Bilateral Laplace Transforms on Time Scales: Convergence, Convolution, and the Characterization of Stationary Stochastic Time Series

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***Abstract*—The convergence of Laplace transforms on time scales is generalized to the bilateral case. The bilateral Laplace transform of a signal on a time scale subsumes the continuous time bilateral Laplace transform, and the discrete time bilateral z-transform as special cases. As in the unilateral case, the regions of convergence (ROCs) time scale Laplace transforms are determined by the time scale's graininess. ROCs for the bilateral Laplace transforms of double sided time scale exponentials are defined by two modified Hilger circles. The ROC is the intersection of points external to modified Hilger circle determined by behavior for positive time and the points internal to the second modified Hilger circle determined by negative time. Since graininess lies between zero and infinity, there can exist conservative ROCs applicable for all time scales. For continuous time ($\mathbb{R}$) bilateral transforms, the circle radii become infinite and results in the familiar ROC between two lines parallel to the imaginary z axis. Likewise, on $\mathbb{Z}$, the ROC is an annulus. For signals on time scales bounded by double sided exponentials, the ROCs are at least that of the double sided exponential. The Laplace transform is used to define the *boxminus* shift through which time scale convolution can be defined. Generalizations of familiar properties of signals on $\mathbb{R}$ and $\mathbb{Z}$ include identification of the identity convolution operator, the derivative theorem, and characterizations of wide sense stationary stochastic processes for an arbitrary time scales including autocorrelation and power spectral density expressions.**

Keywords time scales, Laplace transform, z-transforms, region of convergence, Hilger circle, stationarity, autocorrelation, power spectral density, Hilger delta

I. Introduction

A time scale, $\mathbb{T}$, is any collection of closed intervals on the real line. Continuous time, $\mathbb{R}$, and discrete time $\mathbb{Z}$, are special cases. The calculus of time scales was introduced by Hilger [6]. Time scales have found utility in describing the behavior of dynamic systems [1], [7] and have been applied to control theory [2], [3], [5].

This is the second in a series of monographs outlining regions of convergence and applications of Laplace transforms on time scales. The first paper was dedicated to the causal (or one sided) Laplace transform on a time scale [12]. This paper extends these results to the bilateral Laplace transform on a time scale and its use in defining convolution on an arbitrary time scale. Time scale convolution, in turn, allows modeling of wide sense stationary stochastic processes on time scales using autocorrelation and power spectral density descriptors.

For the convergence problem, there are three cases of bilateral time scales considered.

1) For time scales whose graininess is bounded from above and below over the entire time scale or asymptotically,
2) For time scales whose asymptotic graininess approaches a constant. $\mathbb{R}$ and $\mathbb{Z}$ are special cases. All time scales in this class are also asymptotically a member of the time scales in 1).
3) For all time scales. This can be considered a limiting special case of 1) since all time scales are bounded between zero and infinity.

II. Time Scales

Our introduction to time scales is limited to that needed to establish notation. A more detailed explanation is in our first paper [12] and a complete rigorous treatment is in the text by Bohner and Peterson [1].

1) A *time scale*, $\mathbb{T}$, is any collection of closed intervals on the real line. We will assume the origin is always a component of the time scale.
2) The *graininess* of a time scale at time $t \in \mathbb{T}$ is defined by
$$\mu(t) = \left(\inf_{\tau > t, \tau \in \mathbb{T}} \tau \right) - t.$$
3) The *Hilger derivative* of an image $x(t)$ at $t \in \mathbb{T}$ is
$$x^\Delta(t) := \frac{x(t^\sigma) - x(t)}{\mu(t)}$$
where $t^\sigma := t + \mu(t)$. When $\mu(t) = 0$, the Hilger derivative is interpreted in the limiting sense and
$$x^\Delta(t) = \frac{d}{dt} x(t).$$
4) If $y(t) = x^\Delta(t)$, then the *definite time scale integral* is
$$\int_a^b y(t) \Delta t = x(b) - x(a).$$
5) When $x(0) = 1$, the solution to the *Hilger differential equation*,
$$x^\Delta(t) = z x(t),$$

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is $x(t) = e_z(t)$ where the *generalized exponential* is

$$e_z(t) := \exp\left(\int_{\tau=0}^{t} \frac{\ln(1+z\mu(\tau))}{\mu(\tau)}\Delta\tau\right).$$

As a consequence, for $z = 0$,

$$e_0(t) = 1. \qquad \text{(II.1)}$$

6) The *circle minus* operator is defined by

$$y \ominus z := \frac{y-z}{1+z\mu(t)}.$$

The notation $\ominus z$ in interpreted as $y \ominus z$ with $y = 0$.

7) The generalized exponential has the property that [1]

$$e_{\ominus z}(t) = \frac{1}{e_z(t)}.$$

III. Bilateral Laplace

Let $\mathbb{T}$ denote a bilateral time scale and let $f(t)$ be an image on $\mathbb{T}$. Define the bilateral Laplace transform as

$$F(z) := \int_{-\infty}^{\infty} f(t)e^{\sigma}_{\ominus z}(t)\Delta t \qquad \text{(III.1)}$$

where $e^{\sigma}_{\ominus z}(t) := e_{\ominus z}(t^{\sigma})$. The continuous time bilateral Laplace and discrete time z transforms are special cases.

Here are some properties.

1) *Integration property.*

$$F(0) = \int_{-\infty}^{\infty} f(t)\Delta t.$$

This follows immediately from (III.1) and (II.1).

2) *The derivative theorem.* When $f(t)e_{\ominus z}(t)$ goes to zero as $t \to \pm\infty$ and $F(z)$ converges,

$$f^{\Delta}(t) \quad \to zF(z). \qquad \text{(III.2)}$$

Proof:

$$f^{\Delta}(t) \quad \to \int_{-\infty}^{\infty} f^{\Delta}(t)e^{\sigma}_{\ominus z}(t)\Delta t.$$

Using integration by parts [1]

$$\begin{aligned} f^{\Delta}(t) \quad &\to \quad f(t)e_{\ominus z}(t)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\left[e_{\ominus z}(t)\right]^{\Delta}\Delta t \\ &= \quad -\int_{-\infty}^{\infty} f(t)(\ominus z)e_{\ominus z}(t)\Delta t \end{aligned}$$

Since $-(\ominus z)e_{\ominus z}(t) = ze^{\sigma}_{\ominus z}(t)$, the result follows immediately. ■

3) *Special cases.*

- For continuous time, $\mathbb{T} = \mathbb{R}$, we have

$$e^{\sigma}_{\ominus z}(t) = e^{-zt}$$

and (III.1) becomes the conventional bilateral Laplace transform

$$F(z) = \int_{-\infty}^{\infty} f(t)e^{-zt}dt.$$

- For discrete time, $\mathbb{T} = \mathbb{Z}$, we have

$$e^{\sigma}_{\ominus z}(t_n) = (1+z)^{-(n+1)}$$

and (III.1) becomes

$$F(z) = \sum_{n=-\infty}^{\infty} f(n)(1+z)^{-(n+1)}.$$

The bilateral z transform is

$$F_Z(z) = \sum_{n=-\infty}^{\infty} f(n)z^{-n}. \qquad \text{(III.3)}$$

Thus

$$F(z) = \frac{F_Z(z+1)}{z+1}$$

or, equivalently

$$F_Z(z) = zF(z-1).$$

Note, then, that the conventional z transform is related to a shifted time scale Laplace transform for $\mathbb{T} = \mathbb{Z}$. For example, the unit circle for the z transform is centered about the origin. The time scale version is the same circle now shifted to be centered at $z = -1$.

4) *Variation.* An alternate form of the bilateral transform which will prove useful in the characterization of stationary stochastic processes on a time scale is

$$F_{\ominus}(z) := \int_{-\infty}^{\infty} f(t)e^{\sigma}_{z}(t)\Delta t. \qquad \text{(III.4)}$$

Although the analysis of convergence in this paper is for $F(z)$, the convergence properties for $F_{\ominus}(z)$ are similar and follow immediately.

We can break up the transform definition in (III.1) as

$$F(z) = F_+(z) + F_-(z) \qquad \text{(III.5)}$$

where

$$F_+(z) = \int_{0}^{\infty} f(t)e^{\sigma}_{\ominus z}(t)\Delta t$$

and

$$F_-(z) = \int_{-\infty}^{0} f(t)e^{\sigma}_{\ominus z}(t)\Delta t$$

We recognize that

$$F_+(z) = F_u(z)$$

where $F_u(z)$ is the notation for the causal Laplace transform on a time scale [12]. $F_u(z)$ is alternately referred to as the *unilateral* or *one sided* Laplace transform. The regions of convergence for the causal Laplace transform has been established for causal functions with finite area and for transcendental functions arising from solution of linear time invariant differential equations on time scales [12].

A. Asymptotic Graininess Bounds

Here are some graininess bounds useful in determining the convergence of bilateral transforms. All upper bounds are bounded by infinity and all lower bounds must equal or exceed zero.

(1) *Constant Asymptotic Graininess.* The graininess of some time scales asymptotically approach a constant at $t =$

$\pm\infty$. In such cases, we define the constant asymptotic graininesses as

$$\bar{\mu}_+ = \lim_{t\to\infty} \mu(t) \text{ and } \bar{\mu}_- = \lim_{t\to-\infty} \mu(t)$$

More rigorously, $\bar{\mu}_+$ is the positive constant asymptotic graininess if

$$\lim_{t\to\infty} |\mu(t) - \bar{\mu}_+| = 0.$$

Likewise, the negative constant asymptotic graininess

$$\lim_{t\to-\infty} |\mu(t) - \bar{\mu}_-| = 0.$$

(2) *Bounds.* Graininess on a time scale is asymptotically be bounded from above and below.

- Entire Bounds.
  - The positive upper and lower entire bounds for graininess are

$$\acute{\mu}_+^0 = \sup_{t\in\mathbb{T},\ t\geq 0} \mu(t)$$

and

$$\grave{\mu}_+^0 = \inf_{t\in\mathbb{T},\ t\geq 0} \mu(t).$$

  - The negative upper and lower bounds for graininess are

$$\acute{\mu}_-^0 = \sup_{t\in\mathbb{T},\ t<0} \mu(t)$$

and

$$\grave{\mu}_-^0 = \inf_{t\in\mathbb{T},\ t<0} \mu(t).$$

- Attained Bounds.
  - The positive upper and lower attained bounds for graininess are

$$\acute{\mu}_+^T = \sup_{t\in\mathbb{T},\ t\geq T} \mu(t)$$

and

$$\grave{\mu}_+^T = \inf_{t\in\mathbb{T},\ t\geq T} \mu(t).$$

  - The negative upper and lower attained bounds for graininess are

$$\acute{\mu}_-^T = \sup_{t\in\mathbb{T},\ t<T} \mu(t)$$

and

$$\grave{\mu}_-^T = \inf_{t\in\mathbb{T},\ t<T} \mu(t).$$

- Asymptotic Bounds.
  - The positive upper and lower asymptotic bounds for graininess are

$$\begin{aligned}\acute{\mu}_+^\infty &= \lim_{T\to\infty} \acute{\mu}_+^T \\ &= \lim_{T\to\infty} \left(\sup_{t\in\mathbb{T},\ t\geq T} \mu(t) \right)\end{aligned}$$

and

$$\begin{aligned}\grave{\mu}_+^\infty &= \lim_{T\to\infty} \grave{\mu}_+^T \\ &= \lim_{T\to\infty} \left(\inf_{t\in\mathbb{T},\ t\geq T} \mu(t) \right).\end{aligned}$$

  - The negative upper and lower asymptotic bounds for graininess are

$$\begin{aligned}\acute{\mu}_-^\infty &= \lim_{T\to-\infty} \acute{\mu}_-^T \\ &= \lim_{T\to-\infty} \left(\sup_{t\in\mathbb{T},\ t\leq T} \mu(t) \right)\end{aligned}$$

and

$$\begin{aligned}\grave{\mu}_-^\infty &= \lim_{T\to-\infty} \grave{\mu}_-^T \\ &= \lim_{T\to-\infty} \left(\inf_{t\in\mathbb{T},\ t\leq T} \mu(t). \right) \end{aligned} \quad \text{(III.6)}$$

When the positive or negative asymptotic bounds are equal, a time scale has a constant positive and/or negative constant asymptotic graininess.

(3) *Global Asymptotic Bounds.* Note that, since $\grave{\mu}_+$ and $\grave{\mu}_-$ are nonnegative, and $\acute{\mu}_+ < \infty$ and $\acute{\mu}_- < \infty$, we can always set global upper and lower bounds for both the positive and negative cases as ∞ and 0.

B. *Regions of convergence on the z plane.*

Define a *modified Hilger circle* parameterized by a (possibly complex) number α and a graininess μ, as the locus of points for which

$$\left| z + \frac{1}{\mu} \right| = \left| \alpha + \frac{1}{\mu} \right|.$$

The circle is centered at $-1/\mu$ on the negative real axis on the z plane and passes through the point α [12]. From this definition we offer the following definitions of regions on the z plane.

(1) The region $\mathcal{H}(\alpha,\mu)$ contains all points outside of the modified Hilger circle with parameters α and μ. $\bar{\mathcal{H}}(\alpha,\mu)$ is the region inside of the same circle.

(2) The regions $\mathcal{R}(\alpha,\grave{\mu},\acute{\mu})$ and $\mathcal{L}(\alpha,\grave{\mu},\acute{\mu})$, illustrated in Figure 1, are defined as the intersections

$$\mathcal{R}(\alpha,\grave{\mu},\acute{\mu}) = \mathcal{H}(\alpha,\acute{\mu}) \cap \mathcal{H}(\alpha,\grave{\mu}) \quad \text{(III.7)}$$

and

$$\mathcal{L}(\alpha,\grave{\mu},\acute{\mu}) = \bar{\mathcal{H}}(\alpha,\acute{\mu}) \cap \bar{\mathcal{H}}(\alpha,\grave{\mu}). \quad \text{(III.8)}$$

These regions have the obvious properties

$$\mathcal{R}(\alpha,\mu,\mu) = \mathcal{H}(\alpha,\mu) \quad \text{(III.9)}$$

and

$$\mathcal{L}(\alpha,\mu,\mu) = \bar{\mathcal{H}}(\alpha,\mu). \quad \text{(III.10)}$$

(3) The regions $\mathcal{D}(\alpha)$ and $\mathcal{E}(\alpha)$ are the limiting cases of $\mathcal{R}$ and $\mathcal{L}$. They are illustrated in Figure 2.

- Define

$$\mathcal{D}(\alpha) = \mathcal{R}(\alpha,0,\infty).$$

  $\mathcal{D}(\alpha)$ consists of the intersection of all points external to a circle of radius $|\alpha|$ centered on the z plane and all of the points to the right of the line $z = \operatorname{Re}\alpha$.

- Likewise, let

$$\mathcal{E}(\alpha) = \mathcal{L}(\alpha,0,\infty) \quad \text{(III.11)}$$

These regions have the property that [12]

$$\mathcal{D}(\alpha) \subset \mathcal{R}(\alpha, \grave{\mu}, \acute{\mu}).$$

and

$$\mathcal{E}(\alpha) \subset \mathcal{L}(\alpha, \grave{\mu}, \acute{\mu}).$$

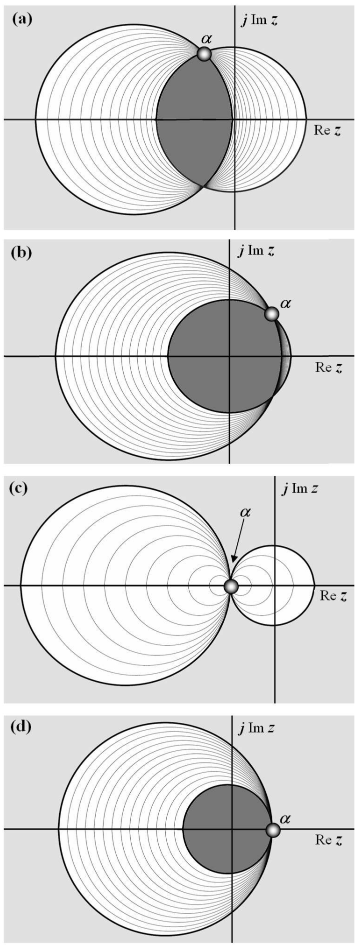

Fig. 1. Illustration of the regions $\mathcal{R}(\alpha, \grave{\mu}, \acute{\mu})$ and $\mathcal{L}(\alpha, \grave{\mu}, \acute{\mu})$. In each case, the leftmost bold circle is centered on the negative real axis at $-1/\grave{\mu}$ and the rightmost bold circle is centered on the negative real axis at $-1/\acute{\mu}$. Both circles pass through the point α. The lightly shaded area, external to all of the circles, are $\mathcal{R}(\alpha, \grave{\mu}, \acute{\mu})$. The darker shaded region, internal to the circles, are $\mathcal{L}(\alpha, \grave{\mu}, \acute{\mu})$. The examples here illustrate the regions for different values of α. (a) $\mathrm{Re}\,\alpha < 0$. (b) $\mathrm{Re}\,\alpha > 0$. (c) α is negative and real. Here, the region $\mathcal{L}$ is empty. (d) α real and positive.

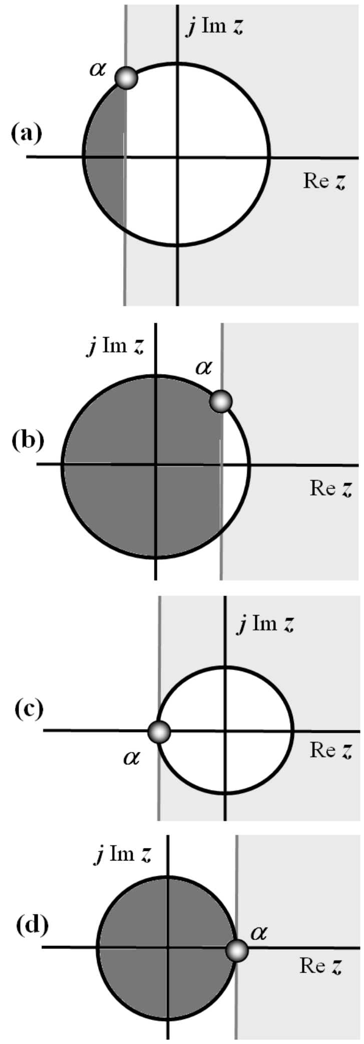

Fig. 2. Illustration of the regions $\mathcal{D}(\alpha)$ and $\mathcal{E}(\alpha)$. In each case, the bold circle is centered at the origin and passes through α. The vertical line is defined by the line $z = \mathrm{Re}\,\alpha$. The lightly shaded area are $\mathcal{D}(\alpha)$. The darker shaded region are $\mathcal{E}(\alpha)$. The examples here illustrate the regions for different values of α. (a) $\mathrm{Re}\,\alpha < 0$. (b) $\mathrm{Re}\,\alpha > 0$. (c) α is negative and real. Here, the region $\mathcal{L}$ is empty. (d) α real and positive.

IV. A Bilateral Laplace Transform

Clearly, if $F_+(u)$ converges in a region $\mathcal{Z}_+$ and $F_-(u)$ converges in a region $\mathcal{Z}_-$, then $F(u)$ converges at least in the region

$$\mathcal{Z} = \mathcal{Z}_+ \cap \mathcal{Z}_-. \tag{IV.1}$$

We can use this to find the region of convergence (ROC) of the double exponential function defined by

$$e_{\beta:\alpha}(t) := \begin{cases} e_\alpha(t) & ; \quad t \geq 0 \\ e_\beta(t) & ; \quad t \leq 0 \end{cases} \tag{IV.2}$$

We will find useful the following shorthand notation

- $\mathbb{T}(\bar{\mu}_+)$ means $\mathbb{T}$ has a constant positive asymptotic graininess of $\bar{\mu}_+$.
- $\mathbb{T}(\bar{\mu}_-)$ means $\mathbb{T}$ has a constant negative asymptotic graininess of $\bar{\mu}_-$.
- $\mathbb{T}(\grave{\mu}_+, \acute{\mu}_+)$ means $\mathbb{T}$ has lower and upper positive t graininess bounds of $\grave{\mu}_+$ and $\acute{\mu}_+$. These can be
  - $(\grave{\mu}_+^0, \acute{\mu}_+^0)$,
  - $(\grave{\mu}_+^T, \acute{\mu}_+^T)$, or
  - $(\grave{\mu}_+^\infty, \acute{\mu}_+^\infty)$.
- $\mathbb{T}(\grave{\mu}_-, \acute{\mu}_-)$ means $\mathbb{T}$ has lower and upper negative t graininess bounds of $\grave{\mu}_-$ and $\acute{\mu}_-$. These can be
  - $(\grave{\mu}_-^0, \acute{\mu}_-^0)$,
  - $(\grave{\mu}_-^T, \acute{\mu}_-^T)$, or
  - $(\grave{\mu}_-^\infty, \acute{\mu}_-^\infty)$.
- $\mathbb{T}(0_+, \infty_+)$ means $\mathbb{T}$ has lower and upper positive t graininess bounds of 0 and ∞.
- $\mathbb{T}(0_-, \infty_-)$ means $\mathbb{T}$ has lower and upper negative t

graininess bounds of 0 and ∞.

Theorem IV.1. *The bilateral Laplace transform of the double exponential function, $e_{\beta:\alpha}(t)$, when it exists, is given by $F(z) = F_+(z) + F_-(z);\ z \in \mathcal{Z}$ where*

$$F_+(z) = \frac{1}{z-\alpha};\ z \in \mathcal{Z}_+, \tag{IV.3}$$

$$F_-(z) = -\frac{1}{z-\beta};\ z \in \mathcal{Z}_- \tag{IV.4}$$

and $\mathcal{Z} = \mathcal{Z}_+ \cap \mathcal{Z}_-$. The component ROC's are

$$\mathcal{Z}_+ = \begin{cases} \mathcal{H}(\alpha, \bar{\mu}_+) & \textit{for} \quad \mathbb{T}(\bar{\mu}_+) \\ \mathcal{R}(\alpha, \grave{\mu}_+, \acute{\mu}_+) & \textit{for} \quad \mathbb{T}(\grave{\mu}_+, \acute{\mu}_+) \\ \mathcal{D}(\alpha) & \textit{for} \quad \mathbb{T}(0_+, \infty_+) \end{cases} \tag{IV.5}$$

and

$$\mathcal{Z}_- = \begin{cases} \bar{\mathcal{H}}(\beta, \bar{\mu}_-) & \textit{for} \quad \mathbb{T}(\bar{\mu}_-) \\ \mathcal{L}(\beta, \grave{\mu}_-, \acute{\mu}_-) & \textit{for} \quad \mathbb{T}(\grave{\mu}_-, \acute{\mu}_-) \\ \mathcal{E}(\beta) & \textit{for} \quad \mathbb{T}(0_-, \infty_-) \end{cases} \tag{IV.6}$$

A. Examples

1) Zero Asymptotic Graininess: Let $\bar{\mu}_+ = \bar{\mu}_- = 0$. Continuous time, $\mathbb{T} = \mathbb{R}$, is a special case as is the log time scale

$$\mathbb{L} = \{t_n | t_n = \text{sgn}(n)\log(|n|+1); -\infty < n < \infty\}.$$

For positive t, the region of convergence is

$$\mathcal{Z}_+ = \mathcal{H}(\alpha, 0).$$

The corresponding Hilger circle has infinite radius and $\mathcal{H}(\alpha, 0)$ is recognized as the set of all points to the right of the line $z = \text{Re}\,\alpha$. Likewise, for negative time,

$$\mathcal{Z}_- = \bar{\mathcal{H}}(\beta, 0).$$

contains all of the points to the left of the line $z = \text{Re}\,\beta$. The resultant slab of convergence, $\mathcal{Z} = \mathcal{Z}_+ \cap \mathcal{Z}_-$, is the familiar region of convergence in the bilateral Laplace transform. It is illustrated on the left of Figure 3,

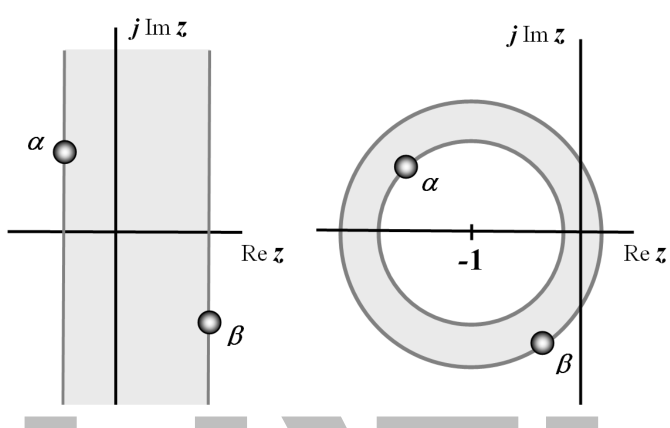

Fig. 3. Left: ROCs for time scales with zero asymptotic graininess for both positive and negative time. Continuous time, $\mathbb{R}$, is a special case. Right: ROCs for time scales with unit asymptotic graininess for both positive and negative time. Discrete time, $\mathbb{Z}$, is a special case.

2) Unit Asymptotic Graininess: Let $\bar{\mu}_+ = \bar{\mu}_- = 1$. Discrete time, $\mathbb{T} = \mathbb{Z}$, is a special case as is the time scale

$$\mathbb{S} = \left\{ t_n \middle| t_n = \text{sgn}(n)(|n| + \sqrt{|n|}); -\infty < n < \infty \right\}.$$

It follows that

$$\mathcal{Z}_+ = \mathcal{H}(\alpha, 1)$$

and

$$\mathcal{Z}_- = \bar{\mathcal{H}}(\beta, 1).$$

The intersection, an annulus region of convergence illustrated on the right of Figure 3, is familiar in bilateral z transforms. Instead of being centered at the origin, however, the circles are centered at $z = -1$. This anomaly is an artifact of inclusion of the bilateral z transform as a special case of the bilateral Laplace transform on time scales. The conventional z transform can be generated from $F_b(u)$ using (III.3).

3) Periodic Graininess: A discrete time scale, $\mathbb{D}$, is said to have *periodic graininess*[1] is there is a period N where $\mu(t_n) = \mu(t_{n+N})$ for all values of n.

(a) *The exponential for time scales with periodic graininess.* In general, for a discrete time scale [12]

$$e_z(t_n) = \begin{cases} \prod_{k=0}^{n-1}(1+\mu(t_k)z) & ; \quad n > 0 \\ 1 & ; \quad n = 0 \\ \prod_{k=n}^{-1}(1+\mu(t_k)z)^{-1} & ; \quad n < 0 \end{cases} \tag{IV.7}$$

We can break the products into periods. For $t > 0$, assume there are

$$P = \left\lfloor \frac{n}{N} \right\rfloor$$

replications of the graininess. Then

$$\begin{aligned} \prod_{k=0}^{n-1} &= \prod_{k=0}^{N-1} \times \prod_{k=N}^{2N-1} \times \prod_{k=2N}^{3N-1} \times \cdots \\ &\quad \times \prod_{k=pN}^{(p+1)N-1} \times \cdots \times \prod_{k=(P-1)N}^{PN-1} \times \prod_{k=PN}^{n-1} \\ &= \left[\prod_{p=0}^{P-1} \left(\prod_{k=pN}^{(p+1)N-1} \right) \right] \times \prod_{k=PN}^{n-1} \\ &= \left[\prod_{p=0}^{P-1} \prod_{q=0}^{N-1} \right) \right] \times \prod_{k=0}^{n-PN-1} \end{aligned}$$

where, in the second product we let $q = k - Np$ and in the third product $q = k - NP$. Imposing the periodicity of the graininess, we conclude that, for $t > 0$,

$$\begin{aligned} e_z(t_n) &= \left[\prod_{p=0}^{P-1} \prod_{q=0}^{N-1}\left(1+\mu(q+pN)\right)\right] \\ &\quad \times \prod_{q=0}^{n-PN-1} (1+\mu(kq+PN)) \\ &= \left[\prod_{p=0}^{P-1} \prod_{q=0}^{N-1}\left(1+\mu(q)\right)\right] \times \prod_{q=0}^{n-PN-1} (1+\mu(kq) \\ &= \prod_{q=0}^{N-1} (1+\mu(q))^P \times \prod_{k=0}^{n-PN-1} (1+\mu(kq) \\ &= \prod_{q=0}^{n-PN-1} (1+\mu(q))^{P+1} \\ &\quad \times \prod_{q=n-PN}^{N-1} (1+\mu(q))^P \qquad \text{(IV.8)} \end{aligned}$$

(b) *ROCs for bilateral Laplace transforms of signals with periodic time scales.* For periodic time scales,

$$\acute{\mu}_+^0 = \acute{\mu}_+^T = \acute{\mu}_+^\infty = \acute{\mu}_-^0 = \acute{\mu}_-^T = \acute{\mu}_-^\infty.$$

We will collectively refer to all of these upper bounds as

$$\acute{\mu} = \max_{m=0}^{N-1} \mu(t_n).$$

Likewise, the lower bound

$$\grave{\mu}_+^0 = \grave{\mu}_+^T = \grave{\mu}_+^\infty = \grave{\mu}_-^0 = \grave{\mu}_-^T = \grave{\mu}_-^\infty.$$

will be referred to collectively as

$$\grave{\mu} = \min_{n=0}^{N-1} \mu(t_n).$$

The ROC for the two sided exponential, $e_{\beta:\alpha}(t)$, follows as

$$\begin{aligned} \mathcal{Z} &= \mathcal{Z}_+ \cap \mathcal{Z}_- \\ &= \mathcal{R}(\alpha, \grave{\mu}, \acute{\mu}) \cap \mathcal{L}(\beta, \grave{\mu}, \acute{\mu}) \end{aligned}$$

Examples of this ROC are shown in Figures 4 through 7. In each of these figures,

- the locations of α and β are labelled and depicted by dots white in the middle fading to black at the dot's edges,
- the center of the modified Hilger circles are shown with a hollow dot, $\circ$, corresponding to the value $-1/\grave{\mu}$ on the negative real axis of the z plane, and a solid dot, $\bullet$, is at $-1/\acute{\mu}$ and is also on the negative real axis of the z plane[2],
- the ROC $\mathcal{Z}$, if not empty, is blackened,
- the region $\mathcal{Z}_+ = \mathcal{R}(\alpha, \grave{\mu}, \acute{\mu})$ not in $\mathcal{Z}$ is shown lightly shaded, and
- the region $\mathcal{Z}_- = \mathcal{L}(\beta, \grave{\mu}, \acute{\mu})$ not in $\mathcal{Z}$ is shown more darkly shaded.

Figures 4 and 5 illustrate scenarios where α and β are real. ROCs for complex α and β are shown in Figures 6 and 7. In all cases, the blackened area, $\mathcal{Z}$, is equal to the intersection of (1) $\mathcal{Z}_+$ corresponding to the area outside the union of the circles passing through α with (2) $\mathcal{Z}_-$ which is the area inside the intersection of the β circles.

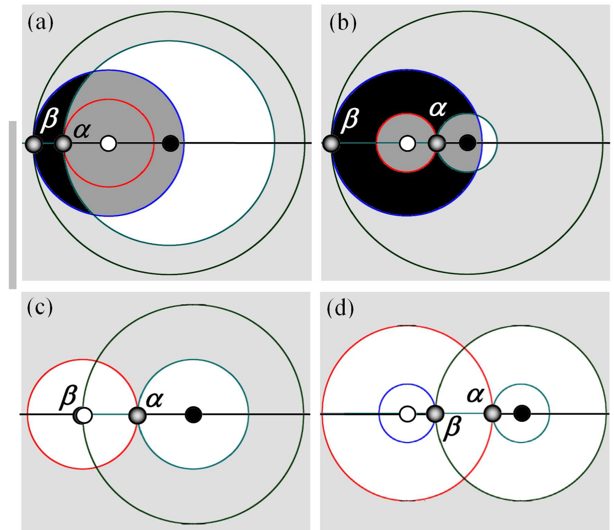

Fig. 4. Illustration of the regions of convergence for time scales with periodic graininess for the double sided exponential $e_{\beta:\alpha}(t)$ in (IV.2) when α and β are real and $\beta < \alpha$. The regions of convergence are shown blackened in (a) and (b). As value of β starting from (b) increases with all other values fixed, the smaller circle passing through β eventually becomes subsumed in the leftmost circle passing through α. When this happens, the $\mathcal{Z}$ ROC is empty. This is shown in (c) where, since $\beta = -1/\grave{\mu}$, the smaller circle passing through β has shrunk to zero. As β and α move to the right with the circle centers fixed, the ROC $\mathcal{Z}$ remains empty.

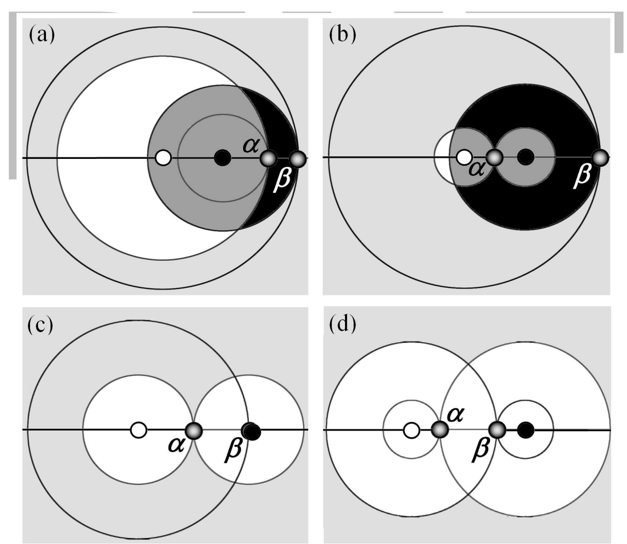

Fig. 5. This is the same case treated in Figure 4 except $\beta > \alpha$. Each illustration is the mirror image of that in Figure 4.

4) Zero Lower Bound and No Upper Bound: An example is shown in Figure 8 using a time scale with no upper graininess bound and a lower graininess bounds of zero.

B. *Proof of Theorem IV.1*

Using the decomposition in (III.5),

$$F_+(z) = \int_0^\infty e_\alpha(t) e_{\ominus z}^\sigma(t) \Delta t$$

This integral is equivalent to the causal time scale Laplace transform of $e_\alpha(t)$ and has been derived elsewhere [12].

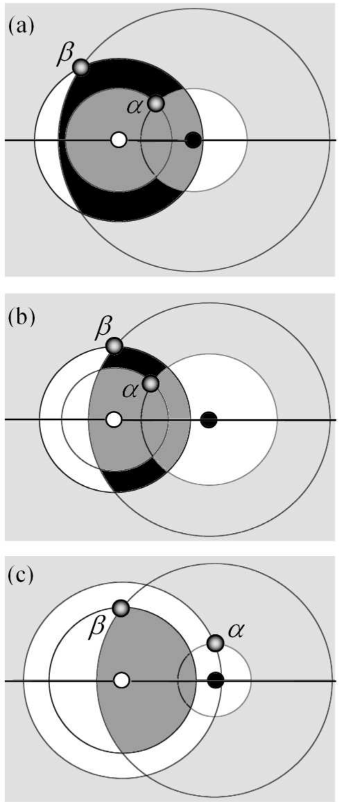

Fig. 6. An example of the ROC, $\mathcal{Z}$, for the double exponential , $e_{\beta:\alpha}(t)$, for $\operatorname{Re}\beta < \operatorname{Re}\alpha$ and $\operatorname{Im}\beta > \operatorname{Im}\alpha > 0$. As the leftmost α circle becomes larger than the leftmost β circle, there is no intersection and $\mathcal{Z}$ is empty. This is illustrated in (c). Note that if α and β are interchanged in (a), (b), or (c), the region $\mathcal{Z}$ will be empty.

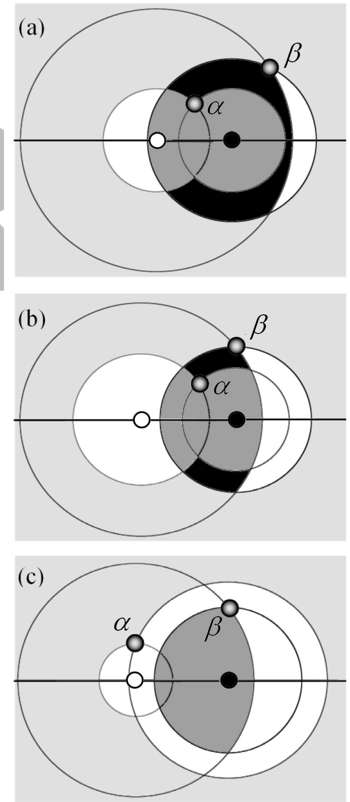

Fig. 7. The ROC's here are the mirror images of those in Figure 6. The ROCs, $\mathcal{Z}$, are for the double exponential $e_{\beta:\alpha}(t)$. Like Figure 6, $\operatorname{Im}\beta > \operatorname{Im}\alpha > 0$. However, in the examples shown here, $\operatorname{Re}\beta < \operatorname{Re}\alpha$. For (c), we see that $\mathcal{Z}$ is empty. Note that if α and β are interchanged in (a), (b), or (c), the region $\mathcal{Z}$ will be empty.

The other component of the Laplace transform is similar.

$$\begin{aligned} F_-(z) &= \int_{-\infty}^{0} e_\beta(t) e^{\sigma}_{\ominus z}(t)\Delta t \\ &= \int_{-\infty}^{0} \frac{1}{1+\mu(t)z} e_\beta(t) e_{\ominus z}(t)\Delta t \\ &= \int_{-\infty}^{0} \frac{1}{1+\mu(t)z} e_{\beta\ominus z}(t)\Delta t \end{aligned} \tag{IV.9}$$

Motivated by the $\ominus$ operator definition, we continue

$$\begin{aligned} F_-(z) &= \frac{1}{\beta - z}\int_{-\infty}^{0} \frac{\beta - z}{1+\mu(t)z} e_{\beta\ominus z}(t)\Delta t \\ &= \frac{1}{\beta - z}\int_{-\infty}^{0} (\beta \ominus z) e_{\beta\ominus z}(t)\Delta t \\ &= \frac{1}{z-\beta}\, e_{\beta\ominus z}(t)\Big|_{-\infty}^{0} \qquad \text{(IV.10)} \\ &= \frac{1}{\beta - z} \qquad \text{(IV.11)} \end{aligned}$$

The step between (IV.10) and (IV.11) is valid if

$$e_{\beta\ominus z}(-\infty) = 0. \tag{IV.12}$$

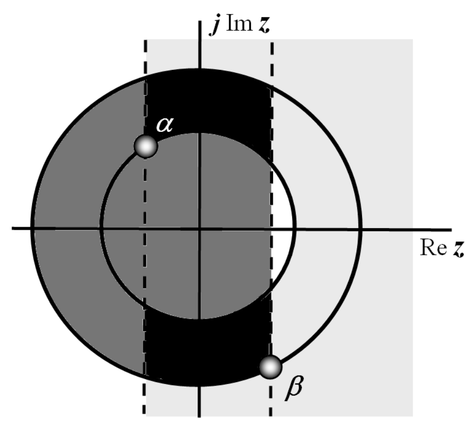

Fig. 8. For the values of α and β shown, the double sided exponential, $e_{\beta:\alpha}(t)$, in Figure IV.2 converges in the ROC, $\mathcal{Z}$, shown shaded black. The region $\mathcal{Z}_+$ is shaded lightly and $\mathcal{Z}_-$ is more darkly shaded. Their intersection $\mathcal{Z}$ given by (IV.1) is shown shaded black.

For discrete time scales for $t < 0$ [12],

$$e_{\beta\ominus z}(t_n) = \prod_{k=n}^{-1} \frac{1+\mu(t_k)z}{1+\mu(t_k)\beta}. \tag{IV.13}$$

- $\mathbb{T}(\grave{\mu}_-^0, \acute{\mu}_-^0)$.
  This product approaches zero if all terms don't exceed

one. This is true if

$$|1+\mu(t_k)z| < |1+\mu(t_k)\beta|$$

or, equivalently

$$\left|z+\frac{1}{\mu(t_k)}\right| < \left|\beta+\frac{1}{\mu(t_k)}\right|.$$

This is the definition of the region $\bar{\mathcal{H}}(\beta,\mu(t_k))$. For $t<0$, the graininess varies over the range $\grave{\mu}_-^0 \le \mu(t_k) \le \acute{\mu}_-^0$, the overall region of convergence is the intersection of all of the $\mathcal{H}$'s in this interval. However, since[3]

$$\bigcap_{\grave{\mu}\le\mu(t_k)\le\acute{\mu}} \bar{\mathcal{H}}(\beta,\mu(t_k)) = \bar{\mathcal{H}}(\beta,\grave{\mu}) \cap \bar{\mathcal{H}}(\beta,\acute{\mu}),$$

we conclude from (III.8) that (IV.12) is true in the ROC

$$\mathcal{Z}_- = \mathcal{L}(\alpha,\grave{\mu}_-^0,\acute{\mu}_-^0). \quad \text{(IV.14)}$$

- $\mathbb{T}(\grave{\mu}_-^T,\acute{\mu}_-^T)$.
  From (IV.13),

$$e_{\beta\ominus z}(-\infty) = \prod_{k=-\infty}^{T} \frac{1+\mu(t_k)z}{1+\mu(t_k)\beta} \times \prod_{k=\sigma(T)}^{-1} \frac{1+\mu(t_k)z}{1+\mu(t_k)\beta}. \quad \text{(IV.15)}$$

  For (IV.12) to be true, only the first product needs to be zero. We can thus deal with grainineses over the interval $-\infty < t \le T$ and the ROC increases from (IV.14) to

$$\mathcal{Z}_- = \mathcal{L}(\alpha,\grave{\mu}_-^T,\acute{\mu}_-^T). \quad \text{(IV.16)}$$

- $\mathbb{T}(\grave{\mu}_-^\infty,\acute{\mu}_-^\infty)$.
  In the limiting case, the ROC is

$$\mathcal{Z}_- = \mathcal{L}(\alpha,\grave{\mu}_-^\infty,\acute{\mu}_-^\infty). \quad \text{(IV.17)}$$

When the lower and upper asymptotic bounds are equal, $\grave{\mu}_-^\infty = \acute{\mu}_-^\infty = \bar{\mu}_-$ and, using (III.10) applied to (IV.17), we have the region of convergence for $\mathbb{T}(\bar{\mu}_-)$.

$$\mathcal{Z}_- = \mathcal{H}(\alpha,\bar{\mu}_-). \quad \text{(IV.18)}$$

Lastly, for $\mathbb{T}(0_-,\infty_-)$, we apply (III.11) to (IV.17) and obtain

$$\mathcal{Z}_- = \mathcal{E}(\alpha).$$

V. Convergence of Signals Bounded by the Double Sided Exponential

In this section, after establishing a sufficient condition for the bilateral Laplace transform of $e_{\beta:\alpha}(t)$ to converge at z (see Lemma V.1), we show under the same condition, the Laplace transform of $f(t)$ will converge at z (see Corollary V.1) when

$$|f(t)| \le |e_{\beta:\alpha}(t)|. \quad \text{(V.1)}$$

The result is a special case of a more general theorem (Theorem V.1) that only requires the bound in (V.1) to be true for $T_- \ge t > T_+$ for any finite T_+ and T_-.

Lemma V.1. *A sufficient condition for the bilateral Laplace transform of $e_{\beta:\alpha}(t)$ to converge at z is*

$$\int_{-\infty}^{\infty} \left|e_{\beta:\alpha}(t)\ e_{\ominus z}^{\sigma}(t)\right| \Delta t < \infty. \quad \text{(V.2)}$$

Proof: We begin with the magnitude of the Laplace transform if of $e_{\beta:\alpha}(t)$ and write

$$\begin{aligned} &\left|\int_{-\infty}^{\infty} e_{\beta:\alpha}(t)\ e_{\ominus z}^{\sigma}(t)\Delta t\right| \\ \le\ & \int_{-\infty}^{\infty} \left|e_{\beta:\alpha}(t)\ e_{\ominus z}^{\sigma}(t)\right| \Delta t \\ <\ & \infty. \end{aligned}$$

■

Theorem V.1. *If there exists*

- *a bounded function $f(t)$ on regressive time scale, $\mathbb{T}$,*
- *finite values $T_- < 0$ and $T_+ > 0$ such that*

$$|f(t)| \le |e_{\beta:\alpha}(t)| \text{ for } t < T_- \text{ and } t > T_+,$$

  and
- *the sufficient condition in Lemma V.2 is true,*

then the bilateral Laplace transform in $f(t)$ converges at z.

Proof: Let $F(z)$ be the Laplace transform of $f(t)$. Then

$$\begin{aligned} |F(z)| &= \left|\int_{-\infty}^{\infty} f(t)e_{\ominus z}^{\sigma}(t)\Delta t\right| \\ &\le \int_{-\infty}^{\infty} \left|f(t)e_{\ominus z}^{\sigma}(t)\right| \Delta t \\ &= \left[\int_{-\infty}^{T_-} + \int_{T_-}^{T_+} + \int_{T_+}^{\infty}\right] \left|f(t)e_{\ominus z}^{\sigma}(t)\right| \Delta t \end{aligned}$$

If $\mathbb{T}$ is regressive and $f(t)$ is bounded, then the middle integral is finite. For the remaining integrals we impose (V.2) and write

$$\begin{aligned} &\left[\int_{-\infty}^{T_-} + \int_{T_+}^{\infty}\right] \left|f(t)e_{\ominus z}^{\sigma}(t)\right| \Delta t \\ \le\ & \left[\int_{-\infty}^{T_-} + \int_{T_+}^{\infty}\right] \left|e_{\beta:\alpha}(t)e_{\ominus z}^{\sigma}(t)\right| \Delta t \\ <\ & \infty \end{aligned}$$

Therefore,

$$|F(z)| < \infty.$$

■

The following corollary follows immediately as a special case.

Corollary V.1. *If (V.1), $\mathbb{T}$ is regressive and (V.2) applies, then the Laplace transform of $f(t)$ converges at z.*

VI. The Box Minus Shift

The *box minus ($\boxminus$) shift* on a time scale can be defined using the bilateral Laplace transform. The box minus shift, $f(t\boxminus\tau)$, reduces to the conventional shift operator, $f(t-\tau)$ on $\mathbb{R}$ and $\mathbb{Z}$.

Definition VI.1. We define the *box minus* operation, $\boxminus$, through

$$f(t\boxminus\tau) \quad \to F(z)e_{\ominus z}(\tau). \quad \text{(VI.1)}$$

Note that, as a consequence,

$$e_z(t\boxminus\tau) = \exp\left(\int_{\xi=\tau}^{t} \frac{\ln(1+\mu(\xi)z)}{\mu(\xi)}\ \Delta\xi\right). \quad \text{(VI.2)}$$

and, interpreting $f(\boxminus\tau) = f(0 \boxminus \tau)$, from the semigroup property,

$$e_z(t \boxminus \tau) = e_z(t)e_z(\boxminus\tau). \tag{VI.3}$$

Lemma VI.1.

$$e_z(\boxminus t) = e_{\ominus z}(t).$$

As a consequence, from (VI.3),

$$e_z(t \boxminus \tau) = e_z(t)e_{\ominus z}(\tau).$$

Proof: Using (VI.2),

$$\begin{aligned} e_z(\boxminus t) &= \exp\left(\int_{\xi=t}^{0} \frac{\ln(1+\mu(\xi)z)}{\mu(\xi)}\,\Delta\xi\right) \\ &= \exp\left(-\int_{\xi=0}^{t} \frac{\ln(1+\mu(\xi)z)}{\mu(\xi)}\,\Delta\xi\right) \\ &= \left[\exp\left(\int_{\xi=0}^{t} \frac{\ln(1+\mu(\xi)z)}{\mu(\xi)}\,\Delta\xi\right)\right]^{-1} \\ &= [e_z(t)]^{-1} \\ &= e_{\ominus z}(t) \end{aligned}$$

■

Definition VI.2. Define the *Hilger delta* as [4]

$$\delta_H(t \boxminus \tau^\sigma) := \begin{cases} \delta[t-\tau]/\mu(t) & ; \quad \mu(t) > 0 \\ \delta(t-\tau) & ; \quad \text{otherwise} \end{cases} \tag{VI.4}$$

where the Kronecker delta, $\delta[t]$, is one for $t = 0$ and is otherwise zero, and $\delta(t)$ is the Dirac delta [11].

For $\mathbb{R}$ and $\mathbb{Z}$, the Hilger delta in (VI.4) becomes $\delta[t-\tau]$ and $\delta(t-\tau)$ respectively.

VII. Time Scale Convolution

The box minus operation allows definition of convolution of two signals on the same time scale.

Definition VII.1. *Convolution on a time scale* is defined as

$$f(t) * h(t) := \int_{\xi\in\mathbb{T}} f(\xi)h(t \boxminus \xi^\sigma)\Delta\xi \tag{VII.1}$$

This definition is consistent with the convolution of transcendental functions defined in Bohner and Peterson [1] and its generalization [4]. It differs, however, from the time scale convolutions defined using the Fourier transform on a time scale [8], [11] which is defined only over a special class of time scales[4]. On $\mathbb{R}$ and $\mathbb{Z}$, (VII.1) becomes conventional convolution that describes the response, $g(t) = f(t) * h(t)$ of a linear time invariant system (LTI) system with *impulse response*, $h(t)$, to a stimulus of $f(t)$ [11]. The Laplace transform of the impulse response is the *system function* or the *transfer function*, $H(z)$, which contains the amplitude and phase changes imposed by the system on the stimulus. This property is generalized to an arbitrary time scale by the following theorem.

Theorem VII.1. System function. *Convolving a function $h(t)$ with a time scale exponential function yields, as a result, the same exponential weighted by the Laplace transform of the impulse response.*

$$e_w(t) * h(t) = e_w(t)H(w). \tag{VII.2}$$

Proof:

$$\begin{aligned} e_w(t) * h(t) &= \int_{-\infty}^{\infty} h(\tau)e_w(t \boxminus \tau^\sigma)\Delta\tau \\ &= e_w(t)\int_{-\infty}^{\infty} h(\tau)e_w(\boxminus\tau^\sigma)\Delta\tau \\ &= e_w(t)\int_{-\infty}^{\infty} h(\tau)e^\sigma_{\ominus w}(\tau)\Delta\tau \end{aligned}$$

from which (VII.2) follows. ■

Theorem VII.2. *Convolution on a time scale corresponds to multiplication in the Laplace domain.*

$$g(t) = f(t) * h(t) \quad \to G(z) = F(z)H(z).$$

Proof:

$$\begin{aligned} g(t) &= f(t) * h(t) \\ &= \int_{\xi\in\mathbb{T}} f(\xi)h(t \boxminus \xi^\sigma)\Delta\xi \\ &\to \int_{t\in\mathbb{T}} \left[\int_{\xi\in\mathbb{T}} f(\xi)h(t \boxminus \xi^\sigma)\Delta\xi\right] e^\sigma_{\ominus z}(t)\Delta t \\ &= \int_{\xi\in\mathbb{T}} f(\xi)\left[\int_{t\in\mathbb{T}} h(t \boxminus \xi^\sigma)e^\sigma_{\ominus z}(t)\Delta t\right]\Delta\xi \\ &= \int_{\xi\in\mathbb{T}} f(\xi)\left[H(z)e^\sigma_{\ominus z}(\xi)\right]\Delta\xi \\ &= H(z)\int_{\xi\in\mathbb{T}} f(\xi)e^\sigma_{\ominus z}(\xi)\Delta\xi \\ &= F(z)H(z) \\ &= G(z) \end{aligned}$$

■

The following results follow immediately.

- Convolution on a time scale is commutative, associative and distributive over addition.
- *The Sifting Property of the Hilger delta.* If we define

  $$f(t) * \delta_H(t) := \int_{t=-\infty}^{\infty} f(\tau)\delta_H(t \boxminus \tau^\sigma)\Delta\tau$$

  it follows that the Hilger delta is the identity operator for convolution on a time scale.

  $$f(t) * \delta_H(t) = f(t).$$

  The sifting properties of the Dirac delta and Kronecker delta on $\mathbb{R}$ and $\mathbb{Z}$ follow as special cases.
- *The Shift Property.* If $g(t) = f(t) * h(t)$, then

  $$f(t \boxminus \xi) * h(t) = f(t) * h(t \boxminus \xi) = g(t \boxminus \xi)$$

- *The Derivative Property.* From the derivative theorem in (III.2), it follows immediately that

  $$g^\Delta(t) = f(t) * h^\Delta(t) = f^\Delta(t) * h(t).$$

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VIII. Wide Sense Stationarity of a Stochastic Process on a Time Scale

Let $x(t)$ be a real stochastic process on a time scale $\mathbb{T}$. Its autocorrelation [11] is

$$R_x(t,\tau) = E\left[x(t)x(\tau)\right]. \tag{VIII.1}$$

Definition VIII.1. A stochastic process, $x(t)$, on a time scale $\mathbb{T}$ is *wide sense stationary* (WSS)[5] [11], [13] if[6]

$$R_x(t,\tau) = R_x(t \boxminus \tau) \tag{VIII.2}$$

As a consequence, the autocorrelation of a wide sense stationary (WSS) stochastic process on a time scale can be represented by a single one dimensional function, $R_x(t)$.

Notes.

1) For $\mathbb{R}$ and $\mathbb{Z}$, (VIII.2) takes on the familiar form

$$R_x(t,\tau) = R_x(t-\tau)$$

2) From (VIII.1), $R_x(t,\tau) = R_x(\tau,t)$, Thus

$$R_x(t \boxminus \tau) = R_x(\tau \boxminus t).$$

Definition VIII.2. The Laplace transform of the autocorrelation is the *power spectral density*, $S_x(z)$.

$$R_x(t) \quad \rightarrow S_x(z)$$

Theorem VIII.1. *On time scale $\mathbb{T}$, let*

$$y(t) = x(t) * h(t). \tag{VIII.3}$$

Then

$$S_y(z) = H(z)H_{\ominus}(z)S_x(z). \tag{VIII.4}$$

Proof: Multiply both sides of (VIII.3) by $x(\tau)$ and expectate to give

$$R_{xy}(\tau,t) = R_x(t \boxminus \tau) \overset{t}{*} h(t) \tag{VIII.5}$$

where $\overset{t}{*}$ denotes convolution with respect to the variable t. Rewrite (VIII.3) as $y(\tau) = x(\tau) \overset{\tau}{*} h(\tau)$, multiply both sides by $y(t)$, and expectate to give

$$R_y(t,\tau) = R_{xy}(\tau,t) \overset{\tau}{*} h(\tau).$$

Substitute (VIII.3) gives

$$R_y(t,\tau) = h(t) \overset{t}{*} R_x(t \boxminus \tau) \overset{\tau}{*} h(\tau).$$

Laplace transform both sides with respect to t gives

$$\begin{aligned}
R_y(t,\tau) \quad &\overset{t}{\rightarrow} \quad \int_{t\in\mathbb{T}} \left[h(t) \overset{t}{*} R_x(t \boxminus \tau) \overset{\tau}{*} h(\tau)\right] e^{\sigma}_{\ominus z}(t)\Delta t \\
&= \int_{t\in\mathbb{T}} \Bigg[\left(\int_{\xi\in\mathbb{T}} R_x(\xi \boxminus \tau) h(t \boxminus \xi^{\sigma})\Delta\xi\right) \\
&\qquad \overset{\tau}{*} h(\tau)\Bigg] e^{\sigma}_{\ominus z}(t)\Delta t \\
&= \int_{\xi\in\mathbb{T}} R_x(\xi \boxminus \tau)\left[\int_{t\in\mathbb{T}} h(t \boxminus \xi^{\sigma}) e^{\sigma}_{\ominus z}(t)\Delta t\right] \\
&\qquad \overset{\tau}{*} h(\tau)\Delta\xi \\
&= \left(\int_{\xi\in\mathbb{T}} R_x(\xi \boxminus \tau)\left[H(z)e^{\sigma}_{\ominus z}(\xi)\right]\Delta\xi\right) \overset{\tau}{*} h(\tau) \\
&= H(z)\left[\int_{\xi\in\mathbb{T}} R_x(\xi \boxminus \tau) e^{\sigma}_{\ominus z}(\xi)\right] \overset{\tau}{*} h(\tau) \\
&= H(z)S_x(z)\left[e_{\ominus z}(\tau) \overset{\tau}{*} h(\tau)\right] \\
&= H(z)S_x(z)\int_{\eta\in\mathbb{T}} h(\eta)e_{\ominus z}(\tau \boxminus \eta^{\sigma})\Delta\eta \\
&= H(z)S_x(z)\int_{\eta\in\mathbb{T}} h(\eta)e_{\ominus z}(\tau)e_{\ominus z}(\boxminus\eta^{\sigma})\Delta\eta \\
&= H(z)S_x(z)e_{\ominus z}(\tau)\int_{\eta\in\mathbb{T}} h(\eta)e_{z}(\eta^{\sigma})\Delta\eta \\
&= H(z)S_x(z)e_{\ominus z}(\tau)\int_{\eta\in\mathbb{T}} h(\eta)e^{\sigma}_{z}(\eta)\Delta\eta \\
&= H(z)S_x(z)H_{\ominus}(z)e_{\ominus z}(\tau)
\end{aligned} \tag{VIII.6}$$

The $e_{\ominus z}(\tau)$ term reveals $R_y(t,\tau)$ is of the form $R_y(t \boxminus \tau)$. Therefore (VIII.4) follows immediately.

■

IX. Conclusion

We have generalized establishment of the ROCs from unilateral to bilateral Laplace transforms. For double sided exponentials, the ROC, when it exists, are the intersection of points outside of a modified Hilger circle defined by behavior for positive time and inside another modified Hilger circle determined by behavior for negative time. The ROCs revert to the familiar horizontal slab ROC for continuous time and annulus for discrete time. Since graininess lies between zero and infinity, there are conservative ROCs applicable for all time scales. Signals bounded by double sided exponentials were shown to converge in at least the ROC of the double sided exponential. The Laplace transform on a time scale is used to define a box minus operator that, in turn, allows definition of time scale convolution. Time scale convolution allows characterization of wide sense stationary stochastic processes on a time scale via its autocorrelation and power spectral density.

Notes

[1]Such time scales arise from *recurrent nonuniform* signal sampling, also called *interlaced* or *bunched* sampling [9], [10], [11], [14], [15], [16], [17].

[2]Since $\grave{\mu} < \acute{\mu}$ the solid dot, •, is always to the right of the hollow dot, ○.

[3]This is graphically evident of in Figure 1 where a sequence of modified Hilger circles are drawn over a range of grainineses. The points internal to all of the modified Hilger circles is equal to the points inside both the leftmost and rightmost circles.

[4]Specifically, *additively idempotent* time scales [8], [11].

[5]An further requirement of a constant first moment accompanies the classic definition of WSS stochastic processes [11], [13], [14]. For the treatment in this paper, however, such an assumption is not needed.

[6]We use here a common abuse of notation [11], [13], [14]. The function R_x cannot simultaneously be a two dimensional function, $R_x(t, \tau)$, and a one dimensional function, $R_x(t)$.

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