\partial f_2} + g_2 \frac{\partial}{\partial g_1} \wedge \frac{\partial}{\partial f_2}, \tag{4.2}$$

As we proved in [33] the system and the system is not splitted.

Remark 4.4. In general we can formulate the condition of an integrable system to be splitted via the Vorobjev data $(\Pi_{Vert}, \Gamma, \mathbb{F})$ associated to the Poisson structure. These data are determined in terms of the Poisson fibration over a symplectic leaf. This is the content of a common project with Camille Laurent-Gengoux.

Indeed the fact, that these systems cannot be splitted is closely related to the non-rigidity of integrable systems in the Poisson context.

That is to say, we can find close integrable systems which are not equivalent.

As an example consider the following one

On $M = \mathbb{R}^4$ consider the Poisson structure

$$\pi = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + z_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}.$$

and consider the two integrable systems.

(1) The integrable system defined by $F = (x, z_1)$.
(2) The integrable system defined by $F_\varepsilon = (x, z_1 + \varepsilon x z_2)$.

Then both integrable systems are close but are not equivalent (the first one is "splitted" and the second one is not. This example is contained in the joint work [46]).

4.3.2. *An action-angle theorem for Poisson manifolds.* Assume that:

(1) The mapping $\mathbf{F} = (f_1, \ldots, f_s)$ defines an integrable system on the Poisson manifold (M, Π) of dimension n and (maximal) rank $2r$.
(2) Suppose that $m \in M$ is a point such that it is regular for the integrable system and the Poisson structure.

(3) Assume further than the integral manifold $\mathcal{F}_m$ of the foliation $X_{f_1},\dots X_{f_s}$ through m is compact (Liouville torus).

The following theorem is contained in [33]:

Theorem 4.5 (Laurent, Miranda, Vanhaecke)**.** *Then there exists* $\mathbf{R}$*-valued smooth functions* $(\sigma_1,\dots,\sigma_s)$ *and* $\mathbf{R}/\mathbf{Z}$*-valued smooth functions* $(\theta_1,\dots,\theta_r)$*, defined in a neighborhood* U *of* $\mathcal{F}_m$ *such that*

(1) *The functions* $(\theta_1,\dots,\theta_r,\sigma_1,\dots,\sigma_s)$ *define a diffeomorphism* $U\simeq\mathbf{T}^r\times B^s$*;*
(2) *The Poisson structure can be written in terms of these coordinates as*

$$\Pi=\sum_{i=1}^{r}\frac{\partial}{\partial\theta_i}\wedge\frac{\partial}{\partial\sigma_i},$$

in particular the functions $\sigma_{r+1},\dots,\sigma_s$ *are Casimirs of* Π *(restricted to* U*);*
(3) *The leaves of the surjective submersion* $\mathbf{F}=(f_1,\dots,f_s)$ *are given by the projection onto the second component* $\mathbf{T}^r\times B^s$*, in particular, the functions* $\sigma_1,\dots,\sigma_s$ *depend only on the functions* $f_1,\dots,f_s$*.*

The proof follows the spirit of Duistermaat in the symplectic case. The steps of the proof are the following:

(1) Topology of the foliation. The fibration in a neighbourhood of a compact connected fiber is a trivial fibration by compact fibers.
(2) These compact fibers are tori: We can recover a $\mathbb{T}^n$-action tangent to the leaves of the foliation. This implies a process of uniformization of periods.

$$\begin{array}{rcccc}\Phi & : & \mathbf{R}^r\times(\mathbf{T}^r\times B^s) & \to & \mathbf{T}^r\times B^s\\ & & ((t_1,\dots,t_r),m) & \mapsto & \Phi^{(1)}_{t_1}\circ\cdots\circ\Phi^{(r)}_{t_r}(m).\end{array}\tag{4.3}$$

(3) We prove that this action is Poisson (we use the fact that if Y is a complete vector field of period 1 and P is a bivector field for which $\mathcal{L}_Y^2P=0$, then $\mathcal{L}_YP=0$).
(4) Finally we use the Poisson cohomology of the manifold and averaging with respect to this action to check that the action is Hamiltonian.
(5) To construct action-angle coordinates we use Darboux-Carathéodory and the constructed Hamiltonian action of $\mathbb{T}^n$ to drag normal forms from a neighbourhood of a point to a neighbourhood of a fiber.

In [33] we also give a version for the non-commutative case.

4.4. **Equivariant theorems for Poisson manifolds.** In this short paragraph we recall very quickly some equivariant results for compact group actions on Poisson manifolds which preserve the Poisson structure.

For a special kind of Poisson structures called tame, we have the following theorem (see [63]) which is an equivariant version of Weinstein's splitting theorem [77].

Theorem 4.6 (Miranda-Zung [63])**.** *Let* (P^n,Π) *be a smooth Poisson manifold,* p *a point of* P*,* $2k=rankPi(p)$*, and* G *a compact Lie group which acts on* P *in such a way that the action preserves* Π *and fixes the point* p*. Assume that the Poisson structure* Π *is tame at* p*. Then there is a smooth canonical local coordinate system*

$(x_1, y_1, \ldots, x_k, y_k,\ z_1, \ldots, z_{n-2k})$ *near* p*, in which the Poisson structure* Π *can be written as*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{4.4}$$

with $f_{ij}(0) = 0$*, and in which the action of* G *is linear and preserves the subspaces* $\{x_1 = y_1 = \ldots x_k = y_k = 0\}$ *and* $\{z_1 = \ldots = z_{n-2k} = 0\}$*.*

This result implies local rigidity for compact Poisson group actions.

By using Conn's linearization theorem [20] for semisimple Lie algebra's of compact type, we can also prove an equivariant linearization theorem which can be found in [63].

Among the set of Poisson actions there is a particular class of Poisson actions which deserves a special attention: The class of Hamiltonian actions.

Recently we have proved a rigidity result for Poisson actions with Philippe Monnier and Nguyen Tien Zung.

Not to distract the reader with many technical details, we give a "sloppy" statement here. The exact statement can be found in [65].

Theorem 4.7 (Miranda-Monnier-Zung [65]). *Let* ρ_0 *and* ρ_1 *two close Hamiltonian actions of compact semisimple type, then they are equivalent.*

In this paper, we also find an application to prove an equivariant normal form result for Poisson structures without assuming that the Poisson structure is *tame.*

References

[1] A. Albouy, *Projective dynamics and classical gravitation* Regular and Chaotic Dynamics, 13 (2008) pp. 525-542.

[2] V. I. Arnold, *Mathematical methods of classical mechanics.* Graduate Texts in Mathematics, **60**.

[3] M. Audin, A. Cannas da Silva and E. Lerman, *Symplectic Geometry of integrable Hamiltonian systems*, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2003.

[4] A. Banyaga, *The geometry surrounding the Arnold-Liouville Theorem* in Advances in geometry, Progress in Mathematics, **172**, Birkhauser, 1999.

[5] A. Banyaga and P. Molino, *Géométrie des formes de contact complètement intégrables de type torique,*, Sminaire Gaston Darboux de Gomtrie et Topologie Différentielle, 1991-1992(Montpellier), Univ. Montpellier II, Montpellier, 1993.

[6] A. V. Bolsinov, B. Jovanovic, *Non-commutative integrability, moment map and geodesic flows*, Annals of Global Analysis and Geometry 23 (2003), no. 4, 305-322.

[7] A. V. Bolsinov, V.S. Matveev, *Singularities of momentum maps of integrable Hamiltonian systems with two degrees of freedom.* (Russian summary) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 235 (1996), Differ. Geom. Gruppy Li i Mekh. 15-2, 54–86, 304; translation in J. Math. Sci. (New York) 94 (1999), no. 4, 1477-1500.

[8] Y. Colin de Verdière and J. Vey, *Le lemme de Morse isochore.*, Topology **18** (1979), 283–293.

[9] Y. Colin de Verdière, *Singular Lagrangian manifolds and semiclassical analysis.* Duke Math. J. **116** (2003), no. 2, 263–298.

[10] Y. Colin de Verdière and S. Vu-Ngoc, *Singular Bohr-Sommerfeld rules for 2D integrable systems*, Ann. Scient. Éc. Norm. Sup. **36**, 2, 1-55, (2003).

[11] C. Currás-Bosch, *Linéarisation symplectique en dimension 2.* Canad. Math. Bull. **44** (2001), no. 2, 129–139.

[12] C. Currás-Bosch, *Sur les feuilletages lagrangiens holonomie linéaire.* C. R. Acad. Sci. Paris Sr. I Math. 317 (1993), no. 6, 605–608.

[13] C. Curras-Bosch and E. Miranda, *Symplectic linearization of singular Lagrangian foliations in M^4*, Differential Geom. Appl. **18** (2003), no. 2 , 195-205.

[14] C. Currás-Bosch and P. Molino, *Holonomie, suspensions et classifications pour les feuilletages lagrangiens.* C. R. Acad. Sci. Paris Sr. I Math. **326** (1998), no. 11, 1317–1320.

[15] C. Currás-Bosch and P. Molino, *Voisinage d'une feuille compacte dans un feuilletage lagrangien: le problme de linarisation symplectique.* Hokkaido Math. J. **23** (1994), no. **2**, 355–360.

[16] C. Curràs-Bosch and P. Molino, *Réduction symplectique d'un feuilletage lagrangien au voisinage d'une feuille compacte.* C. R. Acad. Sci. Paris Sr. I Math. **318** (1994), no. 7, 661–664.

[17] M. Chaperon, *Quelques outils de la théorie des actions différentiables*, in: Third Schnepfenried Geometry Conference, vol. 1 ,Schnepfenried, 1982, in: Astrisque, vols. 107–108, Soc. Math. France, Paris, 1983, pp. 259–275.

[18] C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85–124.

[19] J. Conn, *Normal forms for Analytic Poisson structures*, Ann. of Math(2)119, n3, 577–601.

[20] J. Conn, *Normal forms for smooth Poisson structures,* Ann. of Math. (2) 121 (1985), no. 3, 565–593.

[21] T. Delzant *Hamiltoniens périodiques et images convexes de l'application moment.* Bull. Soc. Math. France 116 (1988), no. 3, 315–339.

[22] B. Khesin and S. Tabaschnikov, *Contact complete integrability*, with B. Khesin. Reg. Chaotic Dynamics, 15 (2010), 504-520

[23] J.P. Dufour, P. Molino and A. Toulet, *Classification des systémes intégrables en dimension 2 et invariants des modèles de Fomenko.* C. R. Acad. Sci. Paris Sr. I Math. 318 (1994), no. 10, 949–952.

[24] J.-P. Dufour and P. Molino, *Compactification d'actions de $\mathbb{R}^n$ et variables action-angle avec singularités*, Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), Springer, New York, 1991, pp. 151–167.

[25] J.J. Duistermaat, *On global action-angle coordinates. Comm. Pure Appl. Math.* **33** (1980), no. 6, 687–706.

[26] L. H. Eliasson, *Normal forms for Hamiltonian systems with Poisson commuting integrals*, Ph.D. Thesis (1984).

[27] L.H. Eliasson, *Normal forms for Hamiltonian systems with Poisson commuting integrals—elliptic case*, Comment. Math. Helv. **65** (1990), no. 1, 4–35.

[28] Fomenko, *Topological classification of all integrable Hamiltonian systems of general types with two degrees of freedom*, MSRI Publ., Vol 22 (1991) 131-340.

[29] G. Geiges, *Contact Geometry*, in: Handbook of Differential Geometry vol. 2 (F.J.E. Dillen and L.C.A. Verstraelen, eds.), to appear.

[30] J.W. Gray, *Some global properties of contact structures*, Annals of Math. (2), **69**, 1959, 421–450.

[31] V. Guillemin and S. Sternberg, *Remarks on a paper of Hermann*, Trans. Amer. Math. Soc. **130** (1968) 110–116.

[32] V. Guillemin, D. Shaeffer, *On a certain class of Fuchsian partial differential equations.* Duke Math. J. **44** (1977), no. 1, 157–199.

[33] C. Laurent-Gengoux, E. Miranda and P. Vanhaecke , *Action-angle coordinates for integrable systems on Poisson manifolds,* Int. Math. Res. Not. IMRN 2011, no. 8, 1839–1869.

[34] V. Guillemin and S. Sternberg, *The Gel'fand-Cetlin system and quantization of the complex flag manifolds, J. Funct. Anal.*, 52(1):106–128, 1983.

[35] V. Guillemin, E. Miranda and A. R. Pires, *Codimension one symplectic foliations and regular Poisson manifolds,* Bull Braz Math Soc, New Series 42(4), 607-623, 2011.

[36] V. Guillemin, E. Miranda and A. R. Pires, *Symplectic and Poisson geometry on b-manifolds* , preprint 2011.

[37] M. Hamilton and E. Miranda, *Geometric quantization of integrable systems with hyperbolic singularities*, Annales de l'Institut Fourier, vol. 60 (1) p. 51-85, 2010.

[38] V. Ginzburg, V. Guillemin and Y. Karshon, Moment maps, cobordisms, and Hamiltonian group actions, AMS, 2004.

[39] H. Ito, *Convergence of Birkhoff normal forms for integrable systems*, Comment. Math. Helv. **64** (1989), no. 3, 412–461.

[40] H. Ito *Action-angle coordinates at singularities for analytic integrable systems.*, Math. Z. **206** (1991), no. 3, 363–407.

[41] E. Lerman, , *Contact toric manifolds*, J. Symplectic Geom. 1 (2003), no. 4, 785–828.
[42] E. Lerman and S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties.* Trans. Amer. Math. Soc. **349** (1997), no. 10, 4201–4230.
[43] B. Kostant, "On the Definition of Quantization," *Géométrie Symplectique et Physique Mathématique,* Coll. CNRS, No. 237, Paris, (1975), 187-210.
[44] El Kacimi-Alaoui, *Sur la cohomologie feuillete*, Compositio Math. 49 (1983), no. 2,195–215.
[45] B. Kruglikov and V. Matveev, *Vanishing of the entropy pseudonorm for certain integrable systems.* (Electron. Res. Announc. Amer. Math. Soc. 12 (2006), 19-28.
[46] C. Laurent-Gengoux and E. Miranda, *Splitting theorem and integrable systems in Poisson manifolds*, work in progress, 2011.
[47] P. Libermann, *Legendre foliations on contact manifolds.* Differential Geom. Appl. **1** (1991), no. 1, 57–76.
[48] P. Libermann, C.M Marle, *Symplectic geometry and analytical mechanics.* Mathematics and its Applications, **35**. D. Reidel Publishing Co., Dordrecht, 1987.
[49] J. Liouville, *Note sur l'intégration des quations differentielles de la dynamique, prsennte au bureau des longitudes le 29 juin 1853*, Journal de Mathmatiques pures et appliques **20** (1855), 137-138.
[50] R. Lutz, *Sur la gémétrie des structures de contact invariantes*, Ann.Inst. Fourier, **29**, 1 (1979), 283-306.
[51] J. Martinet, *Formes de contact sur les variétés de dimension 3*, Springer LNM, 209 (1971),142–163.
[52] V. Matveev, *Integrable Hamiltonian systems with two degrees of freedom. Topological structure of saturated neighborhoods of points of focus-focus and saddle-saddle types.* (Russian. Russian summary) Mat. Sb. 187 (1996), no. 4, 29–58; translation in Sb. Math. 187 (1996), no. 4, 495-524.
[53] H. Mineur, *Reduction des systèmes mecaniques n degr de liberte admettant n intégrales premieres uniformes en involution aux système variable separees.*, J. Math Pure Appl., IX Sr. **15** (1936), 385–389.
[54] H. Mineur, *Sur les systèmes mécaniques dans lesquels figurent des paramètres fonctions du temps. Étude des systèmes admettant n intégrales premieres uniformes en involution. Extension ces systèmes des conditions de quantification de Bohr-Sommerfeld.*, Journal de l'Ecole Polytechnique, Série III, 143me anne (1937), 173–191 and 237–270.
[55] E. Miranda, *Symplectic equivalence of non-degenerate integrable systems*, work in progress, 2011.
[56] E. Miranda, *From action-angle coordinates to geometric quantization: a round trip,* Oberwolfach report, Geometric quantization in the non-compact setting, 2011.
[57] E. Miranda, *On the symplectic classification of singular Lagrangian foliations.* Proceedings of the IX Fall Workshop on Geometry and Physics (Vilanova i la Geltr£, 2000), 239–244, Publ. R. Soc. Mat. Esp., **3**, R. Soc. Mat. Esp., Madrid, 2001.
[58] E. Miranda, *On symplectic linearization of singular Lagrangian foliations*, Ph. D. thesis, Universitat de Barcelona, 2003, ISBN: 9788469412374.
[59] E. Miranda and Nguyen Tien Zung, *Equivariant normal forms for nondegenerate singular orbits of integrable Hamiltonian systems*,Ann. Sci. Ecole Norm. Sup.,37 (2004), no. 6, 819–839 2004.
[60] E. Miranda, *Symplectic linearization of semisimple Lie algebra actions*, preprint 2008.
[61] E. Miranda and S. Vu Ngoc, *A singular Poincaré Lemma* , IMRN, n 1, 27-46, 2005.
[62] E. Miranda, *Some rigidity results for Symplectic and Poisson group actions*, XV International Workshop on Geometry and Physics, 177–183, Publ. R. Soc. Mat. Esp., 11, R. Soc. Mat. Esp., Madrid, 2007.
[63] E. Miranda and N. T. Zung, *A note on equivariant normal forms of Poisson structures*, Math. Research Notes, 2006, vol 13-6, 1001–1012.
[64] P. Monnier and N.T Zung, *Levi decomposition for Poisson structures*, Journal of Differential Geometry, **68**, 347-395, 2004.
[65] E. Miranda, P. Monnier and N.T.Zung, *Rigidity of Hamiltonian actions on Poisson manifolds*, Advances in Mathematics, in press, 2011, doi:10.1016/j.aim.2011.09.013
[66] A. Pelayo, S.V. Ngoc, *Constructing integrable systems of semitoric type.* Acta Math. 206 (2011), no. 1, 93-125.

[67] A. Pelayo, S.V. Ngoc, *Semitoric integrable systems on symplectic 4-manifolds.* Invent. Math. 177 (2009), no. 3, 57–97.
[68] A. Toulet, *Classification des systèmes integrables en dimension 2.* Thèse, Universit de Montpellier II, 1996.
100 (1978), no. 3, 591–614.
[69] J. Vey, *Sur le lemme de Morse.* Invent. Math. **40** (1977), no. 1, 1–9.
[70] S. Vu Ngoc, *On semi-global invariants for focus-focus singularities.* Topology **42** (2003), no. 2, 365–380.
[71] G. Paternain, *On the topology of manifolds with completely integrable geodesic flows. II* , Journ. Geom. Phys. 123 (1994), 289298.
[72] R. Palais, *Equivalence of nearby differentiable actions of a compact group.* Bull. Amer. Math. Soc. 67 1961 362–364.
[73] J. Śniatycki, *On Cohomology Groups Appearing in Geometric Quantization*, *Differential Geometric Methods in Mathematical Physics* (1975).
[74] J. Vey, *Sur certains systèmes dynamiques séparables*, Amer. J. Math.
[75] N.M.J. Woodhouse, *Geometric quantization,* second edition. Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1992.
[76] A. Weinstein, *Lectures on symplectic manifolds.* Regional Conference Series in Mathematics, No.**29**. American Mathematical Society, Providence, R.I., 1977.
[77] A. Weinstein, *The local structure of Poisson manifolds.*, J. Differential Geom. 18 (1983), no. 3, 523–557.
[78] J. Williamson, *On the algebraic problem concerning the normal forms of linear dynamical systems,* Amer. J. Math. **58:1** (1936), 141-163.
[79] Nguyen Tien Zung, *Symplectic topology of integrable Hamiltonian systems. I: Arnold-Liouville with singularities.*, Compos. Math. **101** (1996), no. 2, 179–215.
[80] Nguyen Tien Zung, *Another note on focus-focus singularities.* Lett. Math. Phys. **60** (2002), no. 1, 87–99.

Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Barcelona, Spain

E-mail address: eva.miranda@upc.edu