A CONGRUENCE PROBLEM

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How can we determine by measuring two polyhedra or even two polygons whether they are congruent? We illustrate the ideas of how to determine whether two dimensional polygons are congruent, specifically triangles. In high school geometry we are taught that the minimum amount of measurements required to show two triangles are congruent is three. This is through various combinations of side and angle measures. For example, side-angle-side (SAS), side-side-side (SSS), and angle-side-angle (ASA) are all ways to prove that two triangles are congruent. This same concept is the idea behind the problem of this paper. As with triangles, we are trying to determine the minimum amount of measurements required to determine two polyhedra up to congruence. The tools that are necessary in this problem are linear algebra, multivariable derivatives, and some group theory. More generally, we will look at the case of determining two polyhedra up to congruence as outlined in problem 1. We will use the paper *A Congruence Problem for Polyhedra* [BDH10] as the main source for this paper.

Problem 1. Given two polyhedra in $\mathbb{R}^3$ which have the same combinatorial structure (e.g., both are hexahedra with four-sided faces), determine whether a given set of measurements is sufficient to ensure that the polyhedra are congruent [BDH10].

We will make progress towards this by showing that there is a set of measurements, E, that will ensure congruence. We will list the following theorems and definitions to prove Theorem 8, which outlines what we need to determine congruence for two polyhedra. The following fact will be a basis of our problem.

Theorem 2 (Cauchy, 1839)**.** *Two convex polyhedra with corresponding congruent and similarly situated faces have equal corresponding dihedral angles* [BDH10].

Theorem 2 is not true for polyhedra that are not convex. For example, consider two hexahedra with a shallow tetrahedron added to the top of both of the hexahedra. If we push the top of the

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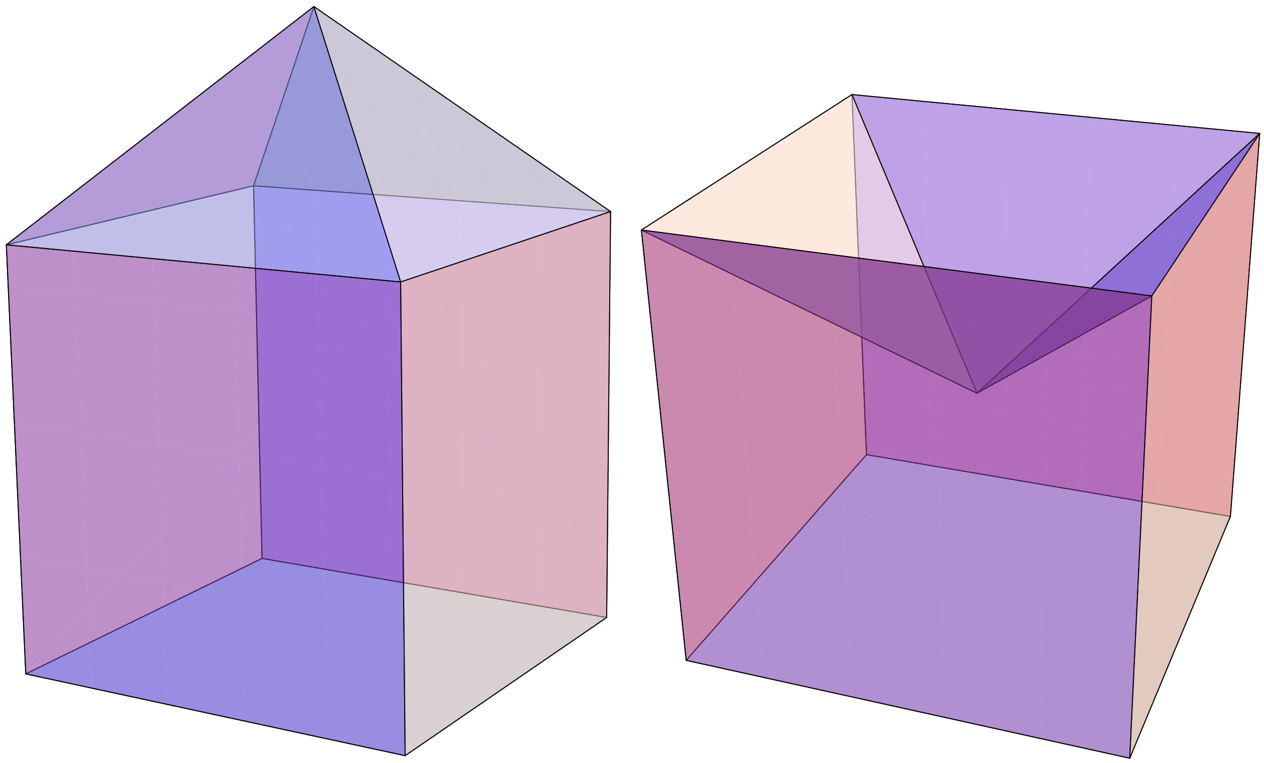

FIGURE 1. The image on the left is a convex polyhedron and the image on the right in a non-convex polyhedron.

tetrahedra down on only one of the hexahedra, it will no longer be convex. The figure below shows an example of a convex polyhedron and a non convex polyhedron. The new polyhedra will not have the same dihedral angles as the original polyhedron. We will define what it means to be a polyhedron in the following definition:

Definition 3. [BDH10] A *closed half-space* is a subset of $\mathbb{R}^3$ of the form

$$\{(x, y, z) \in \mathbb{R}^3 | ax + by + cz + d \geq 0\}$$

with $(a, b, c) \neq (0, 0, 0)$.

We will use the definition of closed half-space to define a convex polyhedron below.

Definition 4. [BDH10] A *convex polyhedron* is a subset P of $\mathbb{R}^3$ which is bounded, does not lie in any plane, and can be expressed as an intersection of finitely many closed half-spaces.

Throughout this paper we will denote the number of edges of our polyhedron by E, the number of vertices by V, and the number of faces by F, as defined in the paper *A Congruence Problem for Polyhedra* [BDH10]. The vertices, edges, and faces of a convex polyhedron P can be defined in terms of intersections of P with suitable closed half-spaces [BDH10]. For example, a face of a polyhedron is created by intersecting a plane, which is the intersection of two half spaces, with P.

Definition 5. [BDH10] The underlying *abstract polyhedron* of a convex polyhedron P is the triple (V_P, F_P, I_P), where V_P is the set of vertices of P, F_P is the set of faces of P, and $I_P \subset V_P \times F_P$ is the incidence relation between vertices and faces, that is, (v, f) is in I_P if and only if vertex v lies on face f.

Consider the following example of a structure for an abstract tetrahedron. We will call this abstact polyhedron Π and the realization of this tetrahedron P. The vertices of tetrahedron P are labeled as follows: vertex 1 is at $(0, 0, 0)$, vertex 2 is at $(1, 0, 0)$, vertex 3 is at $(0, 0, 1)$ and vertex 4 is at $(0, 1, 0)$. We start by labeling $V = \{1, 2, 3, 4\}$ and $F = \{a, b, c, d\}$. We define the incidence relation as follows,

$$I = \{(1, a), (1, b), (1, c), (2, a), (2, c), (2, d), (3, a), (3, b), (3, d), (4, b), (4, c), (4, d)\}.$$

In the definition of our problem, we state that two polyhedra have the same combinatorial structure. The definition below states what it means for two polyhedra to have the same combinatorial structure.

Definition 6. [BDH10] We say that two polyhedra P and Q have the same *combinatorial structure* if their underlying abstract polyhedra are isomorphic; that is, there are bijections $\beta_F : F_P \to F_Q$ and $\beta_V : V_P \to V_Q$ that respect the incidence relation: (v, f) is in I_P if and only if $(\beta_V(\mathrm{v}),\beta_F(f))$ is in I_Q. A *realization* of $\Pi = (V, F, I)$ is a pair of functions (α_V, α_F) where $\alpha_V : V \to \mathbb{R}^3$ gives a point for each v in $V, \alpha_F : F \to \{\text{planes in } \mathbb{R}^3\}$ gives a plane for each f in F, and the point $\alpha_V(v)$ lies on the plane $\alpha_F(f)$ whenever (v, f) is in I.

For an example we will show that the tetrahedron P ha a realization. The realization is the combinatorial structure placed in $\mathbb{R}^3$. An image of the tetrahedron realization is shown below:

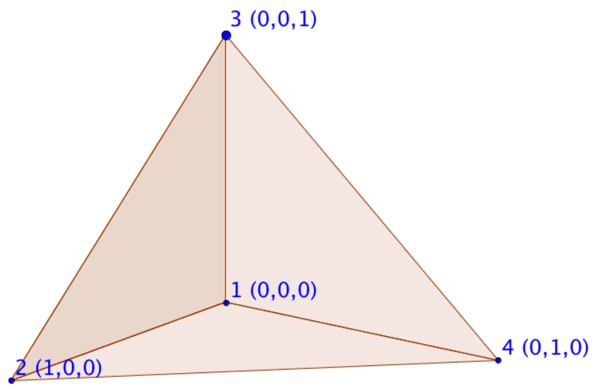

FIGURE 2. A realization, P, of a tetrahedron, Π.

Definition 7. [Rud76] A set $X \subseteq \mathbb{R}^n$ is open if for all $x \in X$, there exists an $\varepsilon > 0$ such that the ball of radius ε about x is a subset of X.

The ball of radius ε about 0 is the set of points with distance ε or less from 0. We use the definition of open to define measurement below.

Definition 8. [BDH10] A *measurement* for P is a smooth (infinitely differentiable) function m defined on an open neighborhood of P in the space of realizations of Π, such that m is invariant under rotations and translations.

The measurement for P is not altered under rotations and translations.

Definition 9. [BDH10] Given two vertices v and w on P that lie on a common face, the *face distance* associated to v and w is the function that maps a realization $Q = (\alpha_V, \alpha_F)$ of Π to the distance from $\alpha_V(v), \alpha_W(w)$. In other words, it corresponds to the measurement of the distance between the vertices of Q corresponding to v and w.

Below, the theorem gives us the answer to the problem we are seeking. The theorems and definitions listed above help us to understand Theorem 10, which helps us understand how to determine that two polyhedra are congruent.

Theorem 10. [BDH10] *Let P be a convex polyhedron with underlying abstract polyhedron* (V, F, I). *Then there is a set S of face distances of P such that (i) S has cardinality E, and (ii) locally near P, the set S completely determines P up to congruence in the following sense: there is a positive real* *number ε such that for any convex polyhedron Q and isomorphism $\beta : (V, F, I) \cong (V_Q, F_Q, I_Q)$ of underlying abstract polyhedra, if*

1. *each vertex v of P is within distance ε of the corresponding vertex $\beta_v(v)$ of Q, and*
2. *m(Q)=m(P) for each measurement m in S,*

then Q is congruent to P.

A more direct way to state Theorem 10 is listed below:

Corollary 11. [BDH10] *Let P be a convex polyhedron with E edges. Then there is a set of E measurements that is sufficient to determine P up to congruence amongst all nearby convex polyhedra with the same combinatorial structure.*

For an example we will look at a set of measurements that do not show that two hexahedra (cubes) are congruent. Consider faces 1, 2, 3, 4, 5, 6 of cube P and 7, 8, 9, 10, 11, 12 of cube Q. Five measurements (the shared edge, and two edges from each face) are necessary to conclude that a face (1) from P and a face (7) from Q are congruent. From there, we know that it will take four new measurements to determine face 2 and face 8 (1 adjacent to 2 and 7 adjacent to 8) are congruent. This is because the adjacent faces share one common side, therefore the next face only requires four measurements instead of five. The last set of measurements comes from showing the faces 3 and 9 are congruent. To show those faces congruent it would only require three measurements since 3 is adjacent to 1 and 2 (it shares two common sides) and 9 is adjacent to 7 and 8. We know that twelve measurements will show P and Q are congruent, but the measurements listed are not enough. So, not every set of E measurements will show P and Q are congruent.

We will look closer at the graphs of polyhedra.

Definition 12. [Joh09] A graph is planar if it can be drawn in the plane without its edges crossing.

Every polyhedron must be a planar graph. The definition for bipartite is given below.

[Joh09]A graph $G = (V, E)$ is bipartite if there exist subsets V_1 and V_2 (either possibly empty) of V such that $V_1 \bigcap V_2 = \emptyset, V_1 \bigcup V_2 = V$, and each edge in E is incident to one vertex in V_1 and one vertex in V_2.

An example of a bipartite planar graph is shown in Figure 2.

Definition 13. [DS06] The order of a graph is the number of vertices in a graph.

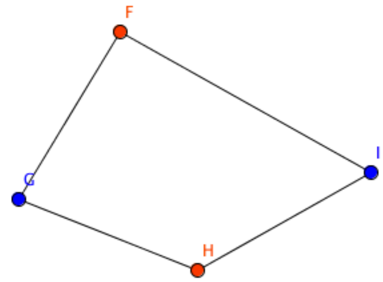

FIGURE 3. A bipartite planar graph

A bipartite graph must have a minimum order of four to be able to create a face. Having only two nodes would not create a face, only an edge. If we were to have three nodes we would have a face, but the graph would not be bipartite. The number of vertices must be even in order to fulfill the requirement of the graph of the face being bipartite.

Definition 14. [Joh09] The degree of a vertex v, $\delta(v)$, is the number of edges incident to v.

Lemma 15. *In a planar bipartite graph, there exists a vertex of degree at most 3.*

Proof. Consider each edge of Γ to be two directed edges that are going opposite directions. In this lemma we will calculate the degree of a face by starting at a vertex and walking around the edges in the same direction counting the number of edges [Poe10]. That means we now have twice the number of edges, or $2e$, where e the number of edges. Each face must consist of at least 4 nodes. This is because only 2 nodes would not make a face and 3 would not allow for the graph the be bipartite. We know that the sum of the degrees of the faces is exactly twice the number of edges in the graph because each edge is counted exactly twice, either two times in the same face or once in two adjoining faces [Poe10]. If each vertex has degree greater than or equal to 4, we know that

$$2e = \sum \deg(F) \geq 4f.$$

In other words, $2e \geq 4f$ or $\frac{1}{2}e \geq f$. Using Euler's Formula, $f = e - r + 2$, we get

$$\frac{1}{2}e \geq e - r + 2.$$

Using algebra we get the following:

$$e \leq 2e - 2r + 4$$

$$2r - 4 \leq e$$

as we stated above. □

We will use Lemma 15 in the proof of the lemma below.

Lemma 16. [BDH10] *Suppose that Γ is a planar bipartite graph of order r. Then there is an ordering n_1, n_2,...,n_r of the nodes of Γ such that each node n_i is adjacent to at most three preceding nodes.*

Proof. By means of induction we wish to show that for Γ, a planar bipartite graph of order r, there is an ordering of nodes $n_1, n_2, \ldots, n_r$ where each node is adjacent to at most three preceding nodes. Base Case: Let $r = 1$. This is obvious if $r = 1$ because there is only one vertex, thus it must be adjacent to at most three proceeding nodes. Inductive Hypothesis: Assume that for any graph Γ of order r there is an ordering of nodes $n_1, n_2, \ldots, n_r$ such that each node is adjacent to at most three preceding nodes. Inductive Step: We need to show that for a graph of order $r + 1$ there is an ordering of nodes $n_1, n_2, \ldots, n_{r+1}$ such that each node is adjacent to at most three preceding nodes. If every node of Γ were adjacent to at least 4 nodes, then there would be at least $2r$ edges in Γ, which contradicts the previous results (from Lemma 15). So we know that all of the nodes will not have order greater than 4. We will name that node n_{r+1}, which has degree at most 3. If we remove the node n_{r+1} we are left with a nonempty bipartite planar graph with r nodes. From our inductive hypothesis we know that there is an ordering of nodes $n_1, n_2, \ldots, n_r$ such that each node is adjacent to at most three preceding nodes. So we add back in the n_{r+1} node which is adjacent to three preceding nodes. Thus, $n_1, n_2, \ldots, n_{r+1}$ gives the necessary ordering to satisfy Lemma 16. □

Lemma 17. [BDH10] *Let P be a convex polyhedron. Consider the set $V \cup F$ consisting of all vertices and all faces of P. It is possible to order the elements of this set such that every vertex or face in this set is incident with at most three earlier elements of $V \cup F$.*

Proof. We will construct a planar graph, Γ, of order $V + F$ in the following way. Γ has one node for each vertex of P and one node for each face of P [BDH10]. Whenever a vertex v of P lies on a face f of P we introduce an edge of Γ connecting the nodes corresponding to v and f. Let P be a convex polyhedron. Since P is convex, we know that it must be planar, meaning no edges will cross. By creating a graph, Γ, we choose a point on the faces of P and connect it with the different nodes. This graph will be planar bipartite because we know from above that none of the edges will cross and we create an edge only when we connect a vertex node and a face node, making this a bipartite planar graph. Applying Lemma 10 from above, every vertex or face in the set is incident with at most three earlier elements of $V \cup F$. □

A plane has equation $a'x + b'y + c'z = d'$. As long as $d' \neq 0$, we can divide both sides by d' to get $ax + by + cz = 1$. To ensure that $d' \neq 0$, we translate the polyhedron if necessary so no plane contains the origin. To describe a realization we need to give the coordinates of each vertex and three numbers to describe each face. Let $\phi_{i,j}$ be the function from $\mathbb{R}^{3V+3F}$ to $\mathbb{R}$ defined by

$$\phi_{i,j}(x_1, y_1, z_1, \ldots, a_1, b_1, c_1, \ldots) = a_j x_i + b_j y_i + c_j z_i - 1.$$

In this function x_i, y_i, z_i are the coordinates of the vertices and a_j, b_j, c_j represent the different faces. In particular, a_j, b_j, c_j are the coefficients of the realization equation. The function $\phi_{i,j}$ determines if a polyhedra is a realization or not. In order for a polyhedra to be a realization, $\phi_{i,j} = 0$. This is because a vertex must be on a face in order for a combinatorial structure to be a realization. If $\phi_{i,j} \neq 0$ then this means that a vertex is not on one of the faces.

Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$. We define Df by:

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & & & \\ \frac{\partial f_3}{\partial x_1} & & & \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & & & \frac{\partial f_n}{\partial x_m} \end{pmatrix}.$$

Pick some ordering of the incidence set. We define $D\phi(P)$ as follows:

$$\left(\begin{array}{ccc|ccc||ccccccccc|ccc}
a_1 & b_1 & c_1 & 0 & \cdots & 0 & x_1 & y_1 & z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 & x_2 & y_2 & z_2 & 0 & 0 & 0 & 0 & \cdots & 0 \\
a_3 & b_3 & c_3 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & y_3 & z_3 & 0 & \cdots & 0 \\
\hline
0 & 0 & 0 & & & & & & & & & & & & & & & \\
\vdots & \vdots & \vdots & & * & & & & & & * & & & & & & * & \\
0 & 0 & 0 & & & & & & & & & & & & & & &
\end{array}\right).$$

$D\phi(P)$ is a $2E$-by-$(3V + 3F)$ matrix. The top left 3 by 3 sub matrix contains the partials with respect to the vertices. The stars in the matrix represent a non-zero block. In order to create the matrix of $D\phi(P)$, we take the partial with respect to the vertices, x_i, y_i, z_i. This gives us the first three columns of the matrix. The nine columns just to the right of the double line are found by taking the partial with respect to the faces, a_j, b_j, c_j . The double bars separate the derivatives of the vertices (left) and the faces (right).

Lemma 18. *The derivative $D\phi(P)$ has rank $2E$.*

We will only provide a proof of Lemma 18 for when all the faces are triangular and each vertex is only on three faces.

Proof. We wish to prove that the rank of matrix $D\phi(P)$ is $2E$. This sub-matrix (the 3-by-3 in the left corner) is nonsingular (the determinant must be non-zero). The equations of faces a_1, a_2, and a_3 have a single point that exists on all three faces, v_1. This is true because the determinant determines how many solutions there are to the equations. If the determinant was zero, then there is exactly one solution, a vertex. The three equations below represent equations of three planes, or faces.

$$a_1x + b_1y + c_1z = 1$$

$$a_2x + b_2y + c_2z = 1$$

$$a_3x + b_3y + c_3z = 1.$$

If the determinant was zero, then there would be infinitely many solutions. That would not be possible because only one vertex lies on three faces, thus just one solution. Because v_1 only lies on the first three faces, all the other entries in the first three columns are zeroes. Thus any nontrivial linear relation of the rows cannot involve the first three rows. So we know that $D\phi(P)$ has full rank (the number of rows is equal to the rank) if and only if the matrix without the first three rows has full rank. The rank will be reduced by three (because we removed the first three rows and they are independent). From there, we can keep deleting rows, knowing that the new matrix (once the rows are deleted) will have full rank as long as $D\phi(P)$ has full rank. By removing rows we are removing vertices from the incidence relation. Thus, if we remove a vertex we remove the three faces that contain that vertex. We continue the process of deleting rows until we are left with a 3-by-$2E$ matrix. Clearly the new matrix is full rank. We know that the new matrix is full rank because we are left with a matrix with 3 rows. Since the rank of the new matrix is 3, we know that the new matrix is full rank because the first three rows are linearly independent. Therefore, the derivative

$D\phi(P)$ has rank $2E$ as long as every vertex lies on only three faces. In general, for any combination of faces and vertices, we know that at least one face or vertex is adjacent to at most three previous faces or vertices. This follows from Lemma 17. □

Lemma 19. *Extension Lemma* [Lee03] *Let M be a smooth manifold, let $A\subset M$ be a closed subset, and let f:$A\to \mathbb{R}^k$ be a smooth function. For any open set U containing A, there exists a smooth function $\tilde{f}$:$U \to \mathbb{R}^k$ such that $\tilde{f}\mid_A = f$ and supp $\tilde{f} \subset U$.*

We define supp$\tilde{f} = \{x \in U | \tilde{f}(x) \neq 0\}$. In the case of the Lemma above, M is $\mathbb{R}^{3V+3F}$, A is the space of all realizations with dimension $E+6$, and f is our measurement, m. Given any measurement m, it follows from Lemma 19 that we can extend m to a smooth function in a neighborhood of P in $\mathbb{R}^{3V+3F}$ [BDH10]. A smooth manifold is "locally" similar to a Euclidean space. Then the derivative $Dm(P)$ is a row vector of length $3V + 3F$ [BDH10].

Theorem 20. [Rud76] *The Inverse Function Theorem*

Suppose f is a $\mathcal{C}^1$-mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, Df(a) is invertible for some $a \in E$, and b=f(a). Then

a.) there exist open sets U and V in $\mathbb{R}^n$ such that $a \in U$, $b \in V$, f is one-to-one on U and f(U) $= V$;

b.) if g is the inverse of f [which exists, by (a)], defined in V by

$$g(f(x)) = x$$

$$(x \in U)$$

then $g \in \mathcal{C}^1(V)$.

We will use The Inverse Function Theorem in the proof of Theorem 21. Now suppose that Q is a polyhedron as in the statement of the theorem.

Theorem 21. [BDH10] *Let S be a finite set of measurements for Π near P. Let $\psi : \mathbb{R}^{3V+3F} \to \mathbb{R}^{|S|}$ be the vector-valued function obtained by combining the measurements in S, and write $D\psi(P)$ for its derivative at P, an |S|-by-(3V+3F) matrix whose rows are the derivatives $Dm(P)$ for m in S. Then the matrix*

$$D(\phi,\psi)(P) = \begin{pmatrix} D\phi(P) \\ D\psi(P) \end{pmatrix}$$

has rank at most 3E, and if it has rank exactly 3E then the measurements in S are sufficient to determine congruence that is, for any realization Q of Π*, sufficiently close to P, if* $m(Q) = m(P)$ *for all m in S then Q is congruent to P.*

Proof. Let $Q(t)$ be any smooth one-dimensional family of realizations of Π such that $Q(0) = P$ and $Q(t)$ is congruent to P for all t. Because ϕ must always be equal to zero in order for a realization to exist, we know that $\phi(Q(t)) = 0$ for all t. Differentiating and applying the chain rule at $t = 0$ gives the matrix equation $D\phi(P)Q'(0) = 0$ where $Q'(0)$ is thought of as a column vector of length $3V + 3F$. $D\phi(P)Q'(0) = 0$ because we are taking the derivative of a constant function. The same argument applies to the map ψ since $Q(t)$ is congruent to P for all t, $\psi(Q(t)) = \psi(P)$ is constant and $D\psi(P)Q'(0) = 0$. $\psi(Q(t)) = \psi(P)$ because ψ determines measurement outputs. Since we know that $Q(t)$ is congruent to P for all t we know that their measurements must be equal, or $\psi(Q(t)) = \psi(P)$. Let G be a $(3V+3F)$-by-6 matrix that consists of the six column vectors that map to zero by $D\phi(P)$ and $D\psi(P)$. We will show the transpose of G below:

$$\left(\begin{array}{ccccccc||ccccccc}
1 & 0 & 0 & 1 & 0 & 0 & \cdots & -a_1^2 & -a_1b_1 & -a_1c_1 & -a_2^2 & -a_2^2 & -a_2b_2 & \cdots \\
0 & 1 & 0 & 0 & 1 & 0 & \cdots & -a_1b_1 & -b_1^2 & -b_1c_1 & -a_2b_2 & -b_2^2 & -b_2c_2 & \cdots \\
0 & 0 & 1 & 0 & 0 & 1 & \cdots & -a_1c_1 & -b_1c_1 & -c_1^2 & -a_2c_2 & -b_2c_2 & -c_2^2 & \cdots \\
0 & -z_1 & y_1 & 0 & -z_2 & y_2 & \cdots & 0 & -c_1 & b_1 & 0 & -c_2 & b_2 & \cdots \\
z_1 & 0 & -x_1 & z_2 & 0 & -x_2 & \cdots & c_1 & 0 & -a_1 & c_2 & 0 & -a_2 & \cdots \\
-y_1 & x_1 & 0 & -y_1 & x_2 & 0 & \cdots & -b_1 & a_1 & 0 & -b_2 & a_2 & 0 & \cdots
\end{array}\right).$$

The left side of the matrix G^T represents altering (translating or rotating) the vertices of a polyhedra $Q(t)$ and the right side of G^T represents altering (translating or rotating) the faces of $Q(t)$. The first three rows of G^T represent translating and the last three rows of G^T represent rotations. We will translate $Q(t)$ t units along the x-axis , y-axis, and z-axis. This will result in the first three rows of the matrix G^T. In addition, we will rotate $Q(t)$ about the x-axis, y-axis, and z-axis by t radians. This will result in the last three rows of the matrix, G^T. In order to obtain the first three rows G^T, we start by translating $Q(t)$ t units along the x-axis, y-axis, and z-axis. We will use the example of translating along the x-axis for sake of verifying the matrix. For all three axes the normal of the plane will be (a_1, b_1, c_1). In order to determine the equation of a plane we need a point and a normal. The point on the normal (a_1, b_1, c_1) will remain the same regardless of what plane is being rotated or translated. We will need to pick a point on the plane of face 1, or more generally the plane that is being translated. We will pick the point $\left(\frac{1}{a_1}, 0, 0\right)$, which is on the x-axis. The point $\left(0, \frac{1}{b_1}, 0\right)$ will be on the y-axis and the point $\left(0, 0, \frac{1}{c_1}\right)$ will be on the z-axis. In the case of this proof we will assume that $a_1 \neq 0, b_1 \neq 0$, and $c_1 \neq 0$. If these points are zero, then we will choose another point. We wish to translate the point to $(\frac{1}{a_1} + t, 0, 0)$. To find the new equation of the plane we will do the following:

$$a_1\left(x - \frac{1}{a_1} - t\right) + b_1(y - 0) + c_1(z - 0) = 0$$
$$a_1x - 1 - a_1t + b_1y + c_1z = 0.$$

Now we rearrange so the right hand side is 1.

$$a_1x + b_1y + c_1z = 1 + a_1t$$
$$\left(\frac{a_1}{a_1t + 1}\right)x + \left(\frac{b_1}{a_1t + 1}\right)y + \left(\frac{c_1}{a_1t + 1}\right)z = 1.$$

Now, taking the derivative with respect to t we get:

$$\frac{d}{dt}\left(\frac{a_1}{a_1t + 1}\right) = \frac{-a_1^2}{(a_1t + 1)^2}$$

$$\frac{d}{dt}\left(\frac{b_1}{a_1t + 1}\right) = \frac{-a_1b_1}{(a_1t + 1)^2}$$
$$\frac{d}{dt}\left(\frac{c_1}{a_1t + 1}\right) = \frac{-a_1c_1}{(a_1t + 1)^2}.$$

Now, we evaluate when $t = 0$ to get:

$$G^T_{1,3V+1} = \frac{-a_1^2}{(a_1(0) + 1)^2} = -a_1^2$$

$$G^T_{1,3V+2} = \frac{-a_1b_1}{(a_1(0) + 1)^2} = -a_1b_1$$

$$G^T_{1,3V+3} = \frac{-a_1c_1}{(a_1t + 1)^2} = -a_1c_1.$$

The rest of the entries for the first three rows of G^T can be verified using a similar process.

In order to translate the vertices the process will be very similar to translating the faces. The coordinate for our first vertex is (x_1, y_1, z_1). In order to translate this vertex t units we will get $(x_1 + t, y_1, z_1)$. Now we take the derivative with respect to t to get the following:

$$\frac{d}{dt}(x_1 + t, y_1, z_1) = (1, 0, 0).$$

This verifies the first three entries of G^T. Verifying the left side of the first three rows of G^T can be done using the same process.

In order to verify the last three rows of the matrix G^T we will rotate $Q(t)$ by t radians about the x-axis, y-axis, and z-axis. We will use the example of rotating about the z-axis for sake of verifying the matrix. Similar to translating each axis t units, the normal will be (a_1, b_1, c_1). We choose a point that is on the z-axis. This point will be $(0, 0, \frac{1}{c_1})$. Again, if we were to chose the x-axis our point would be $(\frac{1}{a_1}, 0, 0)$ and the point would be $(0, \frac{1}{b_1}, 0)$ for the y-axis. We will multiply the column vector of the normal by the rotation about the z-axis by t radians. This will give us the column vector for the normal once $Q(t)$ is rotated about the z-axis by t radians. The matrix multiplication is shown below:

$$\begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}.$$

The following is the new normal:

$$\begin{pmatrix} a_1 \cos(t) - b_1 \sin(t) + 0 \\ a_1 \sin(t) + b_1 \cos(t) + 0 \\ 0 + 0 + c \end{pmatrix}.$$

Or more simply:

$$\begin{pmatrix} a_1 \cos(t) - b_1 \sin(t) \\ a_1 \sin(t) + b_1 \cos(t) \\ c \end{pmatrix}.$$

We then use the new normal to determine the new equation for $Q(t)$. The steps to find this are shown below:

$$(a_1 \cos(t) - b_1 \sin(t))(x - 0) + (a_1 \sin(t) + b_1 \cos(t))(y - 0) + c_1 \left(z - \frac{1}{c_1}\right) = 0$$

$$(a_1\cos(t) - b_1\sin(t))x + (a_1\sin(t) + b_1\cos(t))y + c_1z - 1 = 0$$

$$(a_1\cos(t) - b_1\sin(t))x + (a_1\sin(t) + b_1\cos(t))y + c_1z = 1.$$

Next, we take the derivative with respect to t.

$$\frac{d}{dt}(a_1\cos(t) - b_1\sin(t)) = -a_1\sin(t) - b_1\cos(t)$$

$$\frac{d}{dt}(a_1\sin(t + b_1\cos(t)) = a_1\cos(t) - b_1\sin(t)$$

$$\frac{d}{dt}(c_1) = 0.$$

Now, evaluating where $t = 0$, we get the following:

$$G^T_{6,3V+1} = -a_1\sin(0) - b_1\cos(0) = -b_1$$

$$G^T_{6,3V+2} = a_1\cos(0) - b_1\sin(0) = a_1$$

$$G^T_{6,3V+3} = 0.$$

The last three rows of G^T can be verified using a similar process.

In order to rotate the vertices the process will be very similar to rotating the faces. The coordinate for our first vertex is (x_1, y_1, z_1). In order to rotate this vertex t radians about the z-axis we do the following matrix multiplication:

$$\begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.$$

The following is the new normal:

$$\begin{pmatrix} x_1\cos(t) - y_1\sin(t) + 0 \\ x_1\sin(t) + y_1\cos(t) + 0 \\ 0 + 0 + z_1 \end{pmatrix}$$

Or more simply:

$$\begin{pmatrix} x_1\cos(t) - y_1\sin(t) \\ x_1\sin(t) + y_1\cos(t) \\ z_1 \end{pmatrix}$$

This verifies the first three entries of G. Verifying the left side of the first three rows of G can be done using the same process.

We have the following matrix equation:

$$\begin{pmatrix} D\phi(P) \\ D\psi(P) \end{pmatrix} G = 0.$$

We will now introduce a notion of normalization on the space of realizations of Π. We will say that a realization Q of Π is P-normalized if the following is true:

1.) The first vertices of P and Q are equal. This can be written as $v_1(P) = v_1(Q)$. This can be written as

$$x_1(P) = x_1(Q), y_1(P) = y_1(Q), z_1(P) = z_1(Q).$$

2.) The vector from $v_1(Q)$ to $v_2(Q)$ is in the direction of the positive x-axis [BDH10]. The coordinate of v_2 is in the positive x-direction. This can be written as $x_1(P) = x_1(Q) < x_2(Q)$

3.) $v_1(Q)$, $v_2(Q)$, and $v_3(Q)$ all lie in a plane parallel to the xy-plane, with $v_3(Q)$ lying in the positive $y-$direction from $v_1(Q)$ and $v_2(Q)$. This can be written as $y_2(Q) < y_3(Q)$. The z-coordinate remains the same for all vertices since they all lie in a plane parallel to the xy-plane. This can be written as $z_1(P) = z_1(Q) = z_2(Q) = z_3(Q)$.

Clearly every realization of Q of Π is congruent to a unique P-normalized realization, which we'll refer to as the P-normalization of Q [BDH10]. Note that the normalization operation is a continuous map on a neighborhood of P. Without loss of generality, rotating P around the origin if necessary, we may assume that P itself is normalized [BDH10]. The condition that a realization Q is normalized gives six more conditions on the coordinates of Q, corresponding to six extra functions $\chi_1, \ldots, \chi_6$, which we use to augment the function $(\phi, \psi) : \mathbb{R}^{3V+3F} \to \mathbb{R}^{2E+|S|}$ to a function $(\phi, \psi, \chi) : \mathbb{R}^{3V+3F} \to \mathbb{R}^{2E+|S|+6}$ [BDH10]. These six functions are:

$$\chi_1(Q) = x_1(Q) - x_1(P)$$

$$\chi_2(Q) = y_1(Q) - y_1(P)$$

$$\chi_3(Q) = z_1(Q) - z_1(P)$$

$$\chi_4(Q) = y_2(Q) - y_1(P)$$

$$\chi_5(Q) = z_2(Q) - z_1(P)$$

$$\chi_6(Q) = z_3(Q) - z_1(P).$$

The matrix of $D\chi(P)$ is shown below:

$$\left(\begin{array}{cccccccccc||ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \end{array}\right).$$

As a corollary, the columns of G^T are linearly independent, which proves that the matrix in the statement of the theorem has rank at most $3E$ [BDH10]. Similarly, the rows of $D\chi(P)$ must be linearly independent, and moreover there aren't any nontrivial linear combination of the rows of $D(\phi, \psi)(P)$ [BDH10]. Hence if $D(\phi, \psi)(P)$ has rank exactly $3E$ then the augmented matrix

$$\begin{pmatrix} D\phi(P) \\ D\psi(P) \\ D\chi(P) \end{pmatrix}$$

has rank $3E+6$ [BDH10]. Hence the map (ϕ, ψ, χ) has injective derivative at P, and so by the inverse function theorem the map (ϕ, ψ, χ) is injective on a neighborhood of P in $3V+3F$ [BDH10]. The map $(\phi, \psi, \chi) : \mathbb{R}^{3V+3F} \to \mathbb{R}^{3E+6}$ has the same dimensions for domain and range since $3V + 3F = 3E + 6$ by Euler's Theorem. Let R be the P-normalization of Q [BDH10]. Then $\phi(R) = \phi(Q) = \phi(P) = 0$ because they are all realizations. We know $\psi(R) = \psi(Q) = \psi(P)$ because $\psi(R)$ has the same measurements as $\psi(Q)$. Also, $\chi(R) = \chi(P)$ because P is a normalization of Q. So if Q is sufficiently close to P, then by continuity of the normalization map R is close to P and hence $R = P$ by the inverse function theorem [BDH10]. So Q is congruent to $R = P$ as required [BDH10]. □

Claim 22. The 6-by-6 matrix $D\chi(P)G$ is invertible

Proof. We multiply together the matrix G^T by $D\chi(P)$. This is shown below:

$$\left(\begin{array}{cccccccccc||ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & \cdots & 0
\end{array}\right)$$

$$\cdot\left(\begin{array}{ccccccc||ccccccc}
1 & 0 & 0 & 1 & 0 & 0 & \cdots & -a_1^2 & -a_1b_1 & -a_1c_1 & -a_2^2 & -a_2^2 & -a_2b_2 & \cdots \\
0 & 1 & 0 & 0 & 1 & 0 & \cdots & -a_1b_1 & -b_1^2 & -b_1c_1 & -a_2b_2 & -b_2^2 & -b_2c_2 & \cdots \\
0 & 0 & 1 & 0 & 0 & 1 & \cdots & -a_1c_1 & -b_1c_1 & -c_1^2 & -a_2c_2 & -b_2c_2 & -c_2^2 & \cdots \\
0 & -z_1 & y_1 & 0 & -z_2 & y_2 & \cdots & 0 & -c_1 & b_1 & 0 & -c_2 & b_2 & \cdots \\
z_1 & 0 & -x_1 & z_2 & 0 & -x_2 & \cdots & c_1 & 0 & -a_1 & c_2 & 0 & -a_2 & \cdots \\
-y_1 & x_1 & 0 & -y_1 & x_2 & 0 & \cdots & -b_1 & a_1 & 0 & -b_2 & a_2 & 0 & \cdots
\end{array}\right).$$

The result is the following matrix:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & z_1 & -y_1 \\
0 & 1 & 0 & -z_1 & 0 & x_1 \\
0 & 0 & 1 & y_1 & -x_1 & 0 \\
0 & 1 & 0 & -z_2 & 0 & x_2 \\
0 & 0 & 1 & y_2 & -x_2 & 0 \\
0 & 0 & 1 & y_3 & -x_3 & 0
\end{pmatrix}$$

We start to verify the determinant about the the first column. The work to calculate the determinant follows.

$$det(D\phi(G)) = 1\det\begin{pmatrix} 1 & 0 & -z_1 & 0 & x_1 \\ 0 & 1 & y_1 & -x_1 & 0 \\ 1 & 0 & -z_2 & 0 & x_2 \\ 0 & 1 & y_2 & -x_2 & 0 \\ 0 & 1 & y_3 & -x_3 & 0 \end{pmatrix}$$

$$= 1\left(1\det\begin{pmatrix} 1 & y_1 & -x_1 & 0 \\ 0 & -z_2 & 0 & x_2 \\ 1 & y_2 & -x_2 & 0 \\ 1 & y_3 & -x_3 & 0 \end{pmatrix} + 1\det\begin{pmatrix} 0 & -z_1 & 0 & x_1 \\ 1 & y_1 & -x_1 & 0 \\ 1 & y_2 & -x_2 & 0 \\ 1 & y_3 & -x_3 & 0 \end{pmatrix}\right)$$

$$= 1\left(1\left(x_2\det\begin{pmatrix} 1 & y_1 & -x_1 \\ 1 & y_2 & -x_2 \\ 1 & y_3 & -x_3 \end{pmatrix}\right) + 1\left(x_1\det\begin{pmatrix} 1 & y_1 & -x_1 \\ 1 & y_2 & -x_2 \\ 1 & y_3 & -x_3 \end{pmatrix}\right)\right).$$

Now, multiplying everything and finding the determinant of the 3-by-3 matrices we get:

$$x_2(-x_3y_2 - x_2y_1 - x_1y_3 + x_1y_2 + x_2y_3 + x_3y_1) - x_1(-x_3y_2 - x_2y_1 - x_1y_3 + x_1y_2 + x_2y_3 + x_3y_1)$$

$$= (x_2 - x_1)(-x_3y_2 - x_2y_1 - x_1y_3 + x_1y_2 + x_2y_3 + x_3y_1)$$

As we stated above, we know that $y_1 = y_2$ because of the conditions of normalizing the space of realizations of Π. Using substitution, we get the the determinant, $(y_3 - y_1)(x_2 - x_1)^2$, which is nonzero. □

Definition 23. Call a set S of measurements sufficient for P if the conditions of the above theorem apply: that is, the matrix $D(\phi, \psi)(P)$ has rank $3E$.

Corollary 24. *Given a sufficient set S of measurements, there's a subset of S of size E that's also sufficient.*

Proof. Since the matrix $D(\phi, \psi)(P)$ has rank $3E$ by assumption, and the $2E$ rows coming from $D\phi(P)$ are linearly independent by Lemma 13, we can find E rows corresponding to measurements in S such that $D\phi(P)$ together with those E rows has rank $3E$ [BDH10]. □

Before we finish the proof of our main theorem, we neeed one last definition.

Definition 25. [BDH10] Given two vertices v and w of P that lie on a common face, the face distance associated to v and w is the function that maps a realization $Q = (\alpha_V, \alpha_F)$ of Π to the distance from $\alpha_V(v)$ to $\alpha_V(w)$.

Now to finish the proof of our main theorem, we need to know that there is a set of measurements that is sufficient.

Theorem 26. *The set of all face distances is sufficient.*

We were faced with the problem of determining how many measurements were necessary to determine that two polyhedra were congruent. Looking at two polyhedra P and Q that have the same combinatorial structure, we discover that there is a set of measurements that determine congruence. Through various methods we have determined that there exists a set of measurements that work. In conclusion, we have shown that there is a set of measurements, E, that will determine two polyhedra are congruent.

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