Lecture 12 - Degeneracy pressure

12.1 Key Ideas

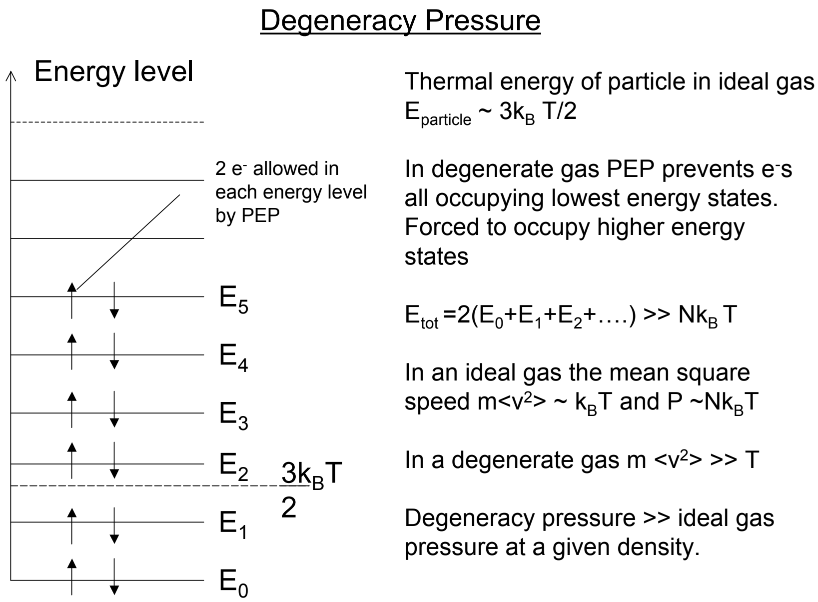
- Degeneracy pressure arises from the quantum mechanical nature of elementary particles
- Degeneracy pressure is form of pressure that withstands gravitational collapse in high density materials.
- White dwarfs supported by electron degeneracy pressure, and neutron stars from neutron degeneracy pressure.
- $P_{deg} \propto \rho^{5/3}$ and independent of temperature (in contrast to ideal gas $P \propto T$).

12.2 Pauli Exclusion Principle

- Electrons and neutrons are particles called **'fermions'**.
  - As well as having electric charge, fermions also have another intrinsic property **'spin'**. 2 possible spin states 'up' $\uparrow$ and 'down' $\downarrow$
- **Pauli exclusion principle** - states that no two electrons (neutrons) can be in the same quantum state
  - Electrons can have exactly the same energy but must have different spins, i.e. 2 electrons can occupy any single energy state.
- A degenerate gas is one in which the density is high enough that electrons behavior is determined by the exclusion principle.

12.3 Wave particle duality

- Already seen that photons can be considered as particles 'photons' ($E = h\nu$) and as a wave with wavelength λ $c = \lambda\nu$.
- de Broglie (10 years after Bohr came up with his model of the atom), came up with an explanation for why the angular momentum is quantized in electron orbits
  - He proposed that particles, e.g. e^-, demonstrate wave like behavior, observable on quantum scales.
  - He showed that the wavelength associated with the particle, the **deBroglie wavelength** λ_{dB} is related to the particle momentum



- In special relativity the energy of a particle is given by

$$E = \sqrt{m_o c^4 + p^2 c^2} \tag{12.1}$$

where m_o is the rest mass energy, and p is the particle momentum.

- For a fast moving particle $m_o c^2 \ll pc$, $E \approx pc$
- We can relate the energy to a wavelength using an analogous equation to that for photon energy

$$E = h\nu = \frac{hc}{\lambda} \tag{12.2}$$

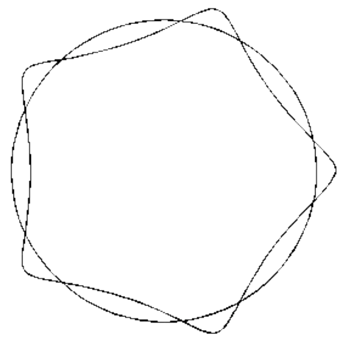
$$\boxed{\lambda_{dB} = \frac{h}{p}} \tag{12.3}$$

- deBroglie proposed that the quantization of electron angular momentum corresponded to requiring the electron orbit to be a whole number of half wavelengths.

- Wave-particle duality was experimentally verified in the Davisson-Germer experiment, in which electrons were fired at a Nickel crystal with atoms separated by distance $\sim \lambda_{dB}$. The electrons underwent diffraction by the atomic lattice, just as X-rays do.

12.4 Heisenberg uncertainty principle

- **Heisenberg uncertainty principle** - we cannot know everything about a particle on quantum scales



- Intrinsic fundamental uncertainty in precisely knowing 2 related quantities at once:

$$e^{-}s\ position \quad \leftrightarrow \quad e^{-}s\ speed \tag{12.4}$$
$$e^{-}s\ energy \quad \leftrightarrow \quad e^{-}s\ time\ in\ energy\ state \tag{12.5}$$

- In a degenerate gas it is the position - speed relationship that we are in interested in. Heisenberg's uncertainty principle places a numerical value on the minimum uncertainty we must have when measuring an electrons position and momentum simultaneously

$$\Delta p \Delta x \geq \frac{h}{2\pi} \tag{12.6}$$

- If the electrons are closely packed, i.e. their position is known precisely, this means that their momentum cannot be known as well.
  - If uncertainty in momentum is high, momentum itself can be high
  - This corresponds to electrons having to populate high energy states (as we saw with the Pauli exclusion principle earlier.

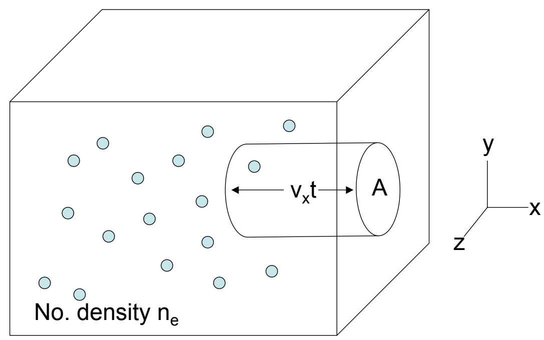
12.5 Electron degeneracy pressure

- Consider electrons moving as a gas in an enclosed box. No. density of electrons, n_e, v_x is average speed of an electron in x-direction, A is surface area on side of box being considered

  No. of particles hitting area A in time t = $n_e v_x A t$
  No. of particles hitting unit area per unit time = $n_e v_x$
  Change in momentum per unit area per unit time= $2 n_e v_x p_x$
  (Assume elastic recoil - no change in total energy $p_x \rightarrow -p_x$ i.e. size of change is $2p_x$)
  Pressure exerted= force/unit area = $2 n_e v_x p_x$

- Each electron takes up average volume = $1/n_e$
  - Average uncertainty in electron position $\Delta x = 1/n_e^{1/3}$
- Momentum of electron given roughly by intrinsic uncertainty (using Heisenberg uncertainty principle)

$$p_x \simeq \Delta p_x \tag{12.7}$$
$$\simeq \frac{h}{2\pi \Delta x} = \frac{h n_e^{1/3}}{2\pi} \tag{12.8}$$



- Assuming electron is non-relativistic

$$p_x = m_e v_x \quad (12.9)$$
$$(12.10)$$

Therefore substituting into equation for pressure we derived

$$P \simeq 2n_e v_x p_x \quad (12.11)$$
$$\simeq 2n_e \frac{p_x^2}{m_e} \quad (12.12)$$
$$\simeq 2n_e \frac{hn_e^{1/3}}{2\pi} \quad (12.13)$$

$$\boxed{P \simeq \frac{h^2 n_e^{5/3}}{2\pi^2} \quad (12.14)}$$

- Can write this in terms of overall density.
  - Assuming electric charge neutrality no. of protons = no. of electrons. If protons are contained in n_Z nuclei with atomic number Z,

$$n_e = Zn_z \quad (12.15)$$

  - Total density

$$\rho = An_Z m_p + m_e n_e \quad (Assume\ m_n = m_p) \quad (12.16)$$
$$\approx An_Z m_p \quad (m_e \ll m_p) \quad (12.17)$$
$$\approx \frac{A}{Z} n_e m_p \quad (12.18)$$
$$n_e \approx \frac{\rho}{m_p}\frac{Z}{A} \quad (12.19)$$

– Substituting into the pressure expression

$$P = \frac{2}{m_e}\left(\frac{h}{2\pi}\right)^2\left(\frac{Z}{A}\right)^{5/3}\left(\frac{\rho}{m_p}\right)^{5/3} \tag{12.20}$$

- We have derived the key result describing the behavior of degenerate gases.

$$P \propto \rho^{5/3} \tag{12.21}$$

in contrast to ideal gas in which $P \propto T$.