

A GENERALIZED POLYGAMMA FUNCTION

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We study the properties of a function $\psi(z, q)$ (the generalized polygamma function), intimately connected with the Hurwitz zeta function and defined for complex values of the variables z and q , which is entire in the variable z and reduces to the usual polygamma function $\psi^{(m)}(q)$ for z a non-negative integer m , and to the balanced negapolygamma function $\psi^{(-m)}(q)$ introduced in Ref. [5] for z a negative integer $-m$.

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1 INTRODUCTION

The Hurwitz zeta function defined by

$$\zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z} \quad (1.1)$$

for $z \in \mathbb{C}$, $\operatorname{Re} z > 1$ and $q \neq 0, -1, -2, \dots$ is a generalization of the Riemann zeta function $\zeta(z) = \zeta(z, 1)$. This function admits a meromorphic continuation into the whole complex plane. The only singularity is a simple pole at $z = 1$ with unit residue. The recent paper [9] presents a motivated discussion of this extension.

The Hurwitz zeta function turns out to be related to the classical gamma function, defined for $\operatorname{Re} q > 0$ by

$$\Gamma(q) = \int_0^{\infty} t^{q-1} e^{-t} dt, \quad (1.2)$$

in several different ways. For example, the *digamma* function

$$\psi(q) = \frac{d}{dq} \log \Gamma(q) \quad (1.3)$$

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appears in the Laurent expansion of $\zeta(z, q)$ at the pole $z = 1$:

$$\zeta(z, q) = \frac{1}{z-1} - \psi(q) + O(z-1). \quad (1.4)$$

A second connection among these functions is given by Lerch's identity

$$\zeta'(0, q) = \log \Gamma(q) + \zeta'(0) = \log \left(\frac{\Gamma(q)}{\sqrt{2\pi}} \right) \quad (1.5)$$

where we have used the classical value $\zeta'(0) = -\log \sqrt{2\pi}$ in the last step.

A third example is the relation between the Hurwitz zeta function and the *polygamma function* defined by

$$\psi^{(m)}(q) = \frac{d^m}{dq^m} \psi(q), \quad m \in \mathbb{N}, \quad (1.6)$$

namely

$$\psi^{(m)}(q) = (-1)^{m+1} m! \zeta(m+1, q). \quad (1.7)$$

These relations are not independent. Both (1.5) and (1.7) can be derived from (1.4), in the limiting case $z \rightarrow 1$, with the aid of the formula

$$\left(\frac{\partial}{\partial q} \right)^m \zeta(z, q) = (-1)^m (z)_m \zeta(z+m, q). \quad (1.8)$$

The digamma ($\psi(q) = \psi^{(0)}(q)$) and polygamma functions are analytic everywhere in the complex q -plane, except for poles (of order $m+1$) at all non-positive integers. The residues at these poles are all given by $(-1)^{m+1} m!$.

Extensions of the polygamma function $\psi^{(m)}(q)$ for m a negative integer have been defined by several authors [1, 6, 5]. These functions have been called *negapolygamma* functions. For example, Gosper [6] defined the family of functions

$$\begin{aligned} \psi_{-1}(q) &:= \log \Gamma(q), \\ \psi_{-k}(q) &:= \int_0^q \psi_{-k+1}(t) dt, \quad k \geq 2, \end{aligned} \quad (1.9)$$

which were later reconsidered by Adamchik [1] in the form

$$\psi_{-k}(q) = \frac{1}{(k-2)!} \int_0^q (q-t)^{k-2} \log \Gamma(t) dt, \quad k \geq 2. \quad (1.10)$$

These negapolygamma functions can be expressed in terms of the derivative (with respect to its first argument) of the Hurwitz zeta function at the negative integers [1, 6]. The definition of the negapolygamma functions in (1.9) can be modified by introducing arbitrary constants of integration at every step. This yields infinitely many different families of negapolygamma

functions, with the property that the corresponding members of any two families differ by a polynomial,

$$\psi_a^{(-m)}(q) - \psi_b^{(-m)}(q) = p_{m-1}(q),$$

where the functions $p_n(q)$ are polynomials in q of degree n , satisfying the property

$$p_n(q) = \frac{d}{dq} p_{n+1}(q).$$

An example of such modified negapolygamma functions has been introduced in Ref. [5], in connection with integrals involving the polygamma and the loggamma functions. These are the *balanced negapolygamma* functions, defined for $m \in \mathbb{N}$ by

$$\psi^{(-m)}(q) := \frac{1}{m!} [A_m(q) - H_{m-1} B_m(q)], \tag{1.11}$$

where $H_r := 1 + 1/2 + \dots + 1/r$ is the harmonic number ($H_0 := 0$), $B_m(q)$ is the m th Bernoulli polynomial, and the functions $A_m(q)$ are defined in terms of the Hurwitz zeta function as

$$A_m(q) := m\zeta'(1 - m, q). \tag{1.12}$$

A function $f(q)$ is defined to be *balanced* (on the unit interval) if it satisfies the two properties

$$\int_0^1 f(q) dq = 0 \quad \text{and} \quad f(0) = f(1).$$

Note that the Bernoulli polynomials, which are related to the Hurwitz zeta function in a way similar to (1.12),

$$B_m(q) = -m\zeta(1 - m, q), \quad m \in \mathbb{N}, \tag{1.13}$$

are themselves balanced functions. In Ref. [5] we have shown that the balanced negapolygamma functions (1.11) satisfy

$$\frac{d}{dq} \psi^{(-m)}(q) = \psi^{(-m+1)}(q), \quad m \in \mathbb{N}. \tag{1.14}$$

This makes them a negapolygamma family, connecting $\psi^{(-1)}(q) = \log \Gamma(q) + \zeta'(0)$ to the digamma function $\psi^{(0)}(q) = d \log \Gamma(q) / dq$.

The goal of this work is to introduce and study a meromorphic function of two complex variables, $\psi(z, q)$, the *generalized polygamma function*, that reduces to the polygamma function $\psi^{(m)}(q)$ for $z = m \in \mathbb{N}_0$ and to the balanced negapolygamma function $\psi^{(-m)}(q)$ for $z = -m \in -\mathbb{N}$. We describe some analytic properties of $\psi(z, q)$ and show they extend those of polygamma and balanced negapolygamma functions. We also present some definite integral formulas involving $\psi(z, q)$ in the integrand. Finally, we compare our generalized polygamma function with a different generalization introduced by Grossman [8].

2 THE GENERALIZED POLYGAMMA FUNCTION

The generalized polygamma function is defined by

$$\psi(z, q) := e^{-\gamma z} \frac{\partial}{\partial z} \left[e^{\gamma z} \frac{\zeta(z+1, q)}{\Gamma(-z)} \right], \quad (2.1)$$

where $z \in \mathbb{C}$ and $q \in \mathbb{C}$, $q \notin -\mathbb{N}_0$. At $z = m \in \mathbb{N}$, where $\Gamma(-z)$ has a pole, and at $z = 0$, where both $\Gamma(-z)$ and $\zeta(z+1, q)$ have poles, we define (2.1) by its corresponding limiting values given in the proof of Theorem 2.4. We show below that, for fixed q , $\psi(z, q)$ is indeed an *entire* function of z .

The alternative representation

$$\psi(z, q) = e^{-\gamma z} \frac{\partial}{\partial z} \left[\frac{e^{\gamma z}}{\Gamma(1-z)} \frac{\partial \zeta(z, q)}{\partial q} \right] \quad (2.2)$$

follows directly from (1.8).

LEMMA 2.1 *The function $\psi(z, q)$ is given by*

$$\psi(z, q) = \frac{1}{\Gamma(-z)} [\zeta'(z+1, q) + \{\gamma + \psi(-z)\} \zeta(z+1, q)] \quad (2.3)$$

and

$$\psi(z, q) = \frac{1}{\Gamma(-z)} [\zeta'(z+1, q) + H(-z-1) \zeta(z+1, q)], \quad (2.4)$$

where H is defined by

$$H(z) := \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z} \right). \quad (2.5)$$

Proof Differentiation of (2.1) yields (2.3). The second representation follows from the identity $H(z) = \gamma + \psi(z+1)$; see Ref. [7], for instance. ■

The function H can be termed the generalized harmonic number function. It has simple poles with residue -1 at all negative integers, and reduces to the n th harmonic number H_k for $z = k \in \mathbb{N}_0$. It satisfies the following reflection formula:

$$H(-z) = H(z-1) + \pi \cot \pi z. \quad (2.6)$$

We show first that, for $m \in \mathbb{N}$, $\psi(-m, q)$ reduces to the balanced negapolygamma function $\psi^{(-m)}(q)$ defined in (1.11).

THEOREM 2.2 *For $m \in \mathbb{N}$, $\psi(-m, q) = \psi^{(-m)}(q)$.*

Proof Lemma 2.1 gives

$$\psi(-m, q) = \frac{1}{\Gamma(m)} [\zeta'(1-m, q) + H_{m-1} \zeta(1-m, q)]. \quad (2.7)$$

The result now follows from (1.11)–(1.13). ■

We show next that the generalized polygamma function has no singularities in the complex z plane and that $\psi(0, q)$ is actually the digamma function $\psi(q)$.

THEOREM 2.3 *For fixed $q \in \mathbb{C}$, the function $\psi(z, q)$ is an entire function of z . Moreover $\psi(0, q) = \psi(q)$.*

Proof In the representation (2.4), the term $1/\Gamma(z)$ is entire and $\zeta(z, q)$ has only a simple pole at $z = 1$ and is analytic for $z \neq 1$. Thus $z = 0$ is the only possible singularity for $\psi(z, q)$. This singularity is removable because for z near 0

$$\begin{aligned} \frac{\zeta'(z+1, q)}{\Gamma(-z)} &= \left(-\frac{1}{z^2} + O(z) \right) (-z + \gamma z^2 + O(z^3)) \\ &= \frac{1}{z} - \gamma + O(z) \end{aligned}$$

and

$$\frac{H(-z-1) \zeta(z+1, q)}{\Gamma(-z)} = -\frac{1}{z} + \gamma + \psi(q) + O(z),$$

so that $\psi(z, q) = \psi(q) + O(z)$. ■

We finally show that, for $m \in \mathbb{N}$, $\psi(m, q)$ reduces to the polygamma function $\psi^{(m)}(q)$ defined in (1.6).

THEOREM 2.4 *The function $\psi(z, q)$ satisfies*

$$\frac{\partial}{\partial q} \psi(z, q) = \psi(z+1, q) \quad (2.8)$$

and

$$\psi(m, q) = \psi^{(m)}(q), \quad m \in \mathbb{N}. \quad (2.9)$$

Proof Use (1.8) to produce

$$\frac{\partial}{\partial q} \psi(z, q) = -e^{-\gamma z} \frac{\partial}{\partial z} \left[e^{\gamma z} \frac{(z+1) \zeta(z+2, q)}{\Gamma(-z)} \right]$$

and then use $\Gamma(-z) = -(z+1)\Gamma(-z-1)$ to obtain (2.8).

The identity (2.9) follows by induction from Theorem 2.3 and (2.8), but we provide an alternative proof. Set $z = m + \varepsilon$ and consider (2.4) as $\varepsilon \rightarrow 0$. The expansions

$$\frac{1}{\Gamma(-m - \varepsilon)} = (-1)^{m+1} m! \varepsilon + O(\varepsilon^2) \quad \text{and} \quad H(-1 - m - \varepsilon) = \frac{1}{\varepsilon} + H_m + O(\varepsilon)$$

are the only terms that produce a nonvanishing contribution in (2.4) as $\varepsilon \rightarrow 0$. We conclude that $\psi(m, q) = (-1)^{m+1} m! \zeta(m + 1, q)$ and the result follows from (1.7). ■

3 FUNCTIONAL RELATIONS

The generalized polygamma function $\psi(z, q)$, as a function of q , satisfies some simple algebraic and analytic relations. These are derived from those of $\Gamma(z)$ and $\zeta(z, q)$.

THEOREM 3.1 *The function $\psi(z, q)$ satisfies*

$$\psi(z, q + 1) = \psi(z, q) + \frac{\ln q - H(-z - 1)}{q^{z+1} \Gamma(-z)}. \quad (3.1)$$

Proof The identity

$$\zeta(z, q + 1) = \zeta(z, q) - \frac{1}{q^z}, \quad (3.2)$$

produces

$$\psi(z, q + 1) = \psi(z, q) - e^{-\gamma z} \frac{\partial}{\partial z} \left[e^{\gamma z} \frac{1}{q^{z+1} \Gamma(-z)} \right].$$

The result now follows from $\gamma + \psi(-z) = H(-z - 1)$. ■

Relation (3.1) generalizes the well-known functional relations for the digamma and polygamma functions,

$$\psi(q + 1) = \psi(q) + \frac{1}{q}, \quad (3.3)$$

$$\psi^{(m)}(q + 1) = \psi^{(m)}(q) + \frac{(-1)^m m!}{q^{m+1}}, \quad (3.4)$$

and the corresponding relation

$$\psi^{(-m)}(q + 1) = \psi^{(-m)}(q) + \frac{q^{m-1}}{(m-1)!} [\ln q - H_{m-1}] \quad (3.5)$$

for the balanced negapolygamma function [5].

Note We have been unable to find a generalization of the other well-known functional relation for the polygamma function,

$$(-1)^m \psi^{(m)}(1 - q) = \psi^{(m)}(q) + \frac{d^m}{dq^m} \pi \cot \pi q. \quad (3.6)$$

The next result establishes a multiplication formula for $\psi(z, q)$. It generalizes the analogous result for the digamma function, [7, (8.365.6)].

THEOREM 3.2 *Let $k \in \mathbb{N}$. Then,*

$$\begin{aligned} k^{z+1}\psi(z, kq) &= \sum_{j=0}^{k-1} \psi\left(z, q + \frac{j}{k}\right) - k^{z+1} \ln k \frac{\zeta(z+1, kq)}{\Gamma(-z)} \\ &= \sum_{j=0}^{k-1} \left[\psi\left(z, q + \frac{j}{k}\right) - \frac{\ln k}{\Gamma(-z)} \zeta\left(z+1, q + \frac{j}{k}\right) \right]. \end{aligned} \quad (3.7)$$

Proof Use the multiplication rule

$$k^z \zeta(z, kq) = \sum_{j=0}^{k-1} \zeta\left(z, q + \frac{j}{k}\right) \quad (3.8)$$

for the Hurwitz zeta function in the Definition (2.1) of $\psi(z, q)$. ■

The case $k = 2$ yields the duplication formula

$$\psi(z, 2q) = \frac{1}{2^{z+1}} \left[\psi(z, q) + \psi\left(z, q + \frac{1}{2}\right) \right] - \ln 2 \frac{\zeta(z+1, 2q)}{\Gamma(-z)}. \quad (3.9)$$

4 SERIES EXPANSIONS OF $\psi(z, q)$

In this section, we present two different series expansions for the generalized polygamma function. The first is a generalization of the well-known expansion of the digamma function,

$$\psi(q+1) = -\gamma + \sum_{k=1}^{\infty} (-1)^{k+1} \zeta(k+1) q^k, \quad |q| < 1. \quad (4.1)$$

THEOREM 4.1 *Let $z \in \mathbb{C}$ and $|q| < 1$. Then*

$$\psi(z, q+1) = \sum_{k=0}^{\infty} \psi(z+k, 1) \frac{q^k}{k!}. \quad (4.2)$$

Proof The Taylor expansion of $\psi(z, q+1)$ around $q = 0$ can be expressed in terms of $\psi(z, q)$ using the iterated version of (2.8),

$$\frac{\partial^k}{\partial q^k} \psi(z, q) = \psi(z+k, q) \quad (4.3)$$

evaluated at $q = 1$. The radius of convergence is computed to be 1 by using the ratio test, the identity

$$\psi(z, 1) = \frac{1}{\Gamma(-z)} [\zeta'(z+1) + H(-z-1)\zeta(z+1)], \quad (4.4)$$

and the fact that $\zeta'(z+1)$ tends to zero faster than the term $H(-z-1)\zeta(z+1)$ as $z \rightarrow \infty$. \blacksquare

Note 4.2 The series expansion (4.1) for the digamma function is the special case $z = 0$ of (4.2). This follows from the values

$$\psi(0, 1) = \psi(1) = -\gamma \quad (4.5)$$

and

$$\psi(k, 1) = \psi^{(k)}(1) = (-1)^{k+1} k! \zeta(k+1), \quad (4.6)$$

for $k \in \mathbb{N}$. Similarly $z = -1$ and the value $\psi(-1, 1) = \zeta'(0)$ yield the well-known expansion of the loggamma function,

$$\log \Gamma(q+1) = -\gamma q + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} q^k, \quad |q| < 1. \quad (4.7)$$

Note 4.3 Riemann's functional equation,

$$\zeta(1-z) = \frac{\zeta(z)(2\pi)^{1-z}}{2\Gamma(1-z)\sin(\pi z/2)} = 2\cos\left(\frac{\pi z}{2}\right) \frac{\zeta(z)\Gamma(z)}{(2\pi)^z}, \quad (4.8)$$

yields the alternate representation

$$\psi(z, 1) = 2(2\pi)^z \cos\left(\frac{\pi z}{2}\right) \left[\left(\gamma + \ln 2\pi - \frac{\pi}{2} \tan \frac{\pi z}{2} \right) \zeta(-z) - \zeta'(-z) \right]. \quad (4.9)$$

Note 4.4 Theorems 3.1 and 4.1 determine the behaviour of $\psi(z, q)$ for small q :

$$\psi(z, q) = -\frac{1}{\Gamma(-z)} \frac{\ln q}{q^{z+1}} + \frac{H(-z-1)}{\Gamma(-z)} \frac{1}{q^{z+1}} + \psi(z, 1) + \psi(z+1, 1)q + O(q^2). \quad (4.10)$$

For $z = m \in \mathbb{N}_0$ the coefficients of the first two terms are

$$\frac{1}{\Gamma(-m)} = 0 \quad \text{and} \quad \frac{H(-m-1)}{\Gamma(-m)} = (-1)^{m+1} m!,$$

so the logarithmic term drops out and we recover the known behaviour of the polygamma function as $q \rightarrow 0$,

$$\psi^{(m)}(q) = \frac{(-1)^{m+1} m!}{q^{m+1}} + \psi^{(m)}(1) + O(q). \quad (4.11)$$

For $z \notin \mathbb{N}$ with $\operatorname{Re} z \geq -1$ the first term in (4.10) determines the leading behaviour, and if $\operatorname{Re} z < -1$ the first two terms in (4.10) vanish as $q \rightarrow 0$ and hence $\psi(z, q)$ tends to the finite value $\psi(z, 1)$ given by (4.4) or (4.9).

We now establish a Fourier series representation for the generalized polygamma function $\psi(z, q)$.

THEOREM 4.5 For $\operatorname{Re} z < -1$ and $0 \leq q \leq 1$:

$$\psi(z, q) = 2(2\pi)^z \left[\sum_{n=1}^{\infty} n^z (\gamma + \ln 2\pi n) \cos\left(2\pi nq + \frac{\pi z}{2}\right) - \frac{\pi}{2} \sum_{n=1}^{\infty} n^z \sin\left(2\pi nq + \frac{\pi z}{2}\right) \right]. \quad (4.12)$$

This result generalizes the Fourier expansion for the balanced negapolygamma function given in Ref. [5]. It implies that $\psi(z, q)$ is itself balanced for any z such that $\operatorname{Re} z < -1$.

Proof Let $s = z + 1$ in the Fourier representation of the Hurwitz zeta function,

$$\zeta(s, q) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \left[\sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi qn)}{n^{1-s}} + \cos\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi qn)}{n^{1-s}} \right], \quad (4.13)$$

which is valid for $\operatorname{Re} s < 0$ and $0 \leq q \leq 1$, and substitute (4.13) into (2.1). ■

5 INTEGRAL REPRESENTATIONS OF $\psi(z, q)$

This section contains integral representations for $\psi(z, q)$ that are derived directly from corresponding integral representations of the Hurwitz zeta function. For instance,

$$\zeta(z, q) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{e^{-qt}}{1-e^{-t}} t^{z-1} dt, \quad (5.1)$$

valid for $\operatorname{Re} z > 1$ and $\operatorname{Re} q > 0$, implies the next result.

THEOREM 5.1 Let $\operatorname{Re} z > 0$ and $\operatorname{Re} q > 0$. Then

$$\psi(z, q) = - \int_0^{\infty} \frac{e^{-qt} t^z}{1-e^{-t}} \left[\cos \pi z + \frac{\gamma}{\pi} \sin \pi z + \frac{\sin \pi z}{\pi} \ln t \right] dt. \quad (5.2)$$

Proof The identity

$$\frac{\zeta(z+1, q)}{\Gamma(-z)} = - \frac{\sin \pi z}{\pi} \int_0^{\infty} \frac{e^{-qt}}{1-e^{-t}} t^z dt \quad (5.3)$$

follows from the integral representation for $\zeta(z, q)$ in (5.1) and the reflection rule for the gamma function. The result now follows from the definition of $\psi(z, q)$. ■

A *Hankel-type* contour is a curve that starts at $+\infty + i0+$, moves to the left on the upper half-plane, encircles the origin once in the positive direction, and returns to $+\infty - i0+$ on the lower half-plane. The Hurwitz zeta function has the following integral representation along a Hankel-type contour [12]:

$$\frac{\zeta(z+1, q)}{\Gamma(-z)} = - \frac{1}{2\pi i} \int_{\infty}^{(0+)} \frac{e^{-qt}}{1-e^{-t}} (-t)^z dt, \quad (5.4)$$

valid for arbitrary complex z and $\operatorname{Re} q > 0$.

THEOREM 5.2 *Let $q, z \in \mathbb{C}$ with $\operatorname{Re} q > 0$. Then*

$$\psi(z, q) = -\frac{1}{2\pi i} \int_{\infty}^{(0+)} \frac{[\gamma + \ln(-t)]e^{-qt}}{1 - e^{-t}} (-t)^z dt. \quad (5.5)$$

Proof The result follows directly from (5.4).

6 DEFINITE INTEGRALS INVOLVING $\psi(z, q)$

Definite integrals of $\psi(z, a + bq)$ can be directly obtained from its primitive,

$$\int \psi(z, a + bq) dq = b^{-1} \psi(z - 1, a + bq), \quad (6.1)$$

according to (2.8). So, for example,

$$\int_0^1 \psi(z, q) dq = \begin{cases} 0, & \text{if } \operatorname{Re} z < 0, \\ \infty, & \text{if } \operatorname{Re} z \geq 0, \end{cases} \quad (6.2)$$

where we have used the result (4.10) to evaluate $\psi(z, q)$ at the origin.

The integral formulas presented below are direct consequences of the corresponding integral formulas for the Hurwitz zeta function. Several of these were derived in Refs. [4, 5].

THEOREM 6.1 *For $\operatorname{Re} z, \operatorname{Re} z' < 0$ and $\operatorname{Re}(z + z') < -1$,*

$$\begin{aligned} \int_0^1 \psi(z, q)\psi(z', q) dq &= 2(2\pi)^{z+z'} \cos \frac{\pi(z-z')}{2} \left\{ \left[\frac{\pi^2}{4} + (\gamma + \ln 2\pi)^2 \right] \zeta(-z-z') \right. \\ &\quad \left. - 2(\gamma + \ln 2\pi) \zeta'(-z-z') + \zeta''(-z-z') \right\}. \end{aligned} \quad (6.3)$$

Proof This is a direct consequence of the following result [4],

$$\int_0^1 \zeta(s, q)\zeta(s', q) dq = \frac{2\Gamma(1-s)\Gamma(1-s')}{(2\pi)^{2-s-s'}} \zeta(2-s-s') \cos \frac{\pi(s-s')}{2},$$

valid for $\operatorname{Re} s < 1, \operatorname{Re} s' < 1, \operatorname{Re}(s + s') < 1$. Set $s = z + 1, s' = z' + 1$, divide by $\Gamma(-z) \times \Gamma(-z')$, and construct the functions $\psi(z, q), \psi(z', q)$ in the integrand according to definition (2.1). \blacksquare

The evaluation (6.3) generalizes Example 5.6 in Ref. [5]: for $k, k' \in \mathbb{N}$,

$$\begin{aligned} \int_0^1 \psi^{(-k)}(q)\psi^{(-k')}(q) dq &= \frac{2\cos(k-k')\pi/2}{(2\pi)^{k+k'}} \left[\zeta''(k+k') - 2(\gamma + \ln 2\pi) \zeta'(k+k') \right. \\ &\quad \left. + \left\{ (\gamma + \ln 2\pi)^2 + \frac{\pi^2}{4} \right\} \zeta(k+k') \right]. \end{aligned}$$

The special case $k = k' = 1$ reduces to

$$\int_0^1 (\ln \Gamma(q))^2 dq = \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{1}{3} \gamma \ln \sqrt{2\pi} + \frac{4}{3} \ln^2 \sqrt{2\pi} - (\gamma + 2 \ln \sqrt{2\pi}) \frac{\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2},$$

given in Ref. [4].

COROLLARY 6.2 *Let $\operatorname{Re} z < -1/2$. Then*

$$\int_0^1 \psi(z, q)^2 dq = 2(2\pi)^{2z} \left\{ \left[\frac{\pi^2}{4} + (\gamma + \ln 2\pi)^2 \right] \zeta(-2z) - 2(\gamma + \ln 2\pi) \zeta'(-2z) + \zeta''(-2z) \right\}. \quad (6.4)$$

For $\operatorname{Re} z < -1$,

$$\int_0^1 \psi(z, q) \psi(z+1, q) dq = 0. \quad (6.5)$$

The same type of argument gives the next evaluation.

THEOREM 6.3 *For $\operatorname{Re} z, \operatorname{Re} z' < 0$, and $\operatorname{Re}(z+z') < -1$,*

$$\int_0^1 \zeta(z+1, q) \psi(z', q) dq = 2(2\pi)^{z+z'} \Gamma(-z) \left\{ \frac{\pi}{2} \zeta(-z-z') \sin \frac{\pi}{2}(z-z') + [(\gamma + \ln 2\pi) \zeta(-z-z') - \zeta'(-z-z')] \cos \frac{\pi}{2}(z-z') \right\}. \quad (6.6)$$

COROLLARY 6.4 *For $\operatorname{Re} z < 0$,*

$$\int_0^1 \zeta(z, q) \psi(z, q) dq = -\frac{1}{2} (2\pi)^{2z} \Gamma(1-z) \zeta(1-2z). \quad (6.7)$$

Our final evaluation computes the Mellin transform of the generalized polygamma function.

THEOREM 6.5 *Let $a, b \in \mathbb{R}^+$, $\alpha, z \in \mathbb{C}$ such that $0 < \operatorname{Re} \alpha < \operatorname{Re} z$. Then*

$$\int_0^\infty q^{\alpha-1} \psi(z, a+bq) dq = \frac{b^{-\alpha} \Gamma(\alpha)}{\sin \pi(z-\alpha)} [(\sin \pi z) \psi(z-\alpha, a) + (\sin \pi \alpha) \Gamma(z+1-\alpha) \zeta(z+1-\alpha, a)]. \quad (6.8)$$

Proof Start from formula (2.3.1.1) of Ref. [10],

$$\int_0^\infty q^{\alpha-1} \zeta(s, a+bq) dq = b^{-\alpha} B(\alpha, s-\alpha) \zeta(s-\alpha, a),$$

valid for $a, b \in \mathbb{R}^+$ and $0 < \operatorname{Re}(\alpha) < \operatorname{Re}(s) - 1$, set $s = z + 1$, and use the Definition (2.1) of $\psi(z, q)$ to evaluate the integral as

$$-\frac{b^{-\alpha} \Gamma(\alpha)}{\pi} e^{-\gamma z} \frac{\partial}{\partial z} [e^{\gamma z} (\sin \pi z) \Gamma(z+1-\alpha) \zeta(z+1-\alpha, a)].$$

The desired evaluation now follows from the reflection formulas for the gamma and digamma functions,

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin \pi x} \quad \text{and} \quad \psi(1-x) = \psi(x) + \pi \cot \pi x,$$

respectively. ■

Note 6.6 The special case $z = m \in \mathbb{N}$ in Theorem 6.5 yields an explicit form for the Mellin transform of the polygamma function $\psi^{(m)}(a + bq)$:

$$\int_0^\infty q^{z-1} \psi^{(m)}(a + bq) dq = (-1)^{m+1} b^{-z} \Gamma(\alpha) \Gamma(1 + m - \alpha) \zeta(1 + m - \alpha, a), \quad (6.9)$$

valid when $0 < \operatorname{Re} \alpha < m$ and $a, b \in \mathbb{R}^+$. This formula generalizes formula (6.473) of Ref. [7] to the case $a, b \neq 1$.

7 RELATION TO GROSSMAN'S GENERALIZATION OF THE POLYGAMMA FUNCTION

In 1975, N. Grossman presented a generalization of polygamma functions to arbitrary complex orders [8] which is different to ours. He was motivated by a problem proposed a year earlier by B. Ross [11] concerning the convergence and evaluation of the integral

$$I = \int_0^q (q-t)^{p-1} \log \Gamma(t) dt. \quad (7.1)$$

For $p \in \mathbb{N}$, this integral corresponds precisely (up to a normalization factor) to the Gosper-Adamchik's negapolygamma functions defined by (1.10). In Ref. [8] the author used the techniques of Liouville's fractional integration and differentiation to obtain a generalization $\psi^{(v)}(q)$ of the polygamma function, with $v \in \mathbb{C}$, in the form

$$\psi^{(v)}(q) = \frac{q^{-v-1}}{\Gamma(-v)} \left\{ \ln \frac{1}{q} + \gamma + \frac{\Gamma'(-v)}{\Gamma(-v)} \right\} - \frac{\gamma q^{-v}}{\Gamma(1-v)} - \frac{q^{-v-1}}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} q^z \frac{\Gamma(z)\zeta(z)}{\Gamma(z-v)} \frac{\pi}{\sin \pi z} dz, \quad (7.2)$$

where the contour of integration is along a vertical line with $1 < \lambda < 2$. The function $\psi^{(v)}(q)$ is an entire function in the v -plane, for each q in the plane cut along the negative real axis [8].

For $v = -m \in -\mathbb{N}_0$, Grossman's generalized polygamma $\psi^{(v)}(q)$ reduces to the Gosper-Adamchik negapolygamma functions $\psi_{-m}(q)$. We showed in Ref. [5] that the latter are related to the balanced negapolygammas $\psi^{(-m)}(q)$ by

$$\psi^{(-m)}(q) = \psi_{-m}(q) + \sum_{r=0}^{m-1} \frac{q^{m-r-1}}{r!(m-r-1)!} [\zeta'(-r) + H_r \zeta(-r)], \quad (7.3)$$

which, in light of (4.4), can be also expressed as

$$\psi^{(-m)}(q) = \psi_{-m}(q) + \sum_{r=0}^{m-1} \frac{q^{m-r-1}}{\Gamma(m-r)} \psi(-r-1, 1). \tag{7.4}$$

In the remainder of this section, we shall explore the relation between the functions $\psi(v, q)$ and $\psi^{(v)}(q)$ for arbitrary values of the complex variable v . Since both of these functions are entire in v , their difference

$$\Psi(v, q) := \psi(v, q) - \psi^{(v)}(q) \tag{7.5}$$

must be an entire function itself. Furthermore, since both $\psi(v, q)$ and $\psi^{(v)}(q)$ reduce to the standard polygamma function when $v \in \mathbb{N}_0$, $\Psi(v, q)$ vanishes identically at $v \in \mathbb{N}_0$. In order to study further properties of the function $\Psi(v, q)$, we shall consider the asymptotic and small- q series expansions of both $\psi(v, q)$ and $\psi^{(v)}(q)$. First, we shall derive the correct asymptotic expansion of Grossman’s polygamma for large q , since this was incorrectly given in Ref. [8]. Let

$$I(v, q) = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} q^z \frac{\Gamma(z)\zeta(z)}{\Gamma(z-v)} \frac{\pi}{\sin \pi z} dz. \tag{7.6}$$

As suggested in Ref. [8], for $q > 1$ we can deform the contour so that it starts at $-\infty - i0+$, runs below the real axis, encircles the point $z = 1$ in the positive sense (crossing the real axis to the left of $z = 2$), and then returns to $-\infty + i0+$ running over the real axis. $I(v, q)$ can then be evaluated along the deformed contour by a residue calculation. The only relevant poles are $z = 1$ and $z = 0, -1, -2, \dots$, coming from $\Gamma(z)$, $\zeta(z)$, and from the zeros of $\sin(\pi z)$. The poles at $z = -2k$ are simple since $\zeta(-2k) = 0$. All the other poles are double. Let $R_\nu(z_0)$ be the residue at the pole $z = z_0$. Then

$$\begin{aligned} R_\nu(1) &= \frac{-q \ln q + q\psi(1-v)}{\Gamma(1-v)}, \\ R_\nu(0) &= \frac{H(-1-v) - \ln 2\pi - \ln q}{2\Gamma(-v)}, \\ R_\nu(-m) &= \frac{1}{m! \Gamma(-m-v) q^m} \left[\zeta'(-m) - \frac{B_{m+1}}{m+1} \{ \ln q + H_m - H(-m-v-1) \} \right], \end{aligned}$$

for $m = 1, 2, 3, \dots$. Using the special values $B_0 = 1$, $B_1 = -1/2$, $\zeta'(0) = -(1/2)\ln 2\pi$, and $H_0 = 0$, we obtain the asymptotic expansion

$$\begin{aligned} \psi^{(v)}(q) \sim q^{-v} & \left\{ \ln q \sum_{k=0}^{\infty} \frac{B_k}{k! \Gamma(1-v-k) q^k} - \sum_{k=1}^{\infty} \frac{k \zeta'(1-k) - B_k H_{k-1}}{k! \Gamma(1-v-k) q^k} \right. \\ & \left. - \sum_{k=0}^{\infty} \frac{B_k H(-k-v)}{k! \Gamma(1-v-k) q^k} \right\}. \end{aligned} \tag{7.7}$$

We observe that Grossman [8] missed most of the logarithmic contribution.

The asymptotic expansion of the generalized polygamma function $\psi(v, q)$ for large q can be obtained from (2.1) and the asymptotic expansion of $\zeta(z, q)$ itself. This yields

$$\psi(v, q) \sim q^{-v} \left\{ \ln q \frac{\sin \pi v}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{B_k \Gamma(k+v)}{k! q^k} - \cos \pi v \sum_{k=0}^{\infty} (-1)^k \frac{B_k \Gamma(k+v)}{k! q^k} - \frac{\sin \pi v}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{B_k H(k+v-1) \Gamma(k+v)}{k! q^k} \right\}.$$

The reflection formula for $\Gamma(z)$ yields

$$\frac{\sin \pi v}{\pi} (-1)^k \Gamma(k+v) = \frac{1}{\Gamma(1-v-k)},$$

and the reflection formula (2.6) for the harmonic number function produces

$$H(k+v-1) = H(-k-v) - \pi \cot \pi v.$$

Thus

$$\psi(v, q) \sim \ln q \sum_{k=0}^{\infty} \frac{B_k}{k! \Gamma(1-v-k) q^{k+v}} - \sum_{k=0}^{\infty} \frac{B_k H(-v-k)}{k! \Gamma(1-v-k) q^{k+v}}. \tag{7.9}$$

We obtain therefore the following asymptotic expansion for the function $\Psi(v, q)$ defined by (7.5):

$$\Psi(v, q) \sim \sum_{k=1}^{\infty} \frac{\psi(-k, 1)}{\Gamma(1-v-k) q^{k+v}}. \tag{7.10}$$

We note that for $v = m \in \mathbb{N}_0$ the formal series on the right-hand side vanishes identically, as it should. For $v = -m \in -\mathbb{N}$, the series above reduces to a polynomial in q , which coincides with the one appearing in (7.4).

On the other hand, for $|q| < 1$, Grossman has proven that his polygamma function has the convergent expansion¹

$$\psi^{(v)}(q) = \frac{q^{-v-1}}{\Gamma(-v)} \left\{ -\ln q + \gamma + \psi(-v) + \frac{\gamma q}{v} + \sum_{k=2}^{\infty} (-1)^k \zeta(k) B(-v, k) q^k \right\}, \tag{7.11}$$

which, on account of the special values for $\psi(z, 1)$ at the non-negative integers given in (4.5) and (4.6), can be written as

$$\psi^{(v)}(q) = \frac{-\ln q + H(-v-1)}{q^{v+1} \Gamma(-v)} + \sum_{k=0}^{\infty} \frac{\psi(k, 1)}{\Gamma(-v+k+1)} q^{k-v}. \tag{7.12}$$

¹There is actually an error in the expression given in Ref. [8].

This is to be compared with the small- q expansion of the generalized polygamma function $\psi(v, q)$ obtained in Theorems 3.1 and 4.1:

$$\psi(v, q) = \frac{-\ln q + H(-v - 1)}{q^{v+1}\Gamma(-v)} + \sum_{k=0}^{\infty} \frac{\psi(k + v, 1)}{\Gamma(k + 1)} q^k. \quad (7.13)$$

Again, both expansions coincide if $v \in \mathbb{N}_0$, since $1/\Gamma(z)$ vanishes at the non-positive integers, and differ by the polynomial in (7.4) if $v = -m \in -\mathbb{N}$.

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