The International Mathematical Olympiad

The International Mathematical Olympiad (IMO) is an interesting annual international mathematics competition for high school students. Here I try to give some idea of why it is interesting, what it is trying to do, and what its influence is on mathematics and mathematicians.

From the start let me say that you can find out more details about the IMO by going to their web site www.imo.math.ca. There you will find all the questions that have been asked about IMOs; how well the different countries have performed over the years; who is hosting the IMO this year; Olympiad sites of those countries that have them; etc.; etc.

If you are interested in taking part in an Olympiad you might get in touch with one of the people below:

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Wherever an IMO is held the New Zealand team has had to travel far. There was one exception to this. In 1988, the IMO was held in Canberra, Australia. That was our first Olympiad and there we added to the ever-growing number of countries that take part in the IMO.

Each host country is allowed to invite countries who have never competed in an IMO before and we entered the IMO family through Australia's invitation. This year (2003) will be our 16^{th} IMO.

Each country is allowed to send a maximum of six students to compete. These students have to be under 20 and not attending a tertiary institution full time. In addition we have always sent a Team Leader, a Deputy Team Leader, and a Manager. The Team Leader quickly gets separated from the team and slaves away choosing problems and marking students' work. The Deputy Team Leader coaches the team up to the last possible moment, has fun, and helps the Team Leader with the marking. The Manager is responsible for travel arrangements and the general well-being of the team.

1. The Team Leader I want to talk about the Team Leader first because this is the team member who gets into action first. All the Team Leaders start work at an Olympiad a few days before the Opening Ceremony.

The Team Leaders are collectively The Jury, and the Jury's first task is to produce the problems that will be used in the upcoming competition. Fortunately the problems don't have to be invented on the spot. Each country has been asked to contribute one or two questions well before the IMO begins and a problem committee has sifted through these to pick out about 20 or so from which the final six questions will be chosen.

I should point out straight away that each student has to work individually on three questions (for 4 ½ hours) on the first day of competition and on a further three questions (for 4 ½ hours) on the second day. So you can see that these problems are hard! I'll say more about the questions later.

Having booked in at the IMO, the Team Leaders are immediately sequestered away and are expected to tackle the 30 odd questions that the problem committee has prepared. The plan is that the Team Leaders will look for a good set of questions that will extend the better students and separate out the Gold Medal winners from the Silver and Bronze.

Over the next couple of days, the Jury then goes about whittling down the problems in a committee meeting with nearly 90 members and speaking over 20 languages. You can see that that is likely to present a problem or two.

Officially there are certain specified official languages: English, French, German, Spanish, Russian and the language of the host country (if it's different from these). In practice most of the business recently appears to have been conducted in English, though any formal motions are generally translated to ensure that all the Jury know what they are voting about. This seems to have been a change from the early days when all discussions were translated into all of the official languages. This certainly slowed things down though.

After a couple of days of horse trading and discussion, six questions are finally chosen with hearty applause. But the Jury's work isn't finished. Now the questions have to be translated into every language represented at the IMO – every student is allowed to have the questions in their own languages. Actually some students ask for questions in more than one language.

So translation begins. But first the wording in English is decided. This is used as the basis for the wording in the other official languages. However, as this process unfolds, it often happens that one language group may see a 'nicer' way to write a given question. In that case, this wording may be accepted by all countries.

After the official language translations have been accepted, the questions are translated into the other languages. These translations are then laid out for everyone to check. It's amazing that you can sometimes find errors in mathematical questions written in a language you know nothing about!

The next day is the Opening Ceremony.

2. The Team

Most teams travel to IMOs without their leader who has gone ahead by several days. But our team frequently travels with the Team Leader. This has two advantages. Generally there is a big time gap between the host country and New Zealand, so an early arrival allows time to recover from jet lag. However, it also gives the team a chance to do just a little more preparation. Depending on where the host country is, our team might train alone or with another team. In the past we've worked with both Ireland and Singapore in the days leading up to an IMO. Whatever the situation, the team's first engagement is the Opening Ceremony. As it is the norm for **everyone** involved in the IMO to attend, this provides certain logistical problems. You see, at this point the Team Leaders know all the questions. What's more important, they know all the answers. So it's important to keep Team Leaders and team separated from each other. This is usually done by having the Ceremony in a huge auditorium with a balcony. The students get to sit in the stalls and the leaders in the balcony.

Naturally there is a lot of waving and shouting but since that is all in public, no vital information is generally shared at this point. Though many Team Leaders are just glad to see that their team has arrived.

Actually the matter of security is one that is taken very seriously. To assist in this, Team Leaders and teams are accommodated in different places until the two days of competitions are over. Given cell phones, it's hard to be sure that information isn't transferred these days. But there are punishments. One team was disqualified a few years ago for cheating. They were discovered because a student produced a perfect answer. It was a perfect answer to a question that had been on the listed 30 or so. By the time it had reached the final six it had been changed just enough to make the old perfect answer not so perfect after all.

But the IMO is overwhelmingly fair and there have been only a very few incidents of the sort above. My first introduction to IMO was as an observer in Cuba, the year before we took part officially in Canberra. At one point when the final 6 questions were almost ready to be accepted the Russian leader got up and said "You can't have these questions". After a minute or two of consternation, he added that three of those six questions had been supplied by Russia. To make sure that the competition was seen to be fair, at least one of the Russian questions ought not to be used.

It's that spirit that generally pervades the IMO.

So we've got everyone, suitably separated, to the Opening Ceremony. This is, of course, generally a very swish affair. The host country wants to show off its national dress, dances, culture, and so forth. There are usually formal speeches to endure but after that it's show time!

And when the ceremony is over, the teams go back to their residences and get into a panic for the two days of competitions that are just ahead. At the same time the Team Leaders are panicking over whether they have chosen the right questions for their team and whether they have provided enough training.

The night after the Opening Ceremony is a sleepless one for adults and students. But this is the culmination of more than a year's work for most of the students, so they ought to be mathematically prepared.

3. The Preparation

All teams do a lot of preparation for an IMO but the amount varies considerably. By and large though, there is usually some sort of selection process, a training camp, and a correspondence programme. The New Zealand programme fits the New Zealand budget. Unlike some countries all the people involved in training students give their time free, of course travel expenses and accommodation are covered but no one earns anything from their contribution to the IMO. We are just glad that our employees, mostly the Ministry of Education or a university, allow us the time off to do the job.

In New Zealand, things begin properly with the September problems, though many students have already come to our notice from a previous year, the Nation Bank Junior Mathematics Competition, the Australian Mathematics Competition, or by some other means.

The September Problems are a group of problems circulated to all secondary schools in, naturally, September each year. The students are given a week to solve the problems and they then send in their answers to be marked. On the basis of these answers 20 or so students are chosen to attend the annual training camp held at Rangi Ruru Girls' School in Christchurch in mid January.

The training camp roughly consists of classes on important aspects of mathematics, and lots and lots of problems. Although it's crucial that students have a good content knowledge, it's vital that they can communicate the answers that they produce. So a lot of time needs to be spent worrying out problems and writing out their solutions.

In the middle of the week-long camp, there is a mini exam. On the Friday of the camp, then is a full-blown 4 ½ hours exam, so that students begin to see how to cope with that time scale and the pressure it induces. While the students go to explore Christchurch on Friday afternoon, the adults do the marking. On the basis of the September Problems, the mini-exam, but mostly the maxi-exam, the team of six students and two reserves is chosen. The names are announced at a closing ceremony on Friday evening.

Now you shouldn't get the idea that the training camp is 100% hard mathematical labour. There is certainly some time available at meal breaks and in the evening to play cards, to play football, to play bowling, or simply to play around. But Friday night is the time for letting hair down. Traditionally pizzas are delivered after the closing ceremony and these, along with student-chosen and adult-vetted videos, are designed to last the students through to breakfast.

The lucky eight students now continue to work. Their mathematical education continues by correspondence up to the week before the IMO. Then they get together for a final preparation camp, either in Auckland or on some foreign field.

4. The Competition

Back in the host country, there are now two consecutive days of sweat and toil for the students. Three questions today; four-and-a-half hours to do them. Three questions tomorrow; four-and-a-half-hours to do them. So what maths questions are there that take about 90 minutes to complete? Do the students have to multiply two enormous numbers together? How do they show that they are the cleverest maths students in the world? What does clever mean?

Well clever here means that answers have to be compiled for devious questions which largely are based on material in most school curricula. (There are no questions on calculus as many countries meet this first at university.) The thing that separates these questions from ones on an end-of-school exam is that they are non-standard. They are not found in text books or in other exams anywhere. They all require at least a little nicety in approach – the use of an old idea in a new and clever way.

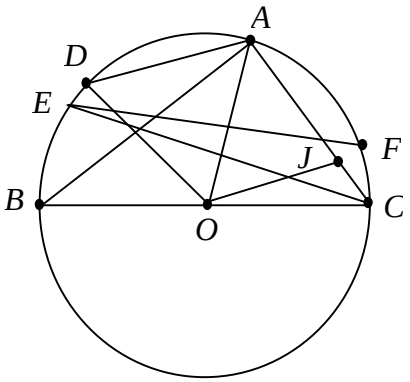
Generally speaking an IMO set of questions will cover geometry (usually two questions of the classical (Euclidean) kind); algebra; combinatorics (discrete maths); number theory; and functional equations. Below I've given the questions from 2002 and indicated which questions come from which area. If you are pretty good at maths, then have a go at them. So that you can check your answers, I've put the official solutions at

the end of this article. Should it make you feel any better, on a good day I **might** get two complete questions out.

Day 1

A1. S is the set of all (h, k) with h, k non-negative integers such that $h + k < n$. Each element of S is colored red or blue, so that if (h, k) is red and $h' \le h$, $k' \le k$, then (h', k') is also red. A type 1 subset of S has n blue elements with different first member and a type 2 subset of S has n blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

A2. BC is a diameter of a circle center O. A is any point on the circle with angle $AOC > 60^o$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incentre of triangle CEF.



A3. Find all pairs of integers $m > 2$, $n > 2$ such that there are infinitely many positive integers k for which $(k^n + k^2 - 1)$ divides $(k_m + k - 1)$.

Day 2

B1. The positive divisors of the integer $n > 1$ are $d_1 < d_2 < ... < d_k$, so that $d_1 = 1$, $d_k = n$. Let $d = d_1d_2 + d_2d_3 + ... + d_{k-1}d_k$. Show that $d < n^2$ and find all n for which d divides n^2.

B2. Find all real-valued functions f on the reals such that $(f(x) + f(y))(f(u) + f(v)) = f(xu - yv) + f(xv + yu)$ for all x, y, u, v.

B3. $n > 2$ circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centres are $O_1, O_2, ... , O_n$. Show that $\sum_{i<j} \frac{1}{O_iO_j} \sum_{i<j} \frac{1}{O_iO_j} \le (n - 1)p/4$.

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On the morning of the first day, the students are summoned and sorted. Generally students from the same team go to different rooms. They can only take writing instruments with them. Calculators are not allowed.

In the first half hour the students can ask anything they like about the questions in front of them. There's no guarantee that the questions will be answered. At least, there's no guarantee that they'll be answered in a way that the students will like or find very useful.

The Jury answers the questions. The questions are relayed to the Jury in another place. They are given to the questioner's Team Leader and the Team Leader suggests an answer. In their turn, the individual team leaders tell the Jury what the question is and what answer they propose. Frequently the answer is an unhelpful "read the question". The Jury has to agree to the answer before it is given to the student.

But there can be interesting situations. A question in Phasi may have missed a vital curl of the pen. Consequently the meaning of a sentence has changed. The Jury will probably allow the curl to be added to the writing of the question.

A student may ask "what does *X* mean?" Perhaps in a given language they don't use the term *X*. In an extreme case, the Jury will decide that *X* should be defined.

When all the students' questions have been answered the team leaders go off for a day of sightseeing. This may be a pleasant lake or river cruise with lunch, a trip to a carpet factory or a barbeque garnished by an ethnic dance group. (Well you know what I mean.)

The students slave away for 270 minutes. To help them survive there is food and drink. Many of them need ideas rather than food.

On the morning of the second day, the whole thing is repeated, except, of course, the questions are different. At the end of that day team leaders and students meet, commiserate, congratulate, relax. Well the students relax. They've done their job. From now on it's all play. They go on the excursions that the Team Leaders have been on and spend the next several days enjoying themselves. The Team Leaders, on the other hand, along with their deputies, go to work again. Now the questions have to be marked.

5. Coordination

And the team leader and deputy team leader mark their own team's work. Is that fair? Well, in the best possible way, yes. And how does that work?

First of all you should know that every question is marked out of seven. That's regardless of difficulty. And every attempt is made to make sure that the third question on the first day is harder than the other two, and the sixth question is the hardest one overall. (Not that the Jury always gets it right, but that's the aim.)

Second, it should be pointed out that **any** complete solution is given seven marks. Now in maths we know that some solutions are nicer and more elegant than others but no attempt is made to judge the relative 'beauty' of a solution. (Trying to do this would be a God-like task.) However, if a student gets a solution to a problem that is outstanding, neat, elegant, whatever, then the Jury may award a special prize. As far as I know though, this special prize has not been awarded for many years.

Third, there is a marking scheme. Sometimes it's a very skimpy marking scheme but it is enough to give the markers some idea of how many marks they should ask for. And that is the clue to the fairness. Because the Team Leaders and their deputies first spend hours, literally, trying to decide how many marks they think their students should get, and then they justify them in the coordination process.

I have no idea where the team coordination comes from. It's an English word that doesn't quite fit unless you say that the process coordinates all the marks. Maybe it is a translation of a Russian word that means more in Russian than in English. I only say this because the Olympiad began behind the Iron Curtain in Roumania in 1957. My guess is that in 1957 they didn't call the process "coordination".

Anyway, here's how it works. Every question has its own coordination team. Before coordination starts they have each spent considerable time working out all of the ways that they can think of that their question might be solved. Then they assign an appropriate number of marks for each step in each solution.

Just in case you think that the Team Leaders could alter the solutions of their team, you should also know that every uncovered part of the students' papers has been over-marked so nothing extra can be added.

So let the coordination begin. Each country is rostered for each question. At the appropriate time the Team Leader and the Deputy Team Leader go to sit opposite two of the coordination team for a given question. The Team Leader then takes the coordinators through each student's question and proposes a mark out of seven. The coordinating pair either agree or disagree, surprisingly. If they agree, so good. If they disagree then … Now occasionally it happens that the coordinators suggest a bigger mark than the one being sought. In that case the country will grab that mark with delight. Otherwise, polite, firm, coaxing, menacing, diplomatic discussions begin. Actually it's often hard to shift the coordinators unless you can find some evidence that they've missed. After all, they only have to appeal to consistency – "we just gave Austria 5 marks for that so you can't have 6". It's difficult to argue against that.

When all discussions are closed, the appropriate numbers are written on a mark sheet, the leaders sign their assent and the mark is put up on the official boards. You can imagine that everyone watches the scores mount with great anticipation or distress.

It is just possible that a country won't accept a mark on offer. In that case it goes to the Jury for a decision. But I've never known the Jury to overturn a coordinator's decision.

When all the marks are up on the board, it begins to be more or less clear who is going to get what medals. There is a simple regulation that spells things out. Roughly speaking, half of the students will gain a medal of some sort. What's more, gold, silver and bronze medals are awarded in the ratio 1 : 2 : 3. Any student who has not obtained a medal but who has got a perfect seven on one of the questions is given an honorable mention.

The reason why it's not certain from looking at the board who is going to get what medals is the "roughly" above. Suppose there are 500 students and 240 get 20 marks or more, while 30 get 19 marks. Is the bronze cut off 20 or 19? Almost certainly 20. Because the regulations say "no more than half of the students will get a medal". $240 + 30 = 270 > 250$. In this example, the students on 19 would theoretically miss out on a bronze.

In fact in some years if it means tht 251 students gain medals, the Jury, whose job it is to decide these things, may relent and allow the rules to be fudged slightly. But it takes

a lot of talking before a vote like that is passed. Tradition is strong at the IMO and a rule is a rule.

Similar rules exist for gold and silver medals, so the Jury meeting that confirms the medal cut off marks can be a tense and exciting one.

6. Home

From here on its just a matter of a Closing Ceremony and the awarding of the medals. Oh there maybe one more excursion while the host country gets itself organized for the big finale. But that's essentially it.

Closing Ceremonies again involve speeches by important people and occasionally the medals are presented by people you know. In Glasgow in 2002, Princess Anne presented the gold medals.

Again at this last ceremony there may be entertainment - they can be great fun. But the finale of any IMO is the banquet. This can be an extravagant affair. In Germany it was held in the grounds of a "stately home". The food wasn't too special but all around the grounds there was continuous entertainment – flamenco dancers, acrobats, ballet and so on. But the most impressive and emotional banquet for me was in Turkey. We were in a palace on the edge of the Bosphorus. It was a warm clear night. The food was great. The teams and their leaders were mingling and talking excitedly. I don't think anybody wanted to leave.

7. Why IMO?

It's clear that the six students in each team get a lot out of an IMO. For many it is their first trip overseas. They get to see new sights and meet new people. They learn a lot of new maths and have a great deal of attention lavished on them. The medal winners have something to treasure for the rest of their lives. In fact they will all have an experience that will last them a long while and something to add to their CV's that will open a few doors.

The adults too will have had a great experience. Sure it will have meant putting in a lot of work both before and during the Olympiad but the company is great and stimulating mathematically, pedagogically, and otherwise. Many people will renew old acquaintances and make new ones. Some of these develop into valuable and lasting professional and personal relationships.

But is that all?

The answer is clearly no. If nothing else the Olympiad raises the profile of mathematics all over the world and makes it OK to be mathematically able. But there are an incredible number of things that happen as a result of the Olympiad that reach thousands of New Zealanders and millions in the world as a whole. To see where the ripples go in New Zealand read the other article on this department's maths site. As a result of all the activities listed there, the ripples go down to the majority of teachers at all levels. When you think about it, it does the same for mathematics that the regular Olympics does for athletics. And surely that has to be good?

8. The Answers

http://www.kalva.demon.co.uk/imo/imo02.html