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On a Construction of Hagge

Christopher J. Bradley and Geoff C. Smith

Abstract. In 1907 Hagge constructed a circle associated with each cevian point P of triangle ABC. If P is on the circumcircle this circle degenerates to a straight line through the orthocenter which is parallel to the Wallace-Simson line of P. We give a new proof of Hagge's result by a method based on reflections. We introduce an axis associated with the construction, and (via an areal analysis) a conic which generalizes the nine-point circle. The precise locus of the orthocenter in a Brocard porism is identified by using Hagge's theorem as a tool. Other natural loci associated with Hagge's construction are discussed.

1. Introduction

One hundred years ago, Karl Hagge wrote an article in *Zeitschrift für Mathematischen und Naturwissenschaftliche Unterricht* entitled (in loose translation) "The Fuhrmann and Brocard circles as special cases of a general circle construction" [5]. In this paper he managed to find an elegant extension of the Wallace-Simson theorem when the generating point is not on the circumcircle. Instead of creating a line, one makes a circle through seven important points. In §2 we give a new proof of the correctness of Hagge's construction, extend and apply the idea in various ways. As a tribute to Hagge's beautiful insight, we present this work as a centenary celebration. Note that the name Hagge is also associated with other circles [6], but here we refer only to the construction just described. Here we present new synthetic arguments to justify Hagge's construction, but the first author has also performed detailed areal calculations which provide an algebraic alternative in [2].

The triangle ABC has circumcircle Γ, circumcenter O and orthocenter H. See Figure 1. Choose P a point in the plane of ABC. The cevian lines AP, BP, CP meet Γ again at D, E and F respectively. Reflect D in BC to a point U, E in CA to a point V and F in AB to a point W. Let UP meet AH at X, VP meet BH at Y and WP meet CH at Z. Hagge proved that there is a circle passing through X, Y, Z, U, V, W and H [5, 7]. See Figure 1. Our purpose is to amplify this observation.

Hagge explicitly notes [5] the similarities between ABC and XYZ, between DEF and UVW, and the fact that both pairs of triangles ABC, DEF and XYZ, UVW are in perspective through P. There is an indirect similarity which carries the points $ABCDEFP$ to $XYZUVWP$.

Peiser [8] later proved that the center $h(P)$ of this Hagge circle is the rotation through π about the nine-point center of ABC of the isogonal conjugate P^* of P. His proof was by complex numbers, but we have found a direct proof by classical

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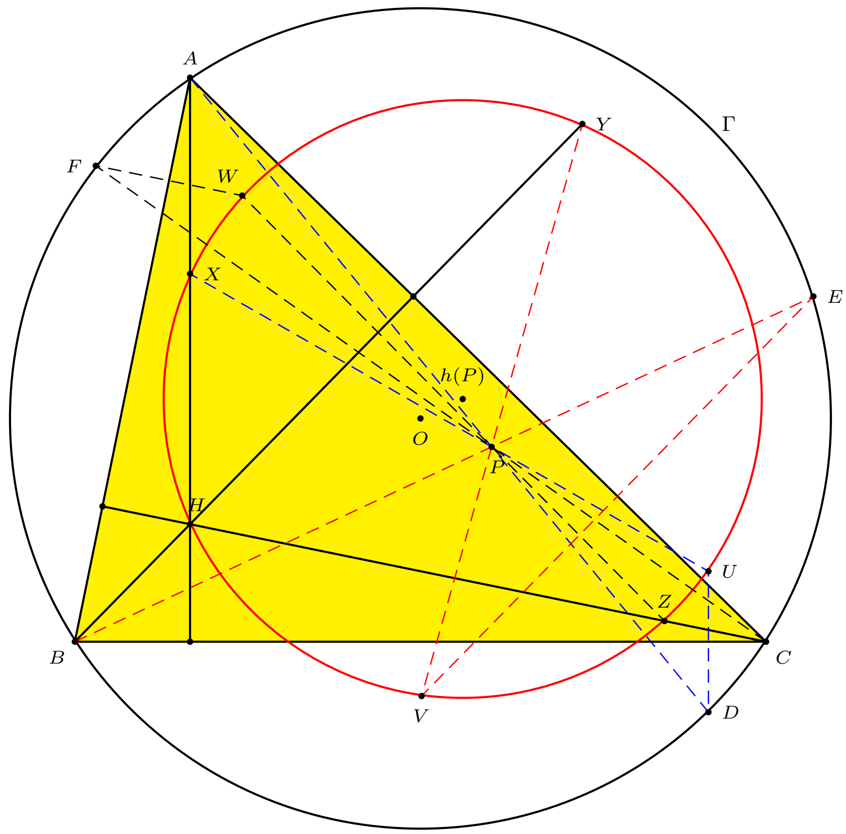

Figure 1. The Hagge construction

means [4]. In our proof of the validity of Hagge's construction we work directly with the center of the circle, whereas Hagge worked with the point at the far end of the diameter through H. This gives us the advantage of being able to study the distribution of points on a Hagge circle by means of reflections in lines through its center, a device which was not available with the original approach.

The point P^* is collinear with G and T, the far end of the diameter from H. The vector argument which justifies this is given at the start of §5.1. Indeed, we show that $P^*G : GT = 1 : 2$.

There are many important special cases. Here are some examples, but Hagge [5] listed even more.

(i) When $P = K$, the symmedian point, the Hagge circle is the orthocentroidal circle. [1]

(ii) When $P = I$, the incenter, the Hagge circle is the Fuhrmann circle.

(iii) When $P = O$, the circumcenter, the Hagge circle and the circumcircle are concentric.

[1]In [5] Hagge associates the name Böklen with the study of this circle (there were two geometers with this name active at around that time), and refers the reader to a work of Prof Dr Lieber, possibly H. Lieber who wrote extensively on advanced elementary mathematics in the *fin de siècle*.

(iv) When $P = H$, the orthocenter, the Hagge circle degenerates to the point H.

(v) The circumcenter is the orthocenter of the medial triangle, and the Brocard circle on diameter OK arises as a Hagge circle of the medial triangle with respect to the centroid G of ABC.

Note that UH is the doubled Wallace-Simson line of D, by which we mean the enlargement of the Wallace-Simson line with scale factor 2 from center D. Similarly VH and WH are the doubled Wallace-Simson lines of E and F. Now it is well known that the angle between two Wallace-Simson lines is half the angle subtended at O by the generating points. This applies equally well to doubled Wallace-Simson lines. A careful analysis (taking care to distinguish between angles and their supplements) will yield the angles between UH, VH and WH, from which it can be deduced that UVW is indirectly similar to DEF. We will not explain the details but rather we present a robust argument for Proposition 2 which does not rely on scrupulous bookkeeping.

Incidentally, if P is on Γ, then the Hagge circle degenerates to the doubled Wallace-Simson line of P. For the rest of this paper, we make the explicit assumption that P is not on Γ. The work described in the rest of this introduction is not foreshadowed in [5]. Since $ABCDEFP$ is similar to $XYZUVWP$, it follows that ABC is indirectly similar to XYZ and the similarity sends DEF to UVW. The point P turns out to be the unique fixed point of this similarity. This similarity must carry a distinguished point H^+ on Γ to H. We will give a geometric recipe for locating H^+ in Proposition 3.

This process admits of extension both inwards and outwards. One may construct the Hagge circle of XYZ with respect to P, or find the triangle RST so that the Hagge circle of RST with respect to P is Γ (with ABC playing the former role of XYZ). The composition of two of these indirect similarities is an enlargement with positive scale factor from P.

Proposition 2 sheds light on some of our earlier work [3]. Let G be the centroid, K the symmedian point, and ω the Brocard angle of triangle ABC. Also, let J be the center of the orthocentroidal circle (the circle on diameter GH). We have long been intrigued by the fact that $\dfrac{OK^2}{R^2} = \dfrac{JK^2}{JG^2}$ since areal algebra can be used to show that each quantity is $1-3\tan^2\omega$. In §3.3 we will explain how the similarity is a geometric explanation of this suggestive algebraic coincidence. In [3] we showed how to construct the sides of (non-equilateral) triangle ABC given only the data O, G, K. The method was based on finding a cubic which had a^2, b^2, c^2 as roots. We will present an improved algebraic explanation in §3.2.

We show in Proposition 4 that there is a point F which when used as a cevian point, generates the same Hagge circle for every triangle in a Brocard porism. Thus the locus of the orthocenter in a Brocard porism must be confined to a circle. We describe its center and radius. We also exhibit a point which gives rise to a fixed Hagge circle with respect to the medial triangles, as the reference triangle ranges over a Brocard porism.

We make more observations about Hagge's configuration. Given the large number of points lying on conics (circles), it is not surprising that Pascal's hexagon theorem comes into play. Let VW meet AH at L, WU meet BH at M, and UV meets CH at N. In §4 we will show that $LMNP$ are collinear, and we introduce the term Hagge axis for this line.

In §5 we will exhibit a *midpoint conic* which passes through six points associated with the Hagge construction. In special case (iv), when $P = H$, this conic is the nine-point circle of ABC. Drawings lead us to conjecture that the center of the midpoint conic is N.

In §6 we study some natural loci associated with Hagge's construction.

2. The Hagge Similarity

We first locate the center of the Hagge circle, but not, as Peiser [8] did, by using complex numbers. A more leisurely exposition of the next result appears in [4].

Proposition 1. *Given a point P in the plane of triangle ABC, the center $h(P)$ of the Hagge circle associated with P is the point such the nine-point center N is the midpoint of $h(P)P^*$ where P^* denotes the isogonal conjugate of P.*

Proof. Let AP meet the circumcircle at D, and reflect D in BC to the point U. The line UH is the doubled Simson line of D, and the reflections of D in the other two sides are also on this line. The isogonal conjugate of D is well known to be the point at infinity in the direction parallel to AP^* . (This is the degenerate case of the result that if D' is not on the circumcircle, then the isogonal conjugate of D' is the center of the circumcircle of the triangle with vertices the reflections of D' in the sides of ABC).

Thus $UH \perp AP^*$. To finish the proof it suffices to show that if OU' is the rotation through π of UH about N, then AP^* is the perpendicular bisector of OU'. However, $AO = R$ so it is enough to show that $AU' = R$. Let A' denote the rotation through π of A about N. From the theory of the nine-point circle it follows that A' is also the reflection of O in BC. Therefore $OUDA'$ is an isosceles trapezium with $OA'//UD$. Therefore $AU' = A'U = OD = R$. □

We are now in a position to prove what we call the Hagge similarity which is the essence of the construction [5].

Proposition 2. *The triangle ABC has circumcircle Γ, circumcenter O and orthocenter H. Choose a point P in the plane of ABC other than A, B, C. The cevian lines AP, BP, CP meet Γ again at D, E, F respectively. Reflect D in BC to a point U, E in CA to a point V and F in AB to a point W. Let UP meet AH at X, VP meet BH at Y and WP meet CH at Z. The points $XYZUVWH$ are concyclic, and there is an indirect similarity carrying $ABCDEFP$ to $XYZUVWP$.*

Discussion. The strategy of the proof is as follows. We consider six lines meeting at a point. Any point of the plane will have reflections in the six lines which are concyclic. The angles between the lines will be arranged so that there is an indirect similarity carrying $ABCDEF$ to the reflections of H in the six lines. The location of the point of concurrency of the six lines will be chosen so that the relevant six reflections of H are $UVWX_1Y_1Z_1$ where X_1, Y_1 and Z_1 are to be determined, but are placed on the appropriate altitudes so that they are candidates to become X, Y and Z respectively. The similarity then ensures that UVW and $X_1Y_1Z_1$ are in perspective from a point P'. Finally we show that $P = P'$, and it follows immediately that $X = X_1$, $Y = Y_1$ and $Z = Z_1$. We rely on the fact that we know where to make the six lines cross, thanks to Proposition 1. This is not the proof given in [5].

Proof of Proposition 2. Let $\angle DAC = a_1$ and $\angle BAD = a_2$. Similarly we define b_1, b_2, c_1 and c_2. We deduce that the angles subtended by A, F, B, D, D and E at O as shown in Figure 2.

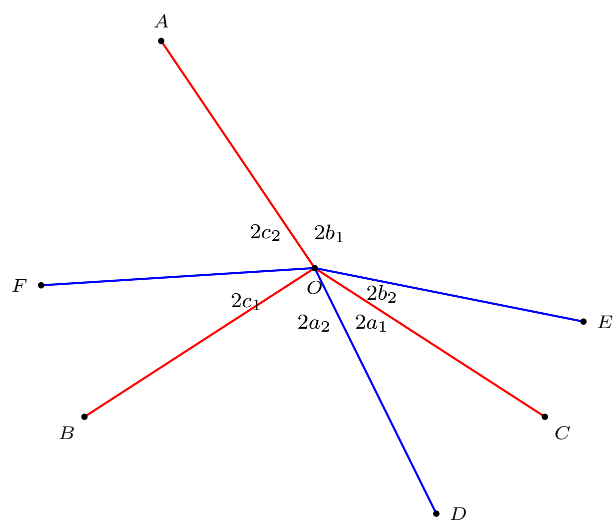

Figure 2. Angles subtended at the circumcenter of ABC

By Proposition 1, $h(P)$ is on the perpendicular bisector of UH which is parallel to AP^* (and similar results by cyclic change).

Draw three lines through $h(P)$ which are parallel to the sides of ABC and three more lines which are parallel to AP^* , BP^* and CP^* . See Figure 3.

Let X_1, Y_1 and Z_1 be the reflections of H in the lines parallel to BC, CA and AB respectively. Also U, V and W are the reflections of H in the lines parallel to AP^*, BP^* and CP^*. Thus $X_1Y_1Z_1UVW$ are all points on the Hagge circle. The angles between the lines are as shown, and the consequences for the six reflections of H are that $X_1Y_1Z_1UVW$ is a collection of points which are indirectly similar to $ABCDEF$. It is not necessary to know the location of H in Figure 3 to deduce this result. Just compare Figures 2 and 4. The point is that $\angle X_1h(P)V = \angle EOA$.

A similar argument works for each adjacent pair of vertices in the cyclic list $X_1VZ_1UY_1W$ and an indirect similarity is established. Let this similarity carrying

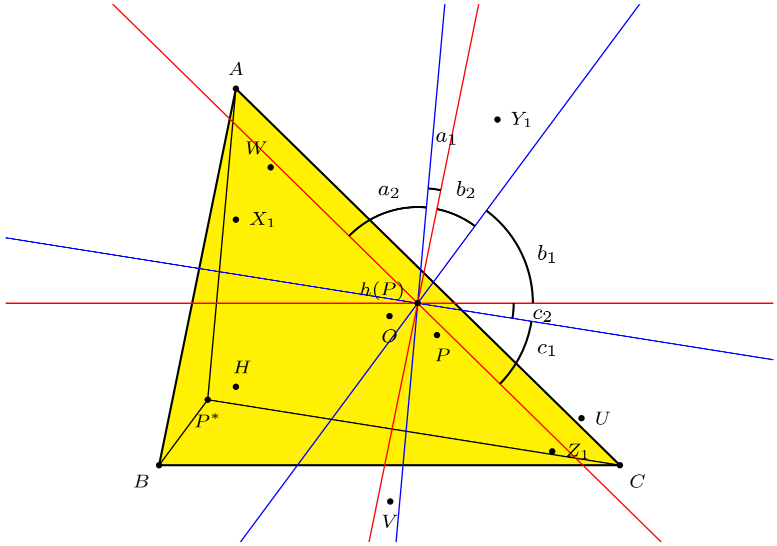

Figure 3. Reflections of the orthocenter

$ABCDEF$ to $X_1Y_1Z_1UVW$ be κ. It remains to show that $\kappa(P) = P$ (for then it will follow immediately that $X_1 = X$, $Y_1 = Y$ and $Z_1 = Z$).

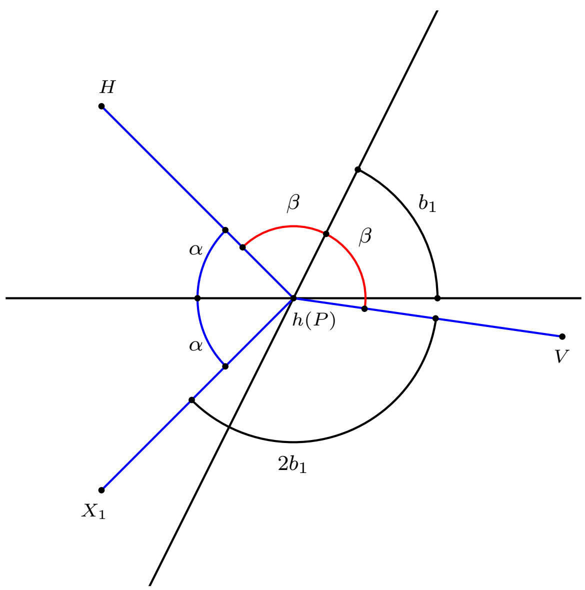

Figure 4. Two reflections of H

Now $X_1Y_1Z_1$ is similar to ABC, and the vertices of $X_1Y_1Z_1$ are on the altitudes of ABC. Also UVW is similar to DEF, and the lines X_1U, Y_1V and Z_1W are concurrent at a point P_1. Consider the directed line segments AD and X_1U which meet at Q. The lines AX_1 and UD are parallel so AX_1Q and DUQ are similar triangles, so in terms of lengths, $AQ : QD = X_1Q : QU$. Since κ carries AD to X_1U, it follows that Q is a fixed point of κ. Now if κ had at least two fixed points, then it would have a line of fixed points, and would be a reflection in that line. However κ takes DEF to UVW, to this line would have to be BC, CA and AB. This is absurd, so Q is the unique fixed point of κ. By cyclic change Q is on AD, BE and CF so $Q = P$. Also Q is on X_1U, Y_1V and Z_1W so $Q = P_1$. Thus X_1U, Y_1V and Z_1W concur at P. Therefore $X_1 = X$, $Y_1 = Y$ and $Z_1 = Z$. □

Proposition 3. *The similarity of Proposition 2 applied to ABC, P carries a point H^+ on Γ to H. The same result applied to XYZ, P carries H to the orthocenter H^- of XYZ. We may construct H^+ by drawing the ray PH^- to meet Γ at H^+.*

Proof. The similarity associated with ABC and P is expressible as: reflect in PA, scale by a factor of λ from P, and rotate about P through a certain angle. Note that if we repeat the process, constructing a similarity using the XYZ as the reference triangle, but still with cevian point P, the resulting similarity will be expressible as: reflect in XP, scale by a factor of λ from P, and rotate about P through a certain angle. Since $XYZP$ is indirectly similar to $ABCP$, the angles through which the rotation takes place are equal and opposite. The effect of composing the two similarities will be an enlargement with center P and (positive) scale factor λ^2. □

Thus in a natural example one would expect the point H^+ to be a natural point. Drawings indicate that when we consider the Brocard circle, H^+ is the Tarry point.

3. Implications for the Symmedian Point and Brocard geometry

3.1. *Standard formulas.* We first give a summary of useful formulas which can be found or derived from many sources, including Wolfram Mathworld [11]. The variables have their usual meanings.

$$abc = 4R\triangle, \tag{1}$$
$$a^2 + b^2 + c^2 = 4\triangle \cot\omega, \tag{2}$$
$$a^2b^2 + b^2c^2 + c^2a^2 = 4\triangle^2 \csc^2\omega, \tag{3}$$
$$a^4 + b^4 + c^4 = 8\triangle^2(\csc^2\omega - 2), \tag{4}$$

where (3) can be derived from the formula

$$\begin{aligned} R_B &= \frac{abc\sqrt{a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2}}{4(a^2 + b^2 + c^2)\triangle} \\ &= \frac{R\sqrt{1 - 4\sin^2\omega}}{2\cos\omega} \end{aligned}$$

for the radius R_B of the Brocard circle given in [11]. The square of the distance between the Brocard points was determined by Shail [9]:

$$\Omega\Omega'^2 = 4R^2\sin^2\omega(1 - 4\sin^2\omega) \tag{5}$$

which in turn is an economical way of expressing

$$\frac{a^2b^2c^2(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}{(a^2b^2+b^2c^2+c^2a^2)^2}.$$

We will use these formulas in impending algebraic manipulations.

3.2. *The symmedian point.* Let G be the centroid, K the symmedian point, and ω be the Brocard angle of triangle ABC. Also let J be the center of the orthocentroidal circle (the circle on diameter GH). It is an intriguing fact that

$$\frac{OK^2}{R^2}=\frac{JK^2}{JG^2} \tag{6}$$

since one can calculate that each quantity is $1-3\tan^2\omega$. The similarity of Proposition 2 explains this suggestive algebraic coincidence via the following paragraph.

We first elaborate on Remark (v) of §1. Let h_{med} denote the function which assigns to a point P the center $h_{\text{med}}(P)$ of the Hagge circle associated with P when the triangle of reference is the medial triangle. The medial triangle is the enlargement of ABC from G with scale factor $-\frac{1}{2}$. Let K_{med} be the symmedian point of the medial triangle. Now K_{med}, G, K are collinear and $K_{\text{med}}G : GK = 1 : 2 = QG : GN$, where Q is the midpoint of ON. Thus, triangle GNK and GQK_{med} are similar and Q is the nine-point center of the medial triangle. By [8], $h\text{med}(G)$ is the reflection in Q of K_{med}. But the line $Qh_{\text{med}}(G)$ is parallel to NK and Q is the midpoint of ON. Therefore, $h_{\text{med}}(G)$ is the midpoint of OK, and so is the center of the Brocard circle of ABC. The similarity of Proposition 2 and the one between the reference and medial triangle, serve to explain (6).

3.3. *The Brocard porism.* A Brocard porism is obtained in the following way. Take a triangle ABC and its circumcircle. Draw cevian lines through the symmedian point. There is a unique conic (the Brocard ellipse) which is tangent to the sides where the cevians cuts the sides. The Brocard points are the foci of the ellipse. There are infinitely many triangle with this circumcircle and this inconic. Indeed, every point of the circumcircle arises as a vertex of a unique such triangle.

These poristic triangles have the same circumcenter, symmedian point, Brocard points and Brocard angle. For each of them, the inconic is their Brocard ellipse. Any geometrical feature of the triangle which can be expressed exclusively in terms of R, ω and the locations of O and K will give rise to a conserved quantity among the poristic triangles.

This point of view also allows an improved version of the algebraic proof that a, b and c are determined by O, G and K [3]. Because of the ratios on the Euler line, the orthocenter H and the orthocentroidal center are determined. Now Equation (6) determines R and angle ω. However, $9R^2-(a^2+b^2+c^2)=OH^2$ so $a^2+b^2+c^2$ is determined. Also the area $\triangle$ of ABC is determined by (2). Now (1) means abc and so $a^2b^2c^2$ is determined. Also, (3) determines $a^2b^2+b^2c^2+c^2a^2$. Thus the polynomial $(X-a^2)(X-b^2)(X-c^2)$ is determined and so the sides of the triangle can be deduced.

As we move through triangles in a Brocard porism using a fixed cevian point P, the Hagge circles of the triangles vary in general, but if P is chosen appropriately, the Hagge circle if each triangle in the porism is the same.

Proposition 4. *Let F be the fourth power point* [2] *of a triangle in a Brocard porism, so that it has areal coordinates (a^4, b^4, c^4). The fourth power point F is the same point for all triangles in the porism. Moreover, when $P = F$, the Hagge circle of each triangle is the same.*

Proof. Our plan is to show that the point $h(F)$ is the same for all triangles in the porism, and then to show that the distance $h(F)H$ is also constant (though the orthocenters H vary). Recall that the nine-point center is the midpoint of O and H, and of F^* and $h(P)$. Thus there is a (variable) parallelogram $Oh(F)HF^*$ which will prove very useful.

The fourth power point F is well known to lie on the Brocard axis where the tangents to the Brocard circle at Ω and Ω' meet. Thus F is the same point for all triangles in the Brocard porism. The isogonal conjugate of F (incidentally the isotomic conjugate of the symmedian point) is $F^* = K_t = \left(\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}\right)$.

In any triangle OK is parallel to F^*H. To see this, note that OK has equation

$$b^2c^2(b^2 - c^2)x + c^2a^2(c^2 - a^2)y + a^2b^2(a^2 - b^2)z = 0.$$

Also F^*H has equation

$$\sum_{\text{cyclic}} b^2c^2(b^2 - c^2)(b^2 + c^2 - a^2)x = 0.$$

These equations are linearly dependent with $x + y + z = 0$ and hence the lines are parallel. (DERIVE confirms that the 3×3 determinant vanishes). In a Hagge circle with $P = F$, $P^* = F^*$ and $F^*Hh(F)O$ is a parallelogram. Thus OK is parallel to F^*H and because of the parallelogram, $h(F)$ is a (possibly variable) point on the Brocard axis OK.

Next we show that the point $h(F)$ us a common point for the poristic triangles. The first component of the normalized coordinates of F^* and H are

$$F^*_x = \frac{b^2c^2}{a^2b^2 + b^2c^2 + c^2a^2}$$

and

$$H_x = \frac{(a^2 + b^2 - c^2)(c^2 + a^2 - b^2)}{16\triangle^2}$$

where $\triangle$ is the area of the triangle in question. The components of the displacement F^*H are therefore

$$\frac{a^2 + b^2 + c^2}{16\triangle^2}(a^2b^2 + b^2c^2 + c^2a^2)(x, y, z)$$

[2] Geometers who speak trilinear rather than areal are apt to call F the third power point for obvious reasons.

where $x = a^2(a^2b^2 + a^2c^2 - b^4 - c^4)$, with y and z found by cyclic change of a, b, c. Using the areal distance formula this provides

$$F^*H^2 = \frac{a^2b^2c^2(a^2+b^2+c^2)^2(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}{16\triangle^2(a^2b^2+b^2c^2+c^2a^2)^2}.$$

Using the formulas of §3.1 we see that

$$Oh(F) = F^*H = 2R\cos\omega\sqrt{1-4\sin^2\omega}$$

is constant for the poristic triangles. The point O is fixed so there are just two candidates for the location of $h(F)$ on the common Brocard axis. By continuity $h(F)$ cannot move between these places and so $h(F)$ is a fixed point.

To finish this analysis we must show that the distance $h(F)H$ is constant for the poristic triangles. This distance is the same as F^*O by the parallelogram. If a point X has good areal coordinates, it is often easy to find a formula for OX^2 using the generalized parallel axis theorem [10] because $OX^2 = R^2 - \sigma_X^2$ and σ_X^2 denotes the mean square distance of the triangle vertices from themselves, given that they carry weights which are the corresponding areal coordinates of X.

In our case $F^* = (a^{-2},\ b^{-2},\ c^{-2})$, so

$$\begin{aligned}\sigma_{F^*}^2 &= \frac{1}{(a^{-2}+b^{-2}+c^{-2})^2}(a^2b^{-2}c^{-2}+a^{-2}b^2c^{-2}+a^{-2}b^{-2}c^2)\\ &= \frac{a^2b^2c^2}{a^2b^2+b^2c^2+c^2a^2}(a^4+b^4+c^4).\end{aligned}$$

This can be tidied using the standard formulas to show that $F^*O = R(1-4\sin^2\omega)$. The distance $Hh(F) = F^*O$ is constant for the poristic triangles and $h(F)$ is a fixed point, so the Hagge circle associated with F is the same for all the poristic triangles. □

Corollary 5. *In a Brocard porism, as the poristic triangles vary, the locus of their orthocenters is contained in a circle with their common center $h(F)$ on the Brocard axis, where F is the* (areal) *fourth power point of the triangles. The radius of this circle is $R(1-4\sin^2\omega)$.*

In fact there is a direct method to show that the locus of H in the Brocard porism is a subset of a circle, but this approach reveals neither center nor radius. We have already observed that $\dfrac{JK^2}{JG^2} = 1 - 3\tan^2\omega$ so for triangles in a Brocard porism (with common O and K) we have $\dfrac{JK^2}{JO^2} = \dfrac{1-3\tan^2\omega}{4}$ is constant. So as you consider the various triangle in the porism, J is constrained to move on a circle of Apollonius with center some point on the fixed line OK. Now the vector $\mathbf{OH}$ is $\frac{2}{3}\mathbf{OJ}$, so H is constrained to move on a circle with its center M on the line OK. In fact H can occupy any position on this circle but we do not need this result (which follows from K ranging over a circle center J for triangles in a Brocard porism [3]).

There is a point which, when used as P for the Hagge construction using medial triangles, gives rise to a common Hagge circle as we range over reference triangles in a Brocard porism. We use dashes to indicate the names of points with respect to the medial triangle $A'B'C'$ of a poristic triangle ABC. We now know that F is a common point for the porism, so the distance OF is fixed. Since O is fixed in the Brocard porism and the locus of H is a circle, it follows that the locus of N is a circle with center half way between O and the center of the locus of H.

Proposition 6. *Let P be the center of the Brocard ellipse (the midpoint of the segment joining the Brocard points of ABC). When the Hagge construction is made for the medial triangle $A'B'C'$ using this point P, then for each ABC in the porism, the Hagge circle is the same.*

Proof. If the areal coordinates of a point are (l, m, n) with respect to ABC, then the areal coordinates of this point with respect to the medial triangle are $(m+n-l, n+l-m, l+n-m)$. The reference areals of P are $(a^2(b^2+c^2), b^2(c^2+a^2), c^2(a^2+b^2))$ so the medial areals are (b^2c^2, c^2a^2, a^2b^2). The medial areals of the medial isogonal conjugate $P^\dagger$ of P are (a^4, b^4, c^4). Now the similarity carrying ABC to $A'B'C'$ takes O to N and F to $P^\dagger$. Thus in terms of distance $OF = 2P^\dagger N$ and moreover OF is parallel to $P^\dagger N$. Now, $OP^\dagger Nh'(P)$ is a parallelogram with center the nine-point center of the medial triangle and $h'(P)$ is the center of the medial Hagge circle. It follows that $h'(P)$ lies on OK at the midpoint of OF. Therefore all triangles in the Brocard porism give rise to a Hagge circle of P (with respect to the medial triangle) which is the circle diameter OF. □

Incidentally, P is the center of the locus of N in the Brocard porism. To see this, note that N is the midpoint of OH, so it suffices to show that $OP = PX$ where X is the center of the locus of H in the Brocard cycle (given that P is on the Brocard axis of ABC). However, it is well known that $OP = R\sqrt{1-4\sin^2\omega}$ and in Proposition 4 we showed that $OX = 2R\cos\omega\sqrt{1-4\sin^2\omega}$. We must eliminate the possibility that X and P are on different sides of O. If this happened, there would be at least one triangle for which $\angle HOK = \pi$. However, K is confined to the orthocentroidal disk [3] so this is impossible.

4. The Hagge axis

Proposition 7. *In the Hagge configuration, let VW meet AH at L, WU meet BH at M and UV meet CH and N. Then the points L, M, N and P are collinear.*

We prove the following more general result. In order to apply it, the letters should be interpreted in the usual manner for the Hagge configuration, and Σ should be taken as the Hagge circle.

Proposition 8. *Let three points X, Y and Z lie on a conic Σ and let l_1, l_2, l_3 be three chords XH, YH, ZH all passing through a point H on Σ. Suppose further that P is any point in the plane of Σ, and let XP, YP, ZP meet Σ again at U, V and W respectively. Now, let VW meet l_1 at L, WU meet l_2 at M, UV meet l_3 at N. Then LMN is a straight line passing through P.*

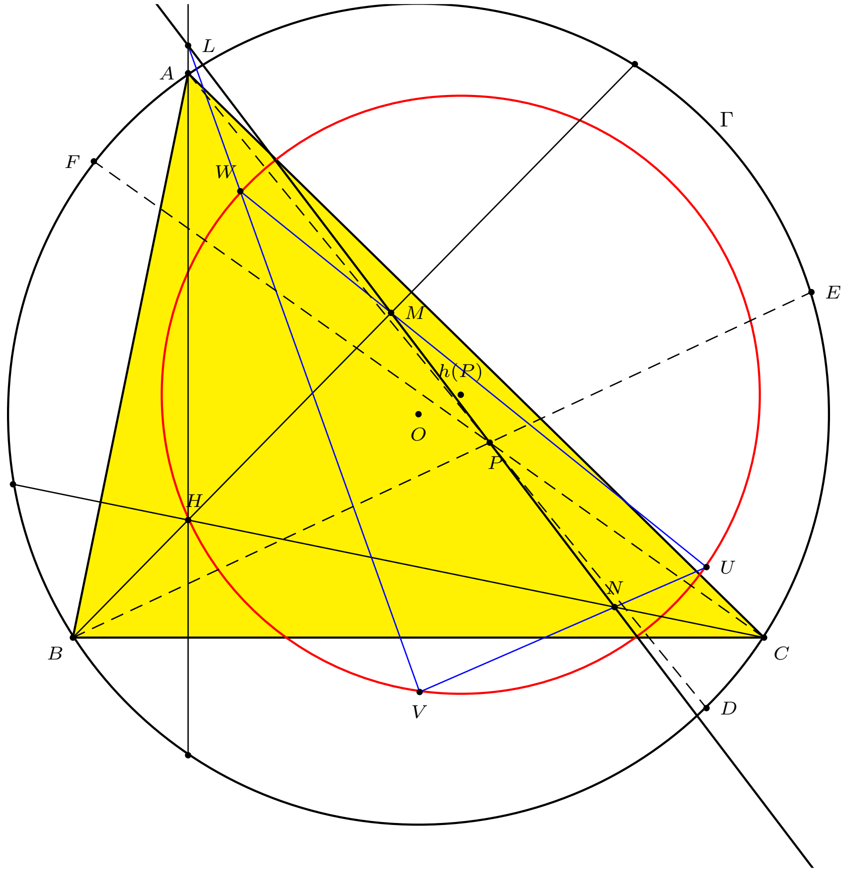

Figure 5. The Hagge axis LMN

Proof. Consider the hexagon $HYVUWZ$ inscribed in Σ. Apply Pascal's hexagon theorem. It follows that M, P, N are collinear. By taking another hexagon N, P, L are collinear. $\square$

5. The Hagge configuration and associated Conics

In this section we give an analysis of the Hagge configuration using barycentric (areal) coordinates. This is both an enterprise in its own right, serving to confirm the earlier synthetic work, but also reveals the existence of an interesting sequence of conics. In what follows ABC is the reference triangle and we take P to have homogeneous barycentric coordinates (u, v, w). The algebra computer package DERIVE is used throughout the calculations.

5.1. *The Hagge circle and the Hagge axis .* The equation of AP is $wy = vz$. This meets the circumcircle, with equation $a^2yz + b^2zx + c^2xy = 0$, at the point D with coordinates $(-a^2vw, v(b^2w + c^2v), w(b^2w + c^2v))$. Note that the sum of these coordinates is $-a^2vw + v(b^2w + c^2v) + w(b^2w + c^2v))$. We now want to find the coordinates of $U(l,m,n)$, the reflection of D in the side BC. It is convenient to take the normalization of D to be the same as that of U so that

$$l+m+n=-a^2vw+v(b^2w+c^2v)+w(b^2w+c^2v)). \tag{7}$$

In order that the midpoint of UD lies on BC the requirement is that $l=a^2vw$. There is also the condition that the displacements $BC(0,-1,1)$ and $UD(-a^2vw-l, v(b^2w+c^2v)-m, w(b^2w+c^2v)-n)$ should be at right angles. The condition for perpendicular displacements may be found in [1, p.180]. When these conditions are taken into account we find the coordinates of U are

$$(l,m,n)=(a^2vw, v(c^2(v+w)-a^2w), w(b^2(v+w)-a^2v)). \tag{8}$$

The coordinates of E, F, V, W can be obtained from those of D, U by cyclic permutations of a, b, c and u, v, w.

The Hagge circle is the circle through U, V, W and its equation, which may be obtained by standard means, is

$$\begin{aligned}&(a^2vw+b^2wu+c^2uv)(a^2yz+b^2zx+c^2xy)\\ &-(x+y+z)(a^2(b^2+c^2-a^2)vwx+b^2(c^2+a^2-b^2)wuy+c^2(a^2+b^2-c^2)uvz)\\ &=0.\end{aligned} \tag{9}$$

It may now be checked that this circle has the characteristic property of a Hagge circle that it passes through H, whose coordinates are

$$\left(\frac{1}{b^2+c^2-a^2}, \frac{1}{c^2+a^2-b^2}, \frac{1}{a^2+b^2-c^2}\right).$$

Now the equation of AH is $(c^2+a^2-b^2)y=(a^2+b^2-c^2)z$ and this meets the Hagge circle with Equation (9) again at the point X with coordinates $(-a^2vw+b^2wu+c^2uv, (a^2+b^2-c^2)vw, (c^2+a^2-b^2)vw)$. The coordinates of Y, Z can be obtained from those of X by cyclic permutations of a, b, c and u, v, w.

Proposition 9. *XU, YV, ZW are concurrent at P.*

This has already been proved in Proposition 2, but may be verified by checking that when the coordinates of X, U, P are placed as entries in the rows of a 3×3 determinant, then this determinant vanishes. This shows that X, U, P are collinear as are Y, V, P and Z, W, P.

If the equation of a conic is $lx^2+my^2+nz^2+2fyz+2gzx+2hxy=0$, then the first coordinate of its center is $(mn-gm-hn-f^2+fg+hf)$ and other coordinates are obtained by cyclic change of letters. This is because it is the pole of the line at infinity. The x-coordinate of the center $h(P)$ of the Hagge circle is therefore $-a^4(b^2+c^2-a^2)vw+(a^2(b^2+c^2)-(b^2-c^2)^2)(b^2wu+c^2uv)$ with y- and z-coordinates following by cyclic permutations of a, b, c and u, v, w.

In §4 we introduced the Hagge axis and we now deduce its equation. The lines VW and AH meet at the point L with coordinates

$$\begin{aligned}&(u(a^2(b^2w(u+v)(w+u-v)+c^2v(w+u)(u+v-w))+b^4w(u+v)(v+w-u)\\&\quad -b^2c^2(u^2(v+w)+u(v^2+w^2)+2vw(v+w))+c^4v(w+u)(v+w-u)),\\&vw(a^2+b^2-c^2)(a^2(u+v)(w+u)-u(b^2(u+v)+c^2(w+u))),\\&vw(c^2+a^2-b^2)(a^2(u+v)(w+u)-u(b^2(u+v)+c^2(w+u)))).\end{aligned}$$

The coordinates of M and N follow by cyclic permutations of a, b, c and u, v, w. From these we obtain the equation of the Hagge axis LMN as

$$\sum_{\text{cyclic}} vw(a^2(u+v)(w+u)-u(b^2(u+v)+c^2(w+u)))(a^2(v-w)-(b^2-c^2)(v+w))x = 0. \tag{10}$$

It may now be verified that this line passes through P.

5.2. *The midpoint Hagge conic* . We now obtain a dividend from the areal analysis in §5.1. The midpoints in question are those of AX, BY, CZ, DU, EV, FW and in Figure 6 these points are labeled X_1, Y_1, Z_1, U_1, V_1, W_1. This notation is not to be confused with the now discarded notation X_1, Y_1 and Z_1 of Proposition 2. We now show these six points lie on a conic.

Proposition 10. *The points X_1, Y_1, Z_1, U_1, V_1, W_1 lie on a conic (the Hagge midpoint conic).*

Their coordinates are easily obtained and are

$$\begin{aligned}&X_1\ (2u(b^2w+c^2v), vw(a^2+b^2-c^2), vw(c^2+a^2-b^2)),\\&U_1\ (0, v(2c^2v+w(b^2+c^2-a^2)), w(2b^2w+v(b^2+c^2-a^2))),\end{aligned}$$

with coordinates of Y_1, Z_1, V_1, W_1 following by cyclic change of letters. It may now be checked that these six points lie on the conic with equation

$$\begin{aligned}&4(a^2vw+b^2wu+c^2uv)\left(\sum_{\text{cyclic}} u^2(-a^2vw+b^2(v+w)w+c^2v(v+w))yz\right)\\&-(x+y+z)\left(\sum_{\text{cyclic}} v^2w^2((a^2+b^2-c^2)u+2a^2v)((c^2+a^2-b^2)u+2a^2w)x\right)=0.\end{aligned} \tag{11}$$

Following the same method as before for the center, we find that its coordinates are $(u(b^2w+c^2v), v(c^2u+a^2w), w(a^2v+b^2u))$.

Proposition 11. *U_1, X_1, P are collinear.*

This is proved by checking that when the coordinates of X_1, U_1, P are placed as entries in the rows of a 3×3 determinant, then this determinant vanishes. This shows that X_1, U_1, P are collinear as are Y_1, V_1, P and Z_1, W_1, P.

Proposition 12. *The center of the Hagge midpoint conic is the midpoint of $Oh(P)$. It divides P^*G in the ratio $3 : -1$.*

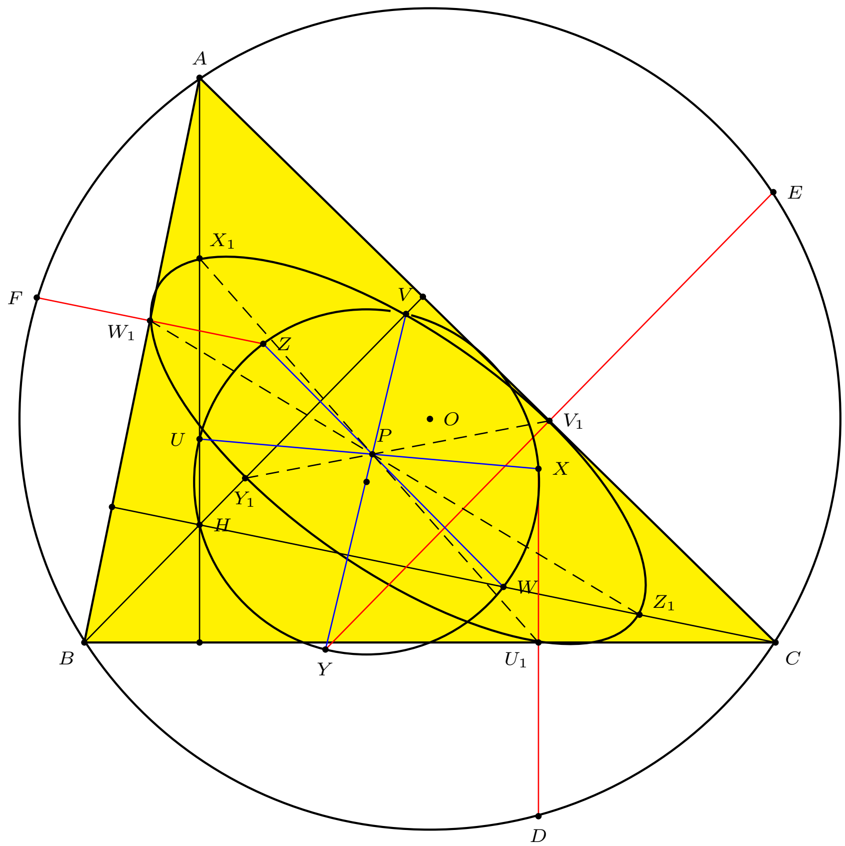

Figure 6.

The proof is straightforward and is left to the reader.

In similar fashion to above we define the six points X_k, Y_k, Z_k, U_k, V_k, W_k that divide the six lines AX, BY, CZ, DU, EV, FW respectively in the ratio $k : 1$ (k real and $\neq 1$).

Proposition 13. *The six points X_k, Y_k, Z_k, U_k, V_k, W_k lie on a conic and the centers of these conics, for all values of k, lie on the line $Oh(P)$ and divide it in the ratio $k : 1$.*

This proposition was originally conjectured by us on the basis of drawings by the geometry software package CABRI and we are grateful to the Editor for confirming the conjecture to be correct. We have rechecked his calculation and for the record the coordinates of X_k and U_k are

$$((1-k)a^2vw + (1+k)u(b^2w + c^2v), k(a^2 + b^2 - c^2)vw, k(c^2 + a^2 - b^2)vw),$$

and

$$(-a^2(1-k)vw, v((1+k)c^2v+(b^2+kc^2-ka^2)w), w((1+k)b^2w+(c^2+kb^2-ka^2)v)),$$

respectively. The conic involved has center with coordinates

$$((a^2(b^2+c^2-a^2)(a^2vw+b^2wu+c^2uv)$$
$$+k(-a^4(b^2+c^2-a^2)vw+(a^2(b^2+c^2)-(b^2-c^2)^2)(u(b^2w+c^2v)),$$
$$\cdots,\cdots).$$

Proposition 14. *U_k, X_k, P are collinear.*

The proof is by the same method as for Proposition 11.

6. Loci of Haggi circle centers

The Macbeath conic of ABC is the inconic with foci at the circumcenter O and the orthocenter H. The center of this conic is N, the nine-point center.

Proposition 15. *The locus of centers of those Hagge circles which are tangent to the circumcircle is the Macbeath conic.*

Proof. We address the elliptical case (see Figure 7) when ABC is acute and H is inside the circumcircle of radius R. The major axis of the Macbeath ellipse Σ is well known to have length R. Suppose that P is a point of the plane. Now $h(P)$ is on Σ if and only if $Oh(P)+h(P)H=R$, but $h(P)H$ is the radius of the Hagge circle, so this condition holds if and only if the Hagge circle is internally tangent to the circumcircle. Note that $h(P)$ is on Σ if and only if P^* is on Σ, and as P^* moves continuously round Σ, the Hagge circle moves around the inside of the circumcircle. The point P moved around the 'deltoid' shape as shown in Figure 7.

The case where ABC is obtuse and the Macbeath conic is a hyperbola is very similar. The associated Hagge circles are externally tangent to the circumcircle. $\square$

Proposition 16. *The locus of centers of those Hagge circles which cut the circumcircle at diametrically opposite points is a straight line perpendicular to the Euler line.*

Proof. Let ABC have circumcenter O and orthocenter H. Choose H' on HO produced so that $HO\cdot OH'=R^2$ where R is the circumradius of ABC. Now if X, Y are diametrically opposite points on S (but not on the Euler line), then the circumcircle S' of XYH is of interest. By the converse of the power of a point theorem, H' lies on each S'. These circles S' form an intersecting coaxal system through H and H' and their centers lie on the perpendicular bisector of HH'. $\square$

References

[1] C. J. Bradley, *Challenges in Geometry*, OUP, 2005.
[2] C. J. Bradley, *The Algebra of Geometry*, Highperception 2007.
[3] C. J. Bradley and G. C. Smith, The locations of triangle centers, *Forum Geom.*, 6 (2006) 57–70.
[4] C. J. Bradley and G. C. Smith, Hagge circles and isogonal conjugation, *Math. Gazette*, 91 (2007) 202–207.
[5] K. Hagge, Der Fuhrmannsche Kreis und der Brocardsche Kreis als Sonderfälle eines allgemeineren Kreises, *Zeitschrift für Math. Unterricht*, 38 (1907) 257–269.

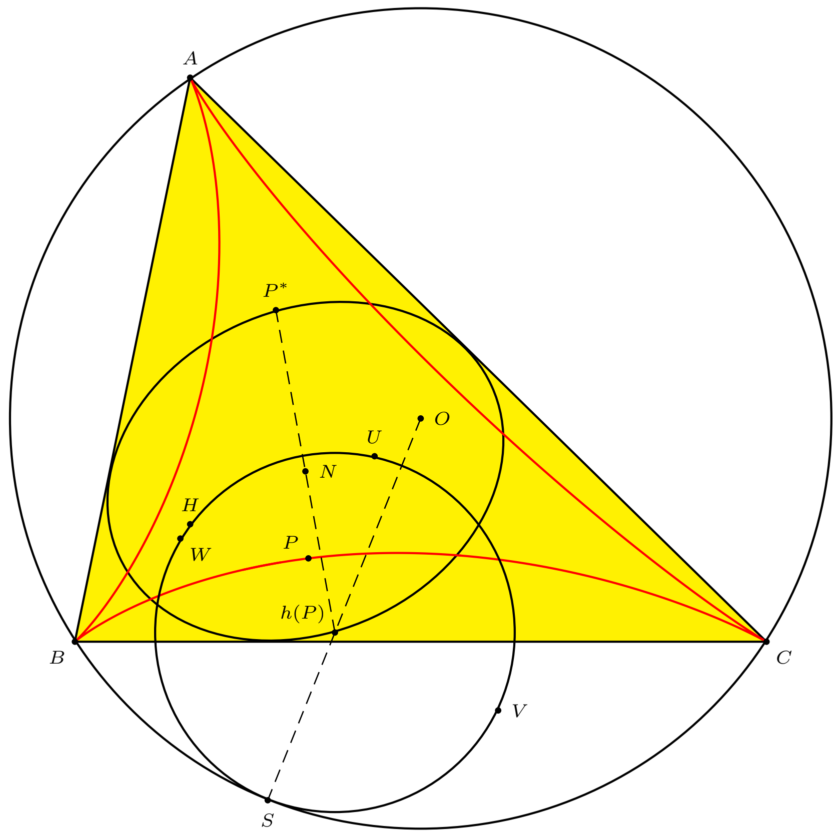

Figure 7.

[6] K. Hagge, Zur Theorie de Lemoineschen Kreise, *Zeitschrift für Math. Unterricht* 39 (1908) 337–341.
[7] R. A. Johnson, *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*, Boston, 1929; reprinted as *Advanced Euclidean Geometry*, Dover, 2007.
[8] A. M. Peiser, The Hagge circle of a triangle, *Amer. Math. Monthly*, 49 (1942) 524–527.
[9] R. Shail, Some properties of Brocard points, *Math. Gazette*. 80 (1996) 485–491.
[10] G. C. Smith, Statics and the moduli space of triangles, *Forum Geom.*, 5 (2005) 181–190.
[11] Wolfram Mathworld, http://mathworld.wolfram.com/

Christopher J. Bradley: c/o Geoff C. Smith, Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, England

Geoff C. Smith: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, England

E-mail address: G.C.Smith@bath.ac.uk