Geometry of multiple zeta values

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Abstract. Many relations are known between multiple zeta values $\zeta(k_1, \dots, k_n)$. A relation coming from the associator condition for the Drinfeld associator, the generating function of multiple zeta values, is a geometric relation. By the theory of mixed motives, we can control the dimension of the rational linear hull of multiple zeta values. The harmonic shuffle relation also comes from geometry, and more strongly, this is implied by the associator relation.

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1. Introduction

Let $k_1, \dots, k_n \geq 1$ be integers such that $k_n \geq 2$. We define a multiple zeta value $\zeta(k_1, \dots, k_n)$ by

$$\zeta(k_1, \dots, k_n) = \sum_{0<m_1<m_2<\cdots<m_n} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

We define the weight w of the index $(k_1, \dots, k_n)$ by $w = k_1 + \cdots + k_n$. Many relations between multiple zeta values are known for a long time; for example,

$$\zeta(2) \cdot \zeta(2) = 4\zeta(1,3) + 2\zeta(2,2), \quad \zeta(3) = \zeta(1,2), \quad \zeta(4) = 4\zeta(1,3).$$

The first one is a quadratic relation and the second and the third ones are linear relations between multiple zeta values. Several systematic methods are known to produce a series of relations for multiple zeta values: iterated integral shuffle relation, duality relation, harmonic shuffle relation, and so on. The iterated integral relation and the duality relation are a part of the associator relation, which is closely related to the Grothendieck–Teichmüller group. These relations produce many linear relations between multiple zeta values. What is very interesting is that all the known rational relations preserve the weights introduced above. So it is natural to expect that all $\mathbb{Q}$-relations come from geometry.

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2. Iterated integral expression

A multiple zeta value has an iterated integral expression, which enables us to study multiple zeta values from a geometric point of view. Let $\omega_1, \dots, \omega_n$ be one-forms on a manifold X and let $\gamma : [0,1] \to X$ be a path starting from a point a and ending with a point b. An iterated integral is defined by

$$\int_\gamma \omega_1\omega_2\cdots\omega_n = \int_{0<t_n<t_{n-1}<\cdots<t_1<1} \mathrm{pr}_1^*\,\omega_1 \wedge \mathrm{pr}_2^*\,\omega_2 \wedge \cdots \wedge \mathrm{pr}_n^*\,\omega_n,$$

where $\mathrm{pr}_i : [0,1]^n \to [0,1]$ is the i-th projection. A multiple zeta value is expressed as

$$\zeta(k_1,\dots,k_n) = \int_{[0,1]} \left(\frac{dx}{x}\right)^{k_n-1} \frac{dx}{1-x} \cdots \left(\frac{dx}{x}\right)^{k_1-1} \frac{dx}{1-x}.$$

To control many relations it is convenient to consider the "generating function" of multiple zeta values which is called the Drinfeld associator ([Dr]). Let $A = \mathbb{C}\langle\langle e_0, e_1\rangle\rangle$ be the non-commutative formal power series ring generated by e_0 and e_1 over $\mathbb{C}$ and $\omega = \frac{e_0 dx}{x} + \frac{e_1 dx}{x-1}$ an A-valued one-form. The Drinfeld associator Φ is defined by

$$\Phi = \Phi(e_0, e_1) = \lim_{t\to 0} t^{-e_1}\left[\exp\int_t^{1-t}\omega\right] t^{e_0},$$

where $\exp\int_a^b \omega = 1 + \sum_{i=1}^\infty \int_a^b \underbrace{\omega\cdots\omega}_{i\text{-times}}$. The multiple zeta value $(-1)^n\zeta(k_1,\dots,k_n)$ appears as the coefficient of $e_0^{k_n-1}e_1\cdots e_0^{k_1-1}e_1$.

3. Period of fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$

The Drinfeld associator Φ describes the period of the fundamental group of $\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$ for tangential base points introduced by Deligne [De]. Let $|\mathcal{M}_{0,4}| = \{\overrightarrow{01}, \overrightarrow{10}, \dots\}$ be a set of tangential points of $\mathcal{M}_{0,4}$ and p, q be elements of $|\mathcal{M}_{0,4}|$. The comparison isomorphism

$$\mathrm{comp}\colon \mathbb{Q}[\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, p, q)]^\wedge \otimes \mathbb{C} \simeq \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle \otimes \mathbb{C} \tag{1}$$

defines a mixed Hodge structure on $\mathbb{Q}[\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, p, q)]^\wedge$. The Drinfeld associator is equal to the image of [0, 1] under the comparison homomorphism. The Drinfeld associator satisfies the associator relations arising from the compatibility condition for homomorphisms of mixed Hodge structures. In general, the associator relations can be formulated via the compatibility condition for isomorphisms between topological and de Rham fundamental groups. To interpret an associator as a functorial isomorphism we introduce the fundamental category (Fund) and the notion of algebroids.

Definition 3.1. Let K be a field. A K-algebroid $\mathcal{U} = \{\mathcal{U}_{ab}\}_{ab}$ over a set B consists of

(1) a family of K-vector spaces $\{\mathcal{U}_{ab}\}_{ab}$ indexed by $a, b \in B$, and
(2) a family of homomorphisms

$$\mathcal{U}_{bc} \otimes \mathcal{U}_{ab} \to \mathcal{U}_{ac},$$

which is associative.

We assume the following properties:

(1) For $a \in B$ the associative ring $\mathcal{U}_{aa}$ has a unit.
(2) The vector space $\mathcal{U}_{ab}$ is a free left $\mathcal{U}_{bb}$-module of rank one and a free right $\mathcal{U}_{aa}$-module of rank one.
(3) The natural homomorphism

$$\mathcal{U}_{bc} \otimes_{\mathcal{U}_{bb}} \mathcal{U}_{ab} \to \mathcal{U}_{ac}$$

is an isomorphism.

We can define the notion of K-Hopf algebroids similarly. For a $\mathbb{Q}$-algebraic variety, by attaching a completion of the $\mathbb{Q}$-linear hull of Betti and de Rham fundamental groupoids, we get two functors

$$\mathcal{U}^{\mathrm{B}}, \mathcal{U}^{\mathrm{DR}}\colon (\mathrm{Var}/\mathbb{Q}) \to (\text{Hopf-alg}/\mathbb{Q})$$

from the category $(\mathrm{Var}/\mathbb{Q})$ of $\mathbb{Q}$-algebraic varieties to the category $(\text{Hopf-alg}/\mathbb{Q})$ of $\mathbb{Q}$-Hopf algebroids. The comparison map obtained by Hodge theory gives a functorial isomorphism of these two functors over $\mathbb{C}$.

The object of the fundamental category (Fund) consists of four spaces $\mathcal{M}_{0,4}$, $\mathcal{M}_{0,5}$, $\overline{\mathcal{M}_{0,4}} - \{0, \infty\}$ and the punctured disc Δ^*, with tangential points. Morphisms are generated by

(1) inclusions $\Delta^* \to \mathcal{M}_{0,4}$ around points 0, 1 or ∞,
(2) "infinitesimal inclusions" $\mathcal{M}_{0,4} \to \mathcal{M}_{0,5}$, and
(3) natural inclusion $\mathcal{M}_{0,4} \to \overline{\mathcal{M}_{0,4}} - \{0, \infty\}$.

We can define two functors of Betti and de Rham fundamental algebroids over a set of tangential points $\mathcal{U}^{\mathrm{B}}(?) = \mathbb{Q}[\pi_1^{\mathrm{B}}(?)]\hat{}$ and $\mathcal{U}^{\mathrm{DR}}(?) = \mathbb{Q}[\pi_1^{\mathrm{DR}}(?)]\hat{}$: (Fund) $\to$ $(\text{Hopf-alg}/\mathbb{Q})$. The set of functorial isomorphisms from $\mathcal{U}^{\mathrm{B}} \otimes \mathbb{C}$ to $\mathcal{U}^{\mathrm{DR}} \otimes \mathbb{C}$ is denoted by $\mathrm{Isom}(\mathcal{U}^{\mathrm{B}} \otimes \mathbb{C}, \mathcal{U}^{\mathrm{DR}} \otimes \mathbb{C})$.

Definition 3.2 (Associator). (1) Let ρ and e be the canonical generators of $\pi_1(\Delta^*, +)$ and the dual of $\frac{dx}{x}$ in $\mathcal{U}^{\mathrm{DR}}(\Delta^*)$. Here $+$ denotes a tangential point of Δ^* defined by the local coordinate x. For an element $\varphi \in \mathrm{Isom}(\mathcal{U}^{\mathrm{B}} \otimes \mathbb{C}, \mathcal{U}^{\mathrm{DR}} \otimes \mathbb{C})$ we define $\lambda(\varphi) \in \mathbb{C}^\times$ by $\lambda(\varphi) = \varphi(\log(\rho))/e$. Thus we have a map

$$\lambda\colon \mathrm{Isom}(\mathcal{U}^{\mathrm{B}} \otimes \mathbb{C}, \mathcal{U}^{\mathrm{DR}} \otimes \mathbb{C}) \to \mathbb{C}^\times.$$

(2) We define the set of associator Ass as the inverse image

$$\lambda^{-1}(2\pi i) \subset \mathrm{Isom}(\mathcal{U}^{\mathrm{B}} \otimes \mathbb{C}, \mathcal{U}^{\mathrm{B}} \otimes \mathbb{C})$$

of $2\pi i$ under the map λ.

Remark 3.3. The functorial isomorphism φ is determined by the element $\Phi = \varphi([0,1]) \in \mathcal{U}^{\mathrm{DR}}(\mathcal{M}_{0,4}) = \mathbb{C}\langle\langle e_0, e_1 \rangle\rangle$. The condition for an element Φ to be able to be continued to a functorial isomorphism is nothing but the classical condition for associators (see [Dr]). By this correspondence, the Drinfeld associator corresponds to the classical comparison map. This has been communicated to the author by M. Matsumoto.

Definition 3.4 (Grothendieck–Teichmüller group). (1) Let X be an algebraic variety over $\mathbb{Q}$. The $\mathbb{Q}_l$ completion of the etale fundamental groupoid of a variety $X \otimes \overline{\mathbb{Q}}$ is denoted by $\pi_1^l(X \otimes \overline{\mathbb{Q}})$. The completion of the $\mathbb{Q}_l$-linear hull of $\pi_1^l(X \otimes \overline{\mathbb{Q}})$ is denoted by $\mathcal{U}^l(X)$. If X is one of $\Delta^* = \mathrm{Spec}(\mathbb{Q}[\![x]\!])$, $\mathcal{M}_{0,4}$ or $\mathcal{M}_{0,5}$, we can similarly define similar tangential base points (see [I]). The Hopf algebroid $\mathcal{U}^l(X)$ gives a functor $\mathcal{U}^l\colon (\mathrm{Fund}) \to (\mathrm{Hopf\text{-}alg}/\mathbb{Q}_l)$.

(2) Let $* = \mathrm{B}$, DR or l. The group $\mathrm{GT}_* = \mathrm{Aut}(\mathcal{U}^*)$ of functorial automorphisms of $\mathcal{U}^*$ is called the $*$-Grothendieck–Teichmüller group. The groups GT_{B}, $\mathrm{GT}_{\mathrm{DR}}$ and GT_l are pro-algebraic groups over $\mathbb{Q}$, $\mathbb{Q}$ and $\mathbb{Q}_l$, respectively. For an element of $\varphi \in \mathrm{Aut}(\mathcal{U}^*)$, by attaching $\varphi(\Delta^*) \in \mathrm{Aut}_{\mathrm{Hopf\text{-}alg}}(\mathcal{U}^*(\Delta^*))$, we have a homomorphism of pro-algebraic groups:

$$\lambda\colon\ \mathrm{GT}_* = \mathrm{Aut}(\mathcal{U}^*) \to \boldsymbol{G}_m,$$

where $\boldsymbol{G}_m$ is the multiplicative group. The kernel of $\lambda\colon\ \mathrm{GT}_* \to \boldsymbol{G}_m$ is denoted by $\mathrm{GT}_*^{(1)}$. It is a pro-nilpotent algebraic group.

(3) By the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have a natural homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GT}_l$.

4. Mixed Tate motives and Grothendieck–Teichmüller group

Following Levine, Voevodsky, Hanamura ([L]) we can define an abelian category $\mathrm{MTM}_{\mathbb{Q}}$ of mixed Tate motives over $\mathbb{Q}$. Goncharov ([G]) defined a full subcategory $\mathrm{MTM}_{\mathbb{Z}}$ of mixed Tate motives over $\mathbb{Z}$. By the classical comparison map (1), $\mathcal{U}^{\mathrm{B+DR}}(\mathbb{P}^1 - \{0,1,\infty\})_{\overrightarrow{01},\overrightarrow{10}}$ becomes a mixed Hodge structure of mixed Tate type. This mixed Hodge structure is obtained by the nearby fiber (= the limit of mixed Hodge structure) at $\overrightarrow{01}$, $\overrightarrow{10}$ of the variation of mixed Hodge structure $\mathcal{U}(\mathbb{P}^1 - \{0,1,\infty\})_{ab}$ of two variables a, b. It is natural to expect that the near by fiber of a family of mixed motives is also a mixed motif, which is not well formulated up to now. In fact, for Hodge realization, a limit of a mixed Hodge structure depends on "the tangential structure", i.e. the choice of a branch of "$\log(t)$". But in our setting, we are very happy to have the following.

Theorem 4.1 (Deligne–Goncharov [DG], Terasoma [T]). *There exists an object* $\mathcal{U}^M(\mathbb{P}^1 - \{0,1,\infty\})_{\vec{01},\vec{10}}$ *in* $\mathrm{MTM}_{\mathbb{Z}}$ *whose Hodge realization is isomorphic to* $\mathcal{U}^{\mathrm{B+DR}}(\mathbb{P}^1 - \{0,1,\infty\})_{\vec{01},\vec{10}}$.

We state a consequence of the above theorem in the language of Tannakian category. Let $H^{\mathrm{DR}}\colon \mathrm{MTM}_{\mathbb{Z}} \to (\mathrm{Vec}_{\mathbb{Q}})$ be the de Rham realization functor, let $\pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^{\mathrm{DR}})$ be the Tannaka fundamental group (see [DM]) of $\mathrm{MTM}_{\mathbb{Z}}$ for the fiber functor H^{DR} and let $\lambda\colon \pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^{\mathrm{DR}}) \to \boldsymbol{G}_m$ be the homomorphism obtained by the functor attaching the associated graded module for weight filtration. The kernel of λ is denoted by $\pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^{\mathrm{DR}})^{(1)}$. Let $\mathcal{U}(\pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^{\mathrm{DR}}))$ be the completion of the $\mathbb{Q}$-linear hull of $\pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^{\mathrm{DR}})$. An object in $\mathrm{MTM}_{\mathbb{Z}}$ corresponding to the representation $\mathcal{U}(\pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^{\mathrm{DR}}))$ of $\pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^{\mathrm{DR}})$ is denoted by $\mathcal{U}^M(\pi_1(\mathrm{MTM}_{\mathbb{Z}}))$. By the above theorem and the definition of Tannaka fundamental group, $H^{\mathrm{DR}}\mathcal{U}^M(\mathbb{P}^1 - \{0,1,\infty\})_{\vec{01},\vec{10}}$ becomes a (homogeneous) $\mathcal{U}(\pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^{\mathrm{DR}}))$-module. Therefore the periods of $\mathcal{U}^M(\mathbb{P}^1 - \{0,1,\infty\})$ are controlled by those of $\mathcal{U}^M(\pi_1(\mathrm{MTM}_{\mathbb{Z}}))$. Taking into account the real structures we have the following corollary.

Corollary 4.2 (Conjectured by Zagier, ([G], [T])). *Let* L_n *be the* $\mathbb{Q}$*-vector space generated by multiple zeta values of weight* n*. Then we have* $\dim_{\mathbb{Q}} L_n \le d_n$*, where* d_n *is defined by the generating function* $\sum_{i=0}^{\infty} d_n t^n = \frac{1}{1-t^2-t^3}$.

Either

(1) by comparing $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^l)$ via the homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^l)$ by using Hain–Matsumoto's result, or
(2) by using the infinitesimal embedding $\mathcal{U}^M(\mathcal{M}_{0,4}) \to \mathcal{U}^M(\mathcal{M}_{0,5})$ of mixed Tate motives according to Deligne–Goncharov,

we have a homomorphism of pro-algebraic groups

$$\mathrm{Rep}\colon \pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^{\mathrm{B}}) \to \mathrm{GT}_{\mathrm{B}}$$

which is compatible with the weight homomorphisms λ. The Deligne–Ihara conjecture asserts

Conjecture 4.3 (Deligne–Ihara). The homomorphism Rep is an isomorphism.

Remark 4.4. The injectivity of Rep is equivalent to the following statement:

Any mixed Tate motif over $\mathbb{Z}$ is generated by $\mathcal{U}^M(\mathbb{P}^1 - \{0,1,\infty\})_{\vec{01},\vec{10}}$.

As a consequence the injectivity of Rep implies that periods of any mixed Tate motif over $\mathbb{Z}$ are $\mathbb{Q}$-linear combinations of multiple zeta values.

Note that $\mathcal{U}(\pi_1(\mathrm{MTM}_{\mathbb{Z}}, H^{\mathrm{B}})^{(1)})$ is known to be a free Lie algebra generated by $c_3, c_5, c_7, \ldots$ ([DG]), and the description of GT_{B} is combinatorial. Therefore the above conjecture is (in principle) a combinatorial question. It is answered in the positive up to a quite high degree.

5. Harmonic shuffle relation

In this section we introduce the harmonic shuffle relation according to Hoffman [H] and its regularized version by Ihara–Kaneko–Zagier and Racinet ([IKZ], [R]). The rearrangement of the product of two multiple zeta values expressed by infinite series leads to a $\mathbb{Z}$-linear combination of multiple zeta values: for example, we have

$$\zeta(2)\zeta(2) = \sum_{0<n} \frac{1}{n^2} \cdot \sum_{0<m} \frac{1}{m^2} = \sum_{0<n} \frac{1}{n^4} + 2 \sum_{0<n<m} \frac{1}{n^2 m^2} = \zeta(4) + 2\zeta(2,2).$$

A relation of this type is called harmonic shuffle relation. Using the technique of regularization we also consider a relation of this type for non-convergent sums. As a consequence Racinet and Ihara–Kaneko–Zagier obtained a wider class of relations, the so-called "regularized harmonic shuffle relation". We briefly recall the formulation by Racinet of the regularized harmonic shuffle relation. We have the following approximation of the partial summation

$$\begin{aligned} \zeta_N(k_1, \dots, k_n) &= \sum_{0<m_1<m_2<\cdots<m_n<N} \frac{1}{m_1^{k_1} \dots m_n^{k_n}} \\ &= P_{k_1,\dots,k_n}(\log N + \gamma) + O(N^{-\epsilon}) \end{aligned} \tag{2}$$

by a real coefficient polynomial $P_{k_1,\dots,k_n}(T)$. Using Boutet de Monvel–Zagier's theorem, the generating series of $P_{k_1,\dots,k_n}(T)$ can be computed from the Drinfeld associator. By rearranging the partial summation up to N and using the asymptotic expression (2), the product $P_{k_1,\dots,k_n}(T) P_{k'_1,\dots,k'_{n'}}(T)$ becomes a $\mathbb{Z}$-linear combination of polynomials of the form $P_{k''_1,\dots,k''_{n''}}(T)$. In order to formulate the result it is convenient to introduce a new coproduct, the harmonic coproduct. Let W be a subalgebra $\mathbb{C} \oplus \mathbb{C}\langle\langle e_0, e_1\rangle\rangle e_1$ of $\mathbb{C}\langle\langle e_0, e_1\rangle\rangle$. This algebra is isomorphic to a weighted completion $\mathbb{C}\langle\langle y_1, y_2 \dots\rangle\rangle$ of the free algebra generated by $y_1 = -e_1$, $y_2 = -e_0 e_1, \dots,$ $y_i = -e_0^{i-1} e_1, \dots$. We define a harmonic coproduct $\Delta_*\colon W \to W \otimes W$ by an algebra homomorphism given by

$$\Delta_*(y_n) = \sum_{i=0}^{n} y_i \otimes y_{n-i},$$

with $y_0 = 1$. Let $\Phi_{\mathrm{DR}} = 1 + \varphi_0 e_0 + \varphi_1 e_1$ be the Drinfeld associator and set $\Phi_{\mathrm{DR},Y} = 1 + \varphi_1 e_1 \in W$. Then there is a unique formal power series $\Gamma_{\mathrm{DR}}(s) \in 1 + s^2\mathbb{C}[[s]]$ such that

$$\Phi^{\mathrm{ab}}_{\mathrm{DR},Y}(e_0, e_1) = \frac{\Gamma_{\mathrm{DR}}(-e_0)\Gamma_{\mathrm{DR}}(-e_1)}{\Gamma_{\mathrm{DR}}(-e_0 - e_1)},$$

where $\Phi^{\mathrm{ab}}_{\mathrm{DR},Y}$ is the image of $\Phi_{\mathrm{DR},Y}$ in the abelianization $\mathbb{C}[[e_0, e_1]]$. We define the modified Y-series $\Phi^{\mathrm{mod}}_{\mathrm{DR},Y}$ by $\Phi^{\mathrm{mod}}_{\mathrm{DR},Y} = \Gamma_{\mathrm{DR}}(y_1)^{-1}\Phi_{\mathrm{DR},Y} \in W$.

Theorem 5.1 (Racinet [R], Ihara–Kaneko–Zagier [IKZ] in a different formalism). *With the above notation, we have*

$$\Delta_*(\Phi^{\mathrm{mod}}_{\mathrm{DR},Y}) = \Phi^{\mathrm{mod}}_{\mathrm{DR},Y} \otimes \Phi^{\mathrm{mod}}_{\mathrm{DR},Y}.$$

In other words, $\Phi^{\mathrm{mod}}_{\mathrm{DR},Y}$ is a group-like element under the harmonic coproduct.

6. Fake Hodge realization and harmonic shuffle relation

As is shown in Sections 3 and 5, the origin of the harmonic shuffle relation comes from the rearrangement of partial series, and that of the associator relation comes essentially from the functoriality for infinitesimal inclusions from $\mathcal{M}_{0,4}$ into $\mathcal{M}_{0,5}$. Though two origins are quite different, we can show that the associator relation implies the harmonic shuffle relation. The contents of this section is a result of collaboration with P. Deligne.

For an associator $\Phi = 1+\varphi_0 e_0+\varphi_1 e_1$ set $\Phi_Y = 1+\varphi_1 e_1 = \Phi_Y(y_1, y_2, \dots) \in W$.

Theorem 6.1 (Deligne–Terasoma). (1) *Let Φ^{ab}_Y be the image of Φ_Y in the abelianization $\mathbb{C}[\![e_0, e_1]\!]$. Then there exists a unique element $\Gamma_\Phi(s)$ in $1 + s^2\mathbb{C}[\![s]\!]$ such that*

$$\Phi^{\mathrm{ab}}_Y(e_0, e_1) = \frac{\Gamma_\Phi(-e_0)\Gamma_\Phi(-e_1)}{\Gamma_\Phi(-e_0 - e_1)}.$$

(2) *Using Γ_Φ obtained in* (1) *we define*

$$\Phi^{\mathrm{mod}}_Y(y_1, y_2, \dots) := \Gamma_\Phi(y_1)^{-1}\Phi_Y(y_1, y_2, \dots) \in W.$$

Then we have

$$\Delta_*(\Phi^{\mathrm{mod}}_Y) = \Phi^{\mathrm{mod}}_Y \otimes \Phi^{\mathrm{mod}}_Y.$$

The proof of the above theorem is divided into two parts.

6.1. Fake Hodge realization attached to an associator. By Definition 3.2 an associator defines Hopf algebroid objects $\mathcal{U}(\mathcal{M}_{0,4})$ and $\mathcal{U}(\mathcal{M}_{0,5})$ in the category

$$\begin{aligned}\mathcal{M} &= \mathrm{Vec}_{\mathbb{Q}} \times_{\mathrm{Vec}_{\mathbb{C}}} \mathrm{Vec}_{\mathbb{Q}} \\ &= \{(V^{\mathrm{B}}, V^{\mathrm{DR}}, \varphi) \mid V^{\mathrm{B}} \text{ and } V^{\mathrm{DR}} \text{ are } \mathbb{Q}\text{-vector spaces, } \varphi\colon V^{\mathrm{B}}\otimes\mathbb{C} \xrightarrow{\simeq} V^{\mathrm{DR}}\otimes\mathbb{C} \\ &\qquad \text{is an isomorphism of } \mathbb{C} \text{ vector spaces}\}.\end{aligned}$$

The i-th projection induces a homomorphism $\mathrm{pr}_i\colon \mathcal{U}(\mathcal{M}_{0,5}) \to \mathcal{U}(\mathcal{M}_{0,4})$ of Hopf algebroids in $\mathcal{M}$. For an algebroid $\mathcal{U}$ in $\mathcal{M}$, we can introduce a notion of "$\mathcal{U}$-modules" in the category $\mathcal{M}$. For a $\mathcal{U}(\mathcal{M}_{0,5})$-module M we can define a $\mathcal{U}(\mathcal{M}_{0,4})$-module $R^j\,\mathrm{pr}_i\, M$ by using cohomological technology. Note that an isomorphism

$$R^j\,\mathrm{pr}_i\, M^{\mathrm{B}} \otimes \mathbb{C} \xrightarrow{\simeq} R^j\,\mathrm{pr}_i\, M^{\mathrm{DR}} \otimes \mathbb{C}$$

is encoded in the object $R^j\,\mathrm{pr}_i\, M$, which can be computed from the associator chosen first. This isomorphism is called a fake comparison map associated to an associator. We can also define the notion of "perverse" sheaf, and so on. These modules are equipped with a mixed Hodge structure which is different from the natural one. They are called a fake Hodge realization.

6.2. Multiplicative convolution. Following Deligne we introduce a geometric interpretation of harmonic shuffle relation. Let $\mathcal{A}$ be the category of topological perverse sheaves on $\overline{\mathcal{M}_{0,4}} - \{0, \infty\} \simeq \boldsymbol{G}_m$ smooth outside of $\{1\}$ with a nilpotent monodromy. We introduce an equivalence relation on $\mathcal{A}$ as follows. A morphism $\mathcal{F}_1 \to \mathcal{F}_2$ in $\mathcal{A}$ is equivalent if the induced homomorphism of vanishing cycles is an isomorphism. The quotient category under this equivalence relation is denoted by $\bar{\mathcal{A}}$. Then $\bar{\mathcal{A}}$ is again an abelian category and we can show that this abelian category is equivalent to the category of W^{B} modules, where

$$W^{\mathrm{B}} = \mathcal{U}^{\mathrm{B}}_{\overrightarrow{10},\overrightarrow{10}} \log \rho_1 \oplus \mathbb{Q},$$

ρ_1 being the canonical generator of local monodromy around $\{1\}$. For two perverse sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$, we can define a biadditive functor $*\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by the multiplicative convolution

$$\mathcal{F}_1 * \mathcal{F}_2 = {}^p\mathcal{H}^1 \mathbb{R}\,\mathrm{pr}_{5*}(\mathrm{pr}_1^*(\mathcal{F}_1) \otimes \mathrm{pr}_2^*(\mathcal{F}_2)).$$

The equivalence class of the above convolution $\mathcal{F}_1 * \mathcal{F}_2$ depends only on the equivalence class of $\mathcal{F}_1$ and $\mathcal{F}_2$ and, as a consequence, the convolution induces a biadditive functor $*\colon \bar{\mathcal{A}} \times \bar{\mathcal{A}} \to \bar{\mathcal{A}}$. The corresponding ring homomorphism $W^{\mathrm{B}} \to W^{\mathrm{B}} \otimes W^{\mathrm{B}}$ is denoted by Δ_*. This picture can be filled into the $\mathcal{M} = \mathrm{Vec}_{\mathbb{Q}} \times_{\mathrm{Vec}_{\mathbb{C}}} \mathrm{Vec}_{\mathbb{Q}}$ world with a comparison isomorphism attached to an associator Φ. By the computation of the cohomology of algebroids we can show that the de Rham realization $\Delta_*\colon W^{\mathrm{DR}} \to W^{\mathrm{DR}} \otimes W^{\mathrm{DR}}$ is equal to the harmonic coproduct. By using the fake comparison isomorphism we have the theorem.

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