An Introduction to the Theory of Quasi-uniform Spaces

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7 May, 2005

1 Introduction

If the symmetry condition in the definition of a pseudometric is deleted, the notion of a quasi-pseudometric is obtained. Asymmetric distance functions already occur in the work of Hausdorff in the beginning of the twentieth century when in his book on set-theory [27] he discusses what is now called the Hausdorff metric of a metric space.

A family of pseudo-metrics on a set generates a uniformity. Similarly, a family of quasi-pseudometrics on a set generates a quasi-uniformity [60].

In 1937 Weil published his booklet on (entourage) uniformities, which is now usually considered as the beginning of the modern theory of uniformities. Three years later Tukey suggested an approach to uniformities via uniform coverings. The study of quasi-uniformities started in 1948 with Nachbin's investigations (recorded in [57]) on uniform preordered spaces, that is, those topological preordered spaces whose preorder is given by the intersection of the entourages of a (filter) quasi-uniformity $\mathcal{U}$ and whose topology is induced by the associated sup-uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$. He proved that the topological ordered spaces of this kind can be characterized by the property that they admit T_2-ordered compactifications.

The filter $\mathcal{U}^{-1}$ of inverse relations of a quasi-uniformity $\mathcal{U}$ is also a quasi-uniformity. Similarly, each quasi-pseudometric has an obvious conjugate by interchanging the order of points. Hence quasi-uniformities and quasi-metrics naturally yield bitopological spaces (in the sense of Kelly [35]), that is, sets endowed with two topologies.

The term quasi-proximity first appeared in the articles of Pervin and Steiner [59, 66]. The connection between totally bounded quasi-uniformities and quasi-proximities generalizes the well-known correspondence between totally bounded uniformities and proximities.

The work of Fox, Junnila and Kofner showed that in the class of T_1-spaces the concepts of a γ-space (= a topological space admitting a local quasi-uniformity with a countable base), a quasi-pseudometrizable space and a non-archimedeanly quasi-pseudometrizable space are all distinct [22, 36] and that the fine quasi-uniformity of metrizable and suborderable (= generalized ordered) spaces has a base consisting of transitive entourages [33, 38].

Császár [4] developed the theory of the bicompletion for quasi-uniform spaces that was later popularized by Fletcher, Lindgren and Salbany [17, 63]. It satisfactorily generalizes the theory of the completion from the metric and uniform setting to the asymmetric context. Since its underlying idea is that of a complete uniformity however, many further attempts have been made to find other theories of asymmetric completions (see for instance the work of Deák and Doitchinov [6, 11]).

Brümmer (see e.g. [3]) was first to consider explicitly the class of all functorial quasi-uniformities on topological spaces, although some basic work on canonical quasi-uniformities was done at about the same time by Fletcher and Lindgren.

In the last decade the interest in the study of quasi-uniform function spaces and hyperspaces increased considerably. Partially this fact is explained by their applications in theoretical computer science (see the work of Smyth and Sünderhauf [65, 67]).

In the spirit of Nachbin several authors tried to develop with the help of quasi-pseudometrics a common generalization of the two well-established theories of metric spaces and partially ordered sets that would unify common classical results like fixed point theorems and completions.

In connection with such investigations also the idea to replace the reals in the definition of a quasi-pseudometric by some more general structure became popular (e.g. in the studies on Kopperman's continuity spaces, e.g. [39]). In these theories a quasi-uniformity is understood as a kind of generalized quasi-pseudometric.

Quasi-uniform structures were also investigated in various kinds of topological algebraic structures (see for instance the recent work of Romaguera and his collaborators [55, 25]). In particular the study of paratopological groups with the help of quasi-uniformities is well known. Furthermore the study of asymmetric norms naturally leads to a theory of asymmetric functional analysis.

Various researchers attempted more or less successfully to extend important classical results about quasi-uniformities to fuzzy mathematics. Based on Lowen's theory of approach spaces [53], the concept of an approach quasi-uniformity was introduced and investigated. In recent years more and more results about quasi-uniformities were also generalized to a pointfree setting (for instance in the work of Picado and his colleagues [15]).

Over the years, numerous tools from various mathematical theories like lattice theory, category theory, nonstandard analysis and descriptive set-theory were employed with great success in the study of quasi-uniform spaces.

In this chapter we wish to introduce the reader to many of the aforementioned basic ideas. Obviously, for the more sophisticated topics and results he/she will often be referred to the literature because of lack of space. Besides two survey articles dealing mainly with historic aspects of our topic [1, 45] we would like to mention the two monographs [56, 19] and the survey articles [3, 5, 6, 37, 41, 42, 43, 46]. Of course, also well-known texts (that contain sections) on uniformities [31, 12, 28, 32] and the short articles on (quasi-)uniformities in the recent Encyclopedia of General Topology [26, 47] should be quite useful to the interested reader.

Throughout we shall make use of the entourage approach to the theory of (quasi-)uniformities. While the theory of covering uniformities [31] is often easier to use than the equivalent theory of entourage uniformities, a theory of quasi-uniformities based on (pair)covers is possible [24], but often rather cumbersome. Nevertheless it should be stressed that the latter approach has its merits in pointfree topology and bitopology.

2 Basic Concepts and Results

2.1 Quasi-uniform and Quasi-pseudometric Spaces

It is well known that for topological spaces there is no obvious way to compare the sizes of neighborhoods of distinct points. Hence important concepts from the theory of metric spaces like total boundedness and uniform continuity cannot be formulated for topological spaces.

Of course, one natural way to overcome this problem is to choose for a given topological space X some sufficiently large index set I and to consider for each point $x \in X$ an indexed neighborhood base $\{N_i(x) : i \in I\}$. However this is not yet what we want. For instance certainly there should still be some order $\leq$ on the index set I so that if $i_1, i_2 \in I$ with $i_1 \leq i_2$ then we have $N_{i_1}(x) \subseteq N_{i_2}(x)$ whenever $x \in X$.

In fact, it has turned out to be useful to put even more structure on the index set I in order to be able to formulate a halving condition for the chosen families of neighborhoods, which is somewhat reminiscent of the triangle inequality. We are then naturally led to the fairly simple, but nevertheless very useful concept of a quasi-uniform space.

Definition 2.1.1 *A* quasi-uniformity $\mathcal{U}$ *on a set* X *is a filter on* $X \times X$ *such that*
(1) Each member U *of* $\mathcal{U}$ *contains the diagonal* $\Delta = \{(x,x) : x \in X\}$ *of* X;
(2) For each $U \in \mathcal{U}$ *there is* $V \in \mathcal{U}$ *such that* $V^2 \subseteq U$.
(Here $V^2 = V \circ V = \{(x,z) \in X \times X :$ *There is* $y \in X$ *such that* $(x,y) \in V$ *and* $(y,z) \in V\}$. *Hence* $\circ$ *is the usual composition of binary relations.)*

The members $U \in \mathcal{U}$ are called the *entourages* of $\mathcal{U}$. The elements of X are called *points*. The tuple $(X, \mathcal{U})$ is called a *quasi-uniform space.*

Example 2.1.1 *Let* T *be a reflexive and transitive relation, that is, a* preorder, *on a set* X. *Then the filter* $X \times X$ *generated by the base* $\{T\}$ *is a quasi-uniformity on* X. *For any quasi-uniformity* $\mathcal{U}$ *on* X, $\bigcap \mathcal{U}$ *is a preorder on* X.

Definition 2.1.2 *Let* $[0, \infty)$ *be the set of nonnegative reals. A quasi-pseudometric* d *on a set* X *is a function* $d : X \times X \to [0, \infty)$ *such that* $d(x,x) = 0$ *whenever* $x \in X$, *and* $d(x,z) \leq d(x,y) + d(y,z)$ *whenever* $x, y, z \in X$.

The quasi-pseudometric $d^{-1} : X \times X \to [0, \infty)$ *defined by* $d^{-1}(x,y) = d(y,x)$ *whenever* $x, y \in X$ *is a called the* quasi-pseudometric conjugate *to* d.

A quasi-pseudometric d *on* X *is called* non-archimedean *if* d *satisfies a strong form of the triangle inequality, namely* $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ *whenever* $x, y, z \in X$.

A quasi-pseudometric is called a pseudo-metric *provided that* $d(x,y) = d(y,x)$ *whenever* $x, y \in X$.

Let d be a quasi-pseudometric on a set X. For each $\epsilon > 0$ set $U_\epsilon = \{(x,y) \in X \times X : d(x,y) < \epsilon\}$. The quasi-uniformity on $X \times X$ generated by the base $\{U_\epsilon : \epsilon > 0\}$ is called the *quasi-pseudometric quasi-uniformity* $\mathcal{U}_d$ induced by d on X.

Definition 2.1.3 *Each quasi-uniformity* $\mathcal{U}$ *on a set* X induces a topology $\tau(\mathcal{U})$ *as follows: For each* $x \in X$ *and* $U \in \mathcal{U}$ *set* $U(x) = \{y \in X : (x,y) \in U\}$. *A subset* G *belongs to* $\tau(\mathcal{U})$ *if and only if it satisfies the following condition: For each* $x \in G$ *there exists* $U \in \mathcal{U}$ *such that* $U(x) \subseteq G$.

It turns out that the neighborhood filter at $x \in X$ with respect to the topology $\tau(\mathcal{U})$ is given by $\mathcal{U}(x) = \{U(x) : U \in \mathcal{U}\}$.

Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $\tau(\mathcal{U})$ is a T_0-topology if and only if $\bigcap \mathcal{U}$ is a partial order. Furthermore $\tau(\mathcal{U})$ is a T_1-topology if and only if $\bigcap \mathcal{U}$ is equal to the diagonal of X.

Note also that given a quasi-pseudometric d on a set X, $\tau(\mathcal{U}_d)$ is the usual quasi-pseudometric topology $\tau(d)$ on X having all the open balls $B_d(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$ $(x \in X, \epsilon \in (0,\infty))$ as a base.

Observe that $\tau(d)$ is a T_0-topology if and only if for all $x, y \in X$, $d(x,y) = d(y,x) = 0$ implies that $x = y$. Furthermore $\tau(d)$ is a T_1-topology if and only if for all $x, y \in X$, $d(x,y) = 0$ implies that $x = y$.

Exercise 2.1.1 *Let d be a quasi-pseudometric on a set X. Show that the weights of the topologies $\tau(d)$ and $\tau(d^{-1})$ are equal.*

Definition 2.1.4 *A map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between two quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is called* uniformly continuous *provided that for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(f \times f)(U) \subseteq V$. Here $f \times f$ is the product map from $X \times X$ into $Y \times Y$ defined by $(f \times f)(x_1, x_2) = (f(x_1), f(x_2))$ $(x_1, x_2 \in X)$.*

Of course the identity map on any quasi-uniform space into itself is always uniformly continuous. Furthermore the composition of two uniformly continuous maps between quasi-uniform spaces is readily seen to be uniformly continuous.

Two quasi-uniform spaces are called *isomorphic* provided that there is a uniformly continuous bijection between them whose inverse map is also uniformly continuous.

It is obvious that each uniformly continuous map between quasi-uniform spaces is continuous with respect to the induced topologies.

Somewhat surprisingly, each topological space (X,τ) is *quasi-uniformizable*, that is, there is a quasi-uniformity $\mathcal{U}$ on X such that $\tau(\mathcal{U}) = \tau$. (If the latter condition is satisfied, we shall say that $\mathcal{U}$ is *compatible* with the topology τ or that (X,τ) *admits* $\mathcal{U}$.)

Indeed for each subset A of X set

$$S_A = [(X \setminus A) \times X] \cup [X \times A].$$

Note that S_A is a preorder. We conclude that the collection $\{S_G : G \in \tau\}$ yields a subbase of a quasi-uniformity on X.

This quasi-uniformity is called the *Pervin quasi-uniformity* $\mathcal{P}(\tau)$ of the topological space (X,τ). One checks that indeed it induces the topology τ on X.

Recall that an open collection $\mathcal{C}$ of a topological space X is called *interior-preserving* provided that $\bigcap \mathcal{E}$ is open whenever $\mathcal{E} \subseteq \mathcal{C}$. Observe that if T is a preorder on a topological space X such that $T(x)$ is open whenever $x \in X$, then $\{T(x) : x \in X\}$ is interior preserving.

We note that for the following construction some authors assume that $\mathcal{C}$ is also a cover of X. Obviously this is merely a matter of taste, since it can always be achieved by just adding the set X to $\mathcal{C}$.

(The Fletcher construction) Let $\mathcal{D}$ be a collection of interior-preserving open collections $\mathcal{C}$ of a given topological space X such that $\bigcup_{\mathcal{C} \in \mathcal{D}} \mathcal{C}$ is a subbase of X. For each $\mathcal{C} \in \mathcal{D}$ set $T_{\mathcal{C}} = \bigcap_{C \in \mathcal{C}} S_C$. Note that for each $x \in X$, $T_{\mathcal{C}}(x) = \bigcap\{C : x \in C \in \mathcal{C}\}$. Hence it is readily checked that $\{T_{\mathcal{C}} : \mathcal{C} \in \mathcal{D}\}$ yields a subbase for a compatible quasi-uniformity $\mathcal{U}_{\mathcal{D}}$ of X.

In fact if $\mathcal{D}$ consists of the collection of all finite open collections of X, then $\mathcal{U}_{\mathcal{D}}$ is the Pervin quasi-uniformity of X.

We recall that a family $\mathcal{L}$ of subsets of a topological space (X,τ) is called *well-monotone* provided that the partial order $\subseteq$ of set inclusion is a well-order on $\mathcal{L}$. The quasi-uniformity $\mathcal{U}_{\mathcal{D}}$ where $\mathcal{D}$ is the collection of all well-monotone open collections of X is called the *well-monotone quasi-uniformity* $\mathcal{M}(\tau)$ of X.

If $\mathcal{U}$ is a quasi-uniformity on a set X, then the filter $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ on $X \times X$ is also a quasi-uniformity on X. (Here and in the following, $R^{-1} = \{(x,y) \in X \times X : (y,x) \in R\}$ is the inverse of the binary relation R on X.)

The quasi-uniformity $\mathcal{U}^{-1}$ is called the *conjugate* of $\mathcal{U}$. A quasi-uniformity that is equal to its conjugate is called a *uniformity.*

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are two quasi-uniformities on a set X, then we say that $\mathcal{U}_1$ is *coarser* than $\mathcal{U}_2$ (or $\mathcal{U}_2$ is *finer* than $\mathcal{U}_1$) provided that $\mathcal{U}_1 \subseteq \mathcal{U}_2$.

The set of all quasi-uniformities on a set X ordered by set-theoretic inclusion turns out to be a complete lattice.

The smallest element of this lattice is the *indiscrete uniformity* $\mathcal{I} = \{X \times X\}$. The largest element is the *discrete uniformity* $\mathcal{D}$ which consists of all reflexive binary relations on X.

Exercise 2.1.2 *[7] Let A be a nonempty subset of X and let $\mathcal{U}$ be the quasi-uniformity generated by the base $\{S_A\}$ on $X \times X$. Show that there is no quasi-uniformity $\mathcal{V}$ on X such that $\mathcal{I} \subset \mathcal{V} \subset \mathcal{U}$, that is, $\mathcal{U}$ is an* atom *in the lattice of quasi-uniformities on X.*

Obviously the supremum of a family $(\mathcal{U}_i)_{i\in I}$ of quasi-uniformities on X is generated by the subbase $\bigcup_{i\in I}\mathcal{U}_i$ on $X \times X$.

It is easy to see that $\tau(\bigvee_{i\in I}\mathcal{U}_i) = \bigvee_{i\in I}\tau(\mathcal{U}_i)$. It follows that each topological space (X,τ) admits a finest compatible quasi-uniformity, which will be called the *fine quasi-uniformity* and is denoted by $\mathcal{FN}(\tau)$ in the following.

In particular the union of a quasi-uniformity $\mathcal{U}$ and its conjugate $\mathcal{U}^{-1}$ yields a subbase of the coarsest uniformity finer than $\mathcal{U}$. It will be denoted by $\mathcal{U}^s$ in the following.

We also note that if a function $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous, then $f : (X,\mathcal{U}^{-1}) \to (Y,\mathcal{V}^{-1})$ and $f : (X,\mathcal{U}^s) \to (Y,\mathcal{V}^s)$ are also uniformly continuous.

Definition 2.1.5 *A quasi-uniform space $(X,\mathcal{U})$ is said to have the* Lebesgue property *provided that for each open cover $\mathcal{C}$ of $(X,\tau(\mathcal{U}))$ there is $U \in \mathcal{U}$ such that $\{U(x) : x \in X\}$ is a refinement of $\mathcal{C}$.*

Exercise 2.1.3 *[19] Let $(X,\mathcal{U})$ be a quasi-uniform space such that $\tau(\mathcal{U})$ is compact. Prove that $(X,\mathcal{U})$ has the Lebesgue property.*

Definition 2.1.6 *A quasi-uniform space $(X,\mathcal{U})$ is called* small-set symmetric *provided that $\tau(\mathcal{U}^{-1}) \subseteq \tau(\mathcal{U})$. (The conjugate quasi-uniformity $\mathcal{U}^{-1}$ is then called* point-symmetric*.)*

Exercise 2.1.4 *[19] Let $(X,\mathcal{U})$ be a quasi-uniform space such that $\tau(\mathcal{U})$ is a compact T_1-topology. Show that $\mathcal{U}$ is point-symmetric.*

Exercise 2.1.5 *Show that a countably compact quasi-pseudometric T_1-space is compact.*

Exercise 2.1.6 *Equip $X = \omega_1$ with the quasi-pseudometric $d(x,y) = 0$ if $x \geq y$ and $d(x,y) = 1$ otherwise. Note that $\tau(d)$ is countably compact, but not compact.*

Exercise 2.1.7 *A quasi-uniform space $(X,\mathcal{U})$ is called* locally symmetric *provided that for all $U \in \mathcal{U}$ and $x \in X$, there is $V \in \mathcal{U}$ such that $V^{-1}(V(x)) \subseteq U(x)$. Observe that each point-symmetric quasi-uniform space is locally symmetric. Show that a topological space X admits a locally symmetric quasi-uniformity if and only if X is regular.*

Proposition 2.1.1 *[54] Let $(X,\mathcal{U})$ be a quasi-uniform space satisfying the Lebesgue property, let $(Y,\mathcal{V})$ be a small-set symmetric quasi-uniform space and let $f : (X,\tau(\mathcal{U})) \to (Y,\tau(\mathcal{V}))$ be a continuous map. Then $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous. (Indeed, $f : (X,\mathcal{U}) \to (Y,\mathcal{V}^s)$ is uniformly continuous.)*

Proof. Let $V \in \mathcal{V}$. Choose $W \in \mathcal{V}$ such that $W^2 \subseteq V$. Then for each $x \in X$ there is $W_x \in \mathcal{V}$ such that $W_x \subseteq W$ and $W_x(f(x)) \subseteq W^{-1}(f(x))$ by small-set symmetry of $(Y,\mathcal{V})$. By continuity of f, $\mathcal{D} = \{\text{int} f^{-1}[W_x(f(x))] : x \in X\}$ is an open cover of X. There is $U \in \mathcal{U}$ such that $\{U(x) : x \in X\}$ refines $\mathcal{D}$ by the Lebesgue property of $(X,\mathcal{U})$. Fix $x \in X$. Then there exists $y \in X$ such that $U(x) \subseteq f^{-1}W_y(f(y))$. Thus $f(U(x)) \subseteq W_y(f(y)) \subseteq W^{-1}(f(y))$ and hence $f(y) \in W(f(x))$. Therefore $f(U(x)) \subseteq W_y(f(y)) \subseteq W^2(f(x)) \subseteq V(f(x))$. Hence $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous. The remark in parenthesis immediately follows from this result, since by small-set symmetry $f : (X,\tau(\mathcal{U})) \to (Y,\tau(\mathcal{V}^s))$ is also continuous and $\mathcal{V}^s$, as any uniformity, is small-set symmetric.

Corollary 2.1.1 *A small-set symmetric quasi-uniformity $\mathcal{U}$ on a set X that induces a compact topology $\tau(\mathcal{U})$ is a uniformity.*

Proof. Consider the map $\text{id}_X : (X,\mathcal{U}) \to (X,\mathcal{U}^s)$. By the preceding result it is uniformly continuous. Therefore $\mathcal{U}^s \subseteq \mathcal{U}$ and thus $\mathcal{U}$ is a uniformity.

There seems to be no way to explicitly describe a subbase of the infimum of a family $(\mathcal{U}_i)_{i\in I}$ of quasi-uniformities in terms of the entourages of the quasi-uniformities $\mathcal{U}_i$. Also in general topological spaces do not admit a coarsest compatible quasi-uniformity.

Nevertheless it is easy to see that conjugation commutes with arbitrary infima and suprema in the lattice of quasi-uniformities. In particular for any quasi-uniformity $\mathcal{U}$, $\mathcal{U} \wedge \mathcal{U}^{-1}$ is a uniformity.

Exercise 2.1.8 *Let R and S be preorders on X. Consider the quasi-uniformities $\mathcal{U}$ resp. $\mathcal{V}$ on X generated by the bases $\{R\}$ resp. $\{S\}$ on $X \times X$. Show that $\mathcal{U} \wedge \mathcal{V}$ is generated by the base $\{C\}$ where C is the transitive closure of $R \cup S$. Observe that $\{R \cap S\}$ is a base of the quasi-uniformity $\mathcal{U} \vee \mathcal{V}$.*

A map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is called *uniformly open* if for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $V(f(x)) \subseteq f(U(x))$ whenever $x \in X$.

Proposition 2.1.2 *Let $(X,\mathcal{U})$ be a compact* uniform *space and $(Y,\mathcal{V})$ a quasi-uniform space. Then any map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ that is open and continuous is uniformly open.*

Proof. Let $U \in \mathcal{U}$. There is $P \in \mathcal{U}$ such that $P^2 \subseteq U$. Since f is open, for each $a \in X$ we find $W_a \in \mathcal{V}$ such that $W_a^2(f(a)) \subseteq f(P(a))$. By continuity of f and since $\mathcal{U}$ is a uniformity, we can consider the open cover $\{\mathrm{int}(P^{-1}(a) \cap f^{-1}W_a(f(a))) : a \in X\}$ of X. Since X is compact, there is a finite subset F of X such that $\bigcup_{a\in F} \mathrm{int}(P^{-1}(a) \cap f^{-1}W_a(f(a))) = X$. Set $W = \bigcap_{a\in F} W_a$ and note that $W \in \mathcal{V}$. Consider $x \in X$. There is $b \in F$ such that $x \in P^{-1}(b) \cap f^{-1}W_b(f(b))$. Therefore $f(x) \in W_b(f(b))$ and $W(f(x)) \subseteq W_b^2(f(b)) \subseteq f(P(b)) \subseteq f(P^2(x)) \subseteq f(U(x))$. We have shown that f is uniformly open.

Definition 2.1.7 *Let X be a set and suppose that for each $i \in I$ there is given a map $f_i : X \to Y_i$ into a quasi-uniform space $(Y_i,\mathcal{V}_i)$. Then there is a coarsest quasi-uniformity $\mathcal{V}$ on X such that all maps f_i are uniformly continuous, the so-called* initial quasi-uniformity. *It is generated by the subbase $\{(f_i \times f_i)^{-1}(V) : i \in I, V \in \mathcal{V}_i\}$ on $X \times X$.*

One checks that the initial quasi-uniformity $\mathcal{V}$ induces the coarsest topology on X such that all maps $f_i : X \to (Y_i, \tau(\mathcal{V}_i))$ $(i \in I)$ are continuous, that is, the corresponding initial topology. Note also that the initial quasi-uniformity $\mathcal{V}$ is a uniformity if all quasi-uniformities $\mathcal{V}_i$ are indeed uniformities. We give two applications for the concept of the initial quasi-uniformity.

Let $((X_i,\mathcal{U}_i))_{i\in I}$ be a family of quasi-uniform spaces. Let $\Pi_{i\in I}X_i$ be the set-theoretic product of the family $(X_i)_{i\in I}$ and let $\pi_j : \Pi_{i\in I}X_i \to X_j$ $(j \in I)$ be the projection onto $(X_j,\mathcal{U}_j)$. Then the coarsest quasi-uniformity on $\Pi_{i\in I}X_i$ that makes all projections uniformly continuous is called the *product quasi-uniformity.* It induces the product topology of the topological spaces $((X_i, \tau(\mathcal{U}_i))_{i\in I}$.

Let $(X,\mathcal{U})$ be a quasi-uniform space and let A be a subset of X. Then the coarsest quasi-uniformity on X such that the set-theoretic embedding $e : A \to X$ is uniformly continuous is called the *subspace quasi-uniformity* $\mathcal{U}|A$: It consists of the entourages $U|A := U \cap (A \times A)$ with $U \in \mathcal{U}$ and induces the subspace topology $(\tau(\mathcal{U}))|A$ of $(X, \tau(\mathcal{U}))$ on A.

Exercise 2.1.9 *Let (X,τ) be a topological space and let A be a subset of X. Show that $(\mathcal{P}(\tau))|A = \mathcal{P}(\tau|A)$ and $(\mathcal{M}(\tau))|A = \mathcal{M}(\tau|A)$.*

Let Y be a set and suppose that for each $i \in I$ there is given a map $f_i : X_i \to Y$ from a quasi-uniform space $(X_i,\mathcal{U}_i)$ to Y. Then there is a finest quasi-uniformity $\mathcal{V}$ on Y such that all maps f_i are uniformly continuous, the so-called *final quasi-uniformity.* Observe that $\mathcal{V}$ is a uniformity if all quasi-uniformities $\mathcal{U}_i$ are uniformities.
The final quasi-uniformity $\mathcal{V}$ need not be compatible with the final topology on Y determined by the family of maps $f_i : (X_i, \tau(\mathcal{U}_i)) \to Y$ $(i \in I)$.

Exercise 2.1.10 *Let $\mathbb{R}$ be the set of the reals equipped with its usual metric uniformity. Consider the partition $\mathbb{R}/\sim = \{(-\infty, 0], (0, \infty)\}$ of $\mathbb{R}$ and the corresponding quotient map $q : \mathbb{R} \to \mathbb{R}/\sim$. Determine the quotient (that is, final) quasi-uniformity on $\mathbb{R}/\sim$ and compare its induced topology with the quotient topology.*

2.2 Transitive Quasi-uniform and Topological Spaces

A quasi-uniform space $(X,\mathcal{U})$ is called *transitive* provided that it has a base consisting of transitive entourages (that is, preorders). Such a base is called a *transitive* base. Note that all quasi-uniformities obtained by the Fletcher construction are transitive. Furthermore observe that if d is a non-archimedean quasi-pseudometric on a set X, then the quasi-pseudometric quasi-uniformity $\mathcal{U}_d$ has a countable transitive base.

Exercise 2.2.1 *Let $\mathcal{U}$ be a quasi-uniformity having a countable base of transitive entourages. Then $\mathcal{U}$ is* non-archimedeanly quasi-pseudometrizable, *that is, there is a non-archimedean quasi-pseudometric d on X such that $\mathcal{U} = \mathcal{U}_d$.*

Corollary 2.2.1 *The topology of a topological space can be induced by a non-archimedean quasi-pseudometric if and only if it has a σ-interior preserving base (that is, a base of open sets that can be written as the union of countably many interior-preserving collections).*

Proof. The statement is a consequence of the preceding exercise and a straightforward application of the Fletcher construction.

It is easy to see that the supremum of any family of transitive quasi-uniformities is transitive. On the other hand the infimum of a transitive quasi-uniformity with its conjugate need not be transitive, as we shall see later (see **??**).

The proof of the following result is crucial, but well known and will not be included here.

Lemma 2.2.1 *Let $(U_n)_{n\in\omega}$ be a sequence of binary reflexive relations on a set X such that for each $n \in \omega$, $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$. Then there is a quasi-pseudometric d on X such that $U_{n+1} \subseteq \{(x, y) : d(x, y) < 2^{-n}\} \subseteq U_n$ whenever $n \in \omega$. If each U_n is a symmetric relation, then d can be constructed to be a pseudo-metric.*

Corollary 2.2.2 *Each (quasi-)uniformity $\mathcal{U}$ on a set X with a countable base is (quasi-)pseudometrizable (that is, there is a (quasi-)pseudometric d on X such that $\mathcal{U}_d = \mathcal{U}$).*

Corollary 2.2.3 *Each (quasi-)uniformity $\mathcal{U}$ can be written as the supremum of a family of (quasi-)pseudometric quasi-uniformities.*

Corollary 2.2.4 *A topological space is completely regular if and only if it is* uniformizable*, that is, can be induced by a uniformity.*

Proof. Let (X, τ) be a topological space that is uniformizable by some uniformity $\mathcal{U}$. Then by the preceding corollary $\mathcal{U}$ can be written as the supremum of pseudometric uniformities. Hence τ is the supremum of pseudometric topologies and thus is completely regular.

In order to prove the converse, assume that X is completely regular. Then X carries the initial topology with respect to the family of all continuous maps from X into the real unit interval $([0, 1], \tau(m))$ where m denotes the usual metric on $[0, 1]$. Consider the initial (quasi-)uniformity $\mathcal{C}^*(\tau)$ on X determined by the family of these maps into the uniform space $([0, 1], \mathcal{U}_m)$. Since $\mathcal{C}^*(\tau)$ induces τ, we have shown that (X, τ) is uniformizable.

Corollary 2.2.5 *Let $\mathcal{V}$ be a compatible quasi-uniformity on a completely regular space X. If $\mathcal{V}$ has the Lebesgue property, then it is finer than the finest compatible uniformity $\mathcal{U}$ on X.*

Proof. The continuous map $\mathrm{id}_X : (X, \mathcal{V}) \to (X, \mathcal{U})$ is uniformly continuous by Proposition 2.1.1.

Corollary 2.2.6 *Let (X, τ) be a compact regular space and let $\mathcal{U}$ be a compatible uniformity and $\mathcal{V}$ a compatible quasi-uniformity on X. Then $\mathcal{U} \subseteq \mathcal{V}$.*

In particular $\mathcal{C}^*(\tau)$ is the unique compatible uniformity on X and the coarsest compatible quasi-uniformity on X.

Proof. Consider the identity map $\mathrm{id}_X : (X, \tau(\mathcal{U})) \to (X, \tau(\mathcal{V}))$. It is continuous and open, so the map $\mathrm{id}_X : (X, \mathcal{U}) \to (X, \mathcal{V})$ is uniformly open. Hence $\mathcal{U} \subseteq \mathcal{V}$.

Definition 2.2.1 *Let X be a set. A filter $\mathcal{U}$ on $X \times X$ consisting of reflexive relations is called a* local quasi-uniformity *provided that for each $U \in \mathcal{U}$ and each $x \in X$ there is $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$. A local quasi-uniformity $\mathcal{U}$ on a set X such that $\mathcal{U} = \mathcal{U}^{-1}$ is called a* local uniformity*. A local quasi-uniformity such that $\mathcal{U}^{-1}$ is a local quasi-uniformity will be called a* pairwise-local quasi-uniformity*.*

In the same way as for a quasi-uniformity, a local quasi-uniformity $\mathcal{U}$ induces a topology $\tau(\mathcal{U})$. A topological space is called a γ-space *provided that its topology can be induced by a local quasi-uniformity having a countable base.*

Exercise 2.2.2 *Let $(X, \mathcal{U})$ be a local quasi-uniform space and $x \in X$. Show that $\mathcal{U}(x) = \{U(x) : U \in \mathcal{U}\}$ is the neighborhood filter at x with respect to the topology $\tau(\mathcal{U})$.*

Exercise 2.2.3 *[70] Prove that a topological space (X, τ) admits a local uniformity if and only if its topology τ is regular.*

Example 2.2.1 *A method to build γ-spaces that are not quasi-pseudometrizable was developed by Fox [22], answering a question that originated in work of Ribeiro [62]. Fox and Kofner [23] constructed a γ-space with a Tychonoff topology that is not quasi-pseudometrizable. The problem whether a γ-space satisfying some additional property is quasi-pseudometrizable often turns out to be quite difficult.*

The following positive result is not deep, but the method is often useful.

Proposition 2.2.1 *A topological space X is quasi-pseudometrizable if and only if its topology can be induced by a pairwise-local quasi-uniformity $\mathcal{U}$ with a countable base.*

Proof. Necessity is obvious. In order to prove sufficiency let $\mathcal{U}$ be a local quasi-uniformity with a countable base $\{U_n : n \in \omega\}$ for X such that $\mathcal{U}^{-1}$ is a local quasi-uniformity. Without loss of generality we may assume that $U_{n+1} \subseteq U_n$ whenever $n \in \omega$. Inductively we define a sequence $(V_n)_{n\in\omega}$ of $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$-neighborhoods of Δ as follows. If $n = 0$, choose for each $x \in X$ an $n(x) \in \omega$ such that $U^3_{n(x)}(x) \subseteq U_0(x)$ and $(U^{-1}_{n(x)})^3(x) \subseteq U_0^{-1}(x)$ and set

$$V_0 = \bigcup\{U^{-1}_{n(x)}(x) \times U_{n(x)}(x) : x \in X\}.$$

If $n > 0$ and V_{n-1} is already defined, choose for each $x \in X$ an $n(x) \in \omega$ such that $U^7_{n(x)}(x) \subseteq (V_{n-1} \cap U_n)(x)$ and $(U^{-1}_{n(x)})^7(x) \subseteq (V_{n-1} \cap U_n)^{-1}(x)$ and set

$$V_n = \bigcup\{U^{-1}_{n(x)}(x) \times U_{n(x)}(x) : x \in X\}.$$

An easy computation shows that $V_0^2 \subseteq U_0^2$ and $V_n^4 \subseteq V^2_{n-1} \cap U_n^2$ for each $n > 0$. Hence the quasi-uniformity $\mathcal{V}$ on X generated by $\{V_n^2 : n \in \omega\}$ satisfies $\tau(\mathcal{V}) = \tau(\mathcal{U})$ and $\tau(\mathcal{V}^{-1}) = \tau(\mathcal{U}^{-1})$.

Corollary 2.2.7 *[70] A topological space is pseudometrizable if and only its topology can be induced by a local uniformity possessing a countable base.*

Problem 2.2.1 *(for research) A lot of work has gone into the problem of characterizing topologically those topological spaces that admit a compatible quasi-pseudometric. Numerous solutions to this problem have been suggested in the literature (see e.g. [40, 29]). While in special cases the obtained criteria turn out to be quite useful, it seems fair to say that no known general characterization of quasi-pseudometrizability is as useful as for instance the Nagata-Smirnov Theorem in the theory of metrizable topological spaces.*

Each quasi-uniformity contains a finest transitive quasi-uniformity coarser than $\mathcal{U}$, often denoted by $\mathcal{U}_t$. Of course, $(\mathcal{U}_t)^{-1} = (\mathcal{U}^{-1})_t$. Note that $\mathcal{U}_t$ and $\mathcal{U}$ may induce distinct topologies, for instance on any uniform space $(X, \mathcal{U})$ with a connected topology $\tau(\mathcal{U})$, the indiscrete uniformity on X is the only transitive quasi-uniformity coarser than $\mathcal{U}$.

For any topological space (X, τ) we shall call $(\mathcal{FN}(\tau))_t$ the *fine transitive quasi-uniformity* and denote it by $\mathcal{FT}(\tau)$. Since each topological space admits a transitive quasi-uniformity, namely its Pervin quasi-uniformity, $\mathcal{FT}(\tau)$ is compatible with τ.

Exercise 2.2.4 *Observe that a topological space is zero-dimensional if and only if it admits a transitive uniformity.*

A topological space is called *transitive* if its fine quasi-uniformity is transitive.

In the following example each $n \in \omega$ is identified with its set of predecessors, that is $n = \{0, \ldots, n-1\}$.

Example 2.2.2 *[51] Let X be the set $\mathbf{Q}^\omega$ (of sequences from ω to the rationals $\mathbf{Q}$) equipped with the topology induced by the quasi-uniformity $\mathcal{U}$ having the following countable base $\{U_n : n \in \omega\}$.*

For each $n \in \omega$ and $x \in X$ the set $U_n(x)$ is the set of all $x' \in X$ which (1) agree with x on the initial segment n, and (2) for which the first coordinate $\Delta := \Delta(x, x')$ in which x and x' differ satisfies $x(\Delta) \leq x'(\Delta) \leq x(\Delta) + 2^{-n}$.

In order to see that $\mathcal{U}$ is a quasi-uniformity, it suffices to show that for any $n \in \omega$ and any $x, x', x'' \in X$, we have that $x' \in U_{n+1}(x)$ and $x'' \in U_{n+1}(x')$ imply $x'' \in U_n(x)$.

Indeed, note first that under our hypothesis x, x', x'' agree on $n+1$. Set $\Delta_0 := \Delta(x', x)$ and $\Delta_1 := \Delta(x'', x')$. We now distinguish three cases.

Case 1: If $\Delta_1 < \Delta_0$, then $\Delta_1 = \Delta(x'', x)$, $x'(\Delta_1) = x(\Delta_1)$, $x'(\Delta_1) \leq x''(\Delta_1) \leq x'(\Delta_1) + 2^{-(n+1)}$, and thus $x'' \in U_{n+1}(x)$.

Case 2: If $\Delta_1 > \Delta_0$, then $\Delta_0 = \Delta(x'', x)$, $x''(\Delta_0) = x'(\Delta_0)$, $x(\Delta_0) \leq x'(\Delta_0) \leq x(\Delta_0) + 2^{-(n+1)}$, and thus $x'' \in U_{n+1}(x)$.

Case 3: If $\Delta_1 = \Delta_0$, then $\Delta_0 = \Delta(x'', x)$, $x(\Delta_0) \leq x'(\Delta_0) \leq x(\Delta_0) + 2^{-(n+1)}$ and $x'(\Delta_0) \leq x''(\Delta_0) \leq x'(\Delta_0) + 2^{-(n+1)}$, and thus $x'' \in U_n(x)$.

Let us note that $\tau(\mathcal{U})$ is a T_1-topology, since $\cap_{n\in\omega} U_n(x) = \{x\}$ whenever $x \in X$.

We want to show that $(X, \tau(\mathcal{U}))$ is not transitive. Indeed we shall show that $\tau(\mathcal{U})$ is not non-archimedeanly quasi-pseudometrizable. In order to reach a contradiction, we assume that it admits a quasi-uniformity with a countable (decreasing) base $\{T_n : n \in \omega\}$ consisting of transitive entourages.

Then for each $x \in X$, choose $k(x) \in \omega$ such that $T_{k(x)}(x) \subseteq U_1(x)$; furthermore fix $n_x \in \omega$ such that $U_{n_x}(x) \subseteq T_{k(x)}(x)$.

Viewing the rationals as having the discrete topology and working in the resulting space ω^ω, for each $m, k \in \omega$ we set $A_{m,k} = \{x \in \omega^\omega : n_x = m \text{ and } k(x) = k\}$. With the help of the Baire Category Theorem we find $m, k \in \omega$ such that $\text{int}\overline{A_{m,k}} \neq \emptyset$ in the space ω^ω. Hence there are $n \in \omega$, $\sigma \in \mathbf{Q}^n$ and a dense $D \subseteq [\sigma]$ such that, for all $x \in D$, $k(x) = k$ and $n_x \leq n$. (Here, $[\sigma] = \{x' \in \mathbf{Q}^\omega : x'|n = \sigma\}$.)

Construct $x_1, \ldots, x_{2^n+1} \in D$ such that for each $i = 1, \ldots, 2^{n+1}$, x_i restricted to n equals σ and such that for each $i = 1, \ldots, 2^n$, $x_i(n) + 2^{-n} = x_{i+1}(n)$. We see that $x_{i+1} \in T_k(x_i)$, because $x_{i+1} \in U_n(x_i) \subseteq U_{n_{x_i}}(x_i)$. By transitivity we conclude that $x_{2^n+1} \in T_k(x_1) \subseteq U_1(x_1)$. But now $x_{2^n+1} \in U_1(x_1)$ requires $x_{2^n+1}(\Delta) \leq x_1(\Delta) + 2^{-1}$ where $\Delta := \Delta(x_1, x_{2^n+1}) = n$. However $x_1(\Delta) + 1 = x_1(\Delta) + 2^{-n}(2^n) = x_{2^n+1}(\Delta)$. This contradiction completes the proof.

Remark 2.2.1 *The first example of a topological space that is quasi-pseudometrizable, but not non-archimedeanly quasi-pseudometrizable was constructed by to Kofner and is now called the* Kofner plane *[36]:*

(The Kofner plane) Let $X = \mathbb{R}^2$. For each $x \in X$ and $\epsilon > 0$ let $C(x, \epsilon)$ be the closed disk of radius ϵ lying above the horizontal line through x and tangent to this line at x. For $x, y \in X$ define $d(x, y)$ as follows: Set $d(x, y) = 1$ if $y \notin C(x, 1)$, set $d(x, y) = r$ if $r \leq 1$, $y \in C(x, r)$ and $y \notin C(x, s)$ for all $s < r$, and set $d(x, y) = 0$ if $x = y$. An application of the Baire Category Theorem similarly to the one given above shows that the quasi-pseudometric space (X, d) is not non-archimedeanly quasi-pseudometrizable. Hence it cannot be transitive, since this would imply that it admits a quasi-uniformity with a countable base of transitive relations. We omit the proof that can be found in [19].

Problem 2.2.2 *(for research) It is not known whether each (regular) quasi-pseudometric space (X, d) such that $\tau(d) \subseteq \tau(d^{-1})$ is non-archimedeanly quasi-pseudometrizable.*

Problem 2.2.3 *(for research) No non-archimedeanly quasi-pseudometrizable (Tychonoff) space seems to be known that is not transitive.*

2.3 Bitopological Spaces

Definition 2.3.1 *A bitopological space (X, τ_1, τ_2) consists of a set X equipped with two topologies τ_1 and τ_2. A map $f : X \to Y$ between two bitopological spaces $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called* bicontinuous *provided that $f : (X, \tau_1) \to (Y, \sigma_1)$ and $f : (X, \tau_2) \to (Y, \sigma_2)$ are both continuous.*

Of course, it is natural to consider a topological space (X, τ) as a special case of a bitopological space, namely the bitopological space (X, τ, τ). A lot of work has been done in extending concepts and results about topological spaces to bitopological spaces. As expected, while in some cases such generalizations are more or less straightforward, in other cases they are quite demanding and often highly controversial. Sometimes they also lead to completely unsatisfactory results, since concepts that in the case of topological spaces are equivalent may extend naturally to concepts that are no longer equivalent in the bitopological setting.

Given a quasi-pseudometric d on a set X, it is natural to study the bitopological space $(X, \tau(d), \tau(d^{-1}))$.

Let X be a set and let A, B subsets of X. We let $T_{A,B}$ denote the binary relation

$$(X \times X) \setminus (A \times B) = [(X \setminus A) \times X] \cup [X \times (X \setminus B)].$$

In particular note that $S_A = T_{A, X \setminus A}$.

In the following suppose that ξ is a collection of pairs $< A, B >$ of subsets of X such that $A \subseteq B$.

If for a topology τ on X, for each $\zeta \subseteq \xi$, we have $\text{cl}_\tau \bigcup\{A :< A, B > \in \zeta\} \subseteq \bigcup\{B :< A, B > \in \zeta\}$, then ξ is called (τ-)*cushioned.*

If for a topology τ on X, for each $\zeta \subseteq \xi$, we have $\bigcap\{A :< A, B >\in \zeta\} \subseteq \text{int}_\tau \bigcap\{B :< A, B >\in \zeta\}$ then ξ is called *(τ-)cocushioned.*

If $\xi = \bigcup_{n\in\omega} \xi_n$ where each ξ_n is τ-cushioned (resp. τ-cocushioned), then ξ is called σ-τ-*cushioned* (resp. σ-τ-*cocushioned*).

Exercise 2.3.1 *Let X be a topological space. Prove that a collection of pairs $< A, B >$ of subsets of X is cushioned if and only if the collection of the corresponding pairs $< X \setminus B, X \setminus A >$ is cocushioned.*

As above, let ξ be a collection of pairs $< A, B >$ of subsets of X such that if $< A, B >\in \xi$ then $A \subseteq B$. A *pairbase* for a topological space X is such a collection ξ satisfying

(i) if $< A, B >\in \xi$, then A is open, (ii) if C is a neighborhood of $x \in X$, then there exists $< A, B > \in \xi$ with $x \in A \subseteq B \subseteq C$.

Proposition 2.3.1 *(Fox [21]) A bitopological space (X, τ_1, τ_2) possesses a quasi-pseudometric inducing its bitopology (that is, (X, τ_1, τ_2) is* quasi-pseudometrizable*) if and only if τ_1 has both a σ-τ_2-cushioned pairbase and a σ-τ_1-cocushioned pairbase, and τ_2 has both a σ-τ_1-cushioned pairbase and a σ-τ_2-cocushioned pairbase.*

Proof. Let (X, d) be a quasi-pseudometric space. Set $B_d(x, 2^{-n}) = \{y \in X : d(x, y) < 2^{-n}\}$ whenever $x \in X$ and $n \in \omega$. Let $\beta_n = \{< B_d(x, 2^{-(n+1)}), B_d(x, 2^{-n}) >: x \in X\}$ and $\delta_n = \{< B_{d^{-1}}(x, 2^{-(n+1)}, B_{d^{-1}}(x, 2^{-n}) >: x \in X\}$. Then β_n is both τ_2-cushioned and τ_1-cocushioned, and $\beta = \bigcup_{n\in\omega} \beta_n$ is a pairbase for τ_1. Also δ_n is both τ_1-cushioned and τ_2-cocushioned, and $\delta = \bigcup_{n\in\omega} \delta_n$ is a pairbase for τ_2.

In order to prove sufficiency we shall make use of the following lemmas due to Fox:

Lemma 2.3.1 *If (X, τ_1, τ_2) is a bitopological space where τ_1 has a σ-τ_2-cushioned pairbase μ and a σ-τ_1-cocushioned pairbase ξ, and where τ_2 has a σ-τ_1-cushioned pairbase λ and a σ-τ_2-cocushioned pairbase ζ, then the hypothesis of the next lemma is satisfied.*

Proof. If $\beta = \{< A \cap C, B \cup D >:< A, B >\in \xi$ and $< C, D >\in \mu\}$ and $\delta = \{< A \cap C, B \cup D >:< A, B >\in \zeta$ and $< C, D >\in \lambda\}$, then β is a σ-τ_2-cushioned, σ-τ_1-cocushioned pairbase for τ_1, and δ is a σ-τ_1-cushioned, σ-τ_2-cocushioned pairbase for τ_2.

Lemma 2.3.2 *Let (X, τ_1, τ_2) be a bitopological space, where τ_1 has a σ-τ_2-cushioned, σ-τ_1-cocushioned pairbase β and where τ_2 has a σ-τ_1-cushioned and σ-τ_2-cocushioned pairbase δ. Then (X, τ_1, τ_2) has a compatible pairwise-local quasi-uniformity with a countable base.*

Proof. Without loss of generality we may assume $\beta = \bigcup_{n\in\omega} \beta_n$ and $\delta = \bigcup_{n\in\omega} \delta_n$, where $\beta_n \subseteq \beta_{n+1}$ and $\delta_n \subseteq \delta_{n+1}$, and where each β_n is both τ_2-cushioned and τ_1-cocushioned and each δ_n is both τ_1-cushioned and τ_2-cocushioned. We define for any $n \in \omega$,

$$V_n = \bigcap\{T_{A,X\setminus B} :< A, B >\in \beta_n\} \cap \bigcap\{(T_{A,X\setminus B})^{-1} :< A, B >\in \delta_n\}.$$

Then $V_n(x)$ is a τ_1-neighborhood of x as β_n is τ_1-cocushioned and δ_n is τ_1-cushioned. Also $V^{-1}(x)$ is a σ_2-neighborhood of x as β_n is τ_2-cushioned and δ_n is τ_2-cocushioned. Given any τ_1-neighborhood S of a point $x \in X$, as β is a pairbase for τ_1 we may find $n \in \omega$ and $< A, B >, < C, D >\in \beta_n$ such that $x \in A \subseteq B \subseteq C \subseteq D \subseteq S$. Then $x \in A$ implies $V_n(x) \subseteq B \subseteq C$, which implies $V_n^2(x) \subseteq D \subseteq S$. Similarly, given any τ_2-neighborhood R of x, we may find $n \in \omega$ such that $V_n^{-2}(x) \subseteq R$. It follows that $\{V_n : n \in \omega\}$ is a countable base for a compatible pairwise-local quasi-uniformity on (X, τ_1, τ_2).

The assertion now follows by the proof of Proposition 2.2.1.

Corollary 2.3.1 *[35] A bitopological space (X, τ_1, τ_2) is quasi-pseudometrizable if X is pairwise regular (that is, each point has a τ_i neighborhood base consisting of τ_j-closed sets $(i, j \in \{1, 2\}, i \neq j)$) and both topologies have a countable base.*

Exercise 2.3.2 *Show that a topological space (X, τ) admits a local quasi-uniformity with a countable base if and only if it possesses a σ-τ-cocushioned pairbase.*

Corollary 2.3.2 *A topological space is pseudometrizable if and only if it a γ-space that has a σ-cushioned pairbase. (Note that a topological T_1-space that has a σ-cushioned pairbase is called* stratifiable.*)*

Exercise 2.3.3 *(compare [19, Theorem 2.32]) Let $(X, \mathcal{U})$ be a locally symmetric quasi-uniform space with a countable base. Show that $\tau(\mathcal{U})$ is pseudometrizable. (Hint: Observe that $\tau(\mathcal{U})$ has a σ-cushioned pairbase.)*

Exercise 2.3.4 *Let d be a quasi-pseudometric on a set X. For fixed $x \in X$ consider the function $f_x : X \to \mathbb{R}$ defined by $f_x(y) = d(x, y)$ whenever $y \in X$. Then f_x is (in the usual sense, see e.g. [12, p. 61]) $\tau(d)$-upper semi-continuous and $\tau(d^{-1})$-lower semi-continuous.*

Evidently, each quasi-uniform space $(X, \mathcal{U})$ induces the bitopological space $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$. The obvious question which bitopological spaces can be induced by a quasi-uniformity has the expected answer.

Example 2.3.1 *Let $[0, 1]$ be the real unit interval. For all $x, y \in [0, 1]$ set $r(x, y) = \max\{0, x - y\}$. Then r is a quasi-pseudometric on $[0, 1]$ and $\tau(r)$ is the so-called* upper topology *$\{(a, 1] : a \in [0, 1]\} \cup \{\emptyset, [0, 1]\}$ on $[0, 1]$. Similarly $\tau(r^{-1})$ is the* lower topology *$\{[0, a) : a \in [0, 1]\} \cup \{\emptyset, [0, 1]\}$ on $[0, 1]$.*

A bitopological space (X, τ_1, τ_2) is called *pairwise completely regular* if it carries the coarsest bitopology that makes all the bicontinuous maps $f : (X, \tau_1, \tau_2) \to ([0, 1], \tau(r), \tau(r^{-1}))$ bicontinuous (that is, τ_1 and τ_2 are both initial with respect to this collection of maps and the topologies $\tau(r)$ and $\tau(r^{-1})$, respectively).

Proposition 2.3.2 *A bitopological space (X, τ_1, τ_2) is quasi-uniformizable if and only if it is pairwise completely regular.*

Proof. The proof is completely analogous to the proof of Corollary 2.2.4. Of course, we have to equip $[0, 1]$ with the quasi-uniformity $\mathcal{U}_r$ and its induced bitopology, and consider bicontinuous maps instead of continuous maps.

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