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Open Problems and Conjectures

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In this section, we present some open problems and conjectures about some interesting types of difference equations. Please submit your problems and conjectures with all relevant information to G. Ladas.

The N-Number Ducci Game

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The Ducci map has engaged the mathematics community for over a century and long-standing questions remain open regarding the map's dynamics. This article introduces the Ducci map acting on the vector spaces $\mathbb{Z}_2^n$ and $\mathbb{R}^n$. Open questions on the transient and cyclic behavior of the map are posed.

Keywords: Ducci; N-number; Finite fields; Iterations; Weightings

In the late 1800s, E. Ducci studied iterations of the map $\tilde{D} : \mathbb{Z}^n \to \mathbb{Z}^n$,

$$\tilde{D}_n(\mathbf{x}) = \left(|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_n - x_1|\right) \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ [7]. It was found that iterates of Eq. (1) converge in finite time to strings of the form $k(x_1, \ldots, x_n)$, where $x_i \in \mathbb{Z}_2$, $i = 1, \ldots n$ and k is a positive integer [9], thus the dynamics of forward iterates of the map can be understood on the vector space $\mathbb{Z}_2^n$. Considered over $\mathbb{Z}_2^n$, the Ducci map becomes linear:

$$D_n(\mathbf{x}) = (x_1 + x_2, x_2 + x_3, \ldots, x_n + x_1)$$

where the addition is modulo 2.

The behavior of D_n and $\tilde{D}_n$ have been examined for special cases of n; see Refs. [5,8,10]. In addition, many interesting results have been developed for arbitrary n; see Refs. [1–3,6]. We first pose some open problems concerning D_n, then for more general maps.

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THE NUMBER OF CYCLES

Since D_n is considered over a finite set, the iterates of D_n must eventually cycle. Most of the work on iterates of Eq. (1) has focused on understanding the maximal cycle for arbitrary values of n, and its length as a function of n.

It has been known that one can produce submaximal cycles for composite n in the following manner. Suppose that $n = ds$ for positive integers d, $s > 1$, and that $y = (y_1, y_2, \ldots, y_d)$ belongs to a cycle of length c_d in $\mathbb{Z}_2^d$. Then the vector $\mathbf{x} = (y, y, \ldots, y)$ formed by copying y for a number of s times is also in a cycle of length c_d in $\mathbb{Z}_2^n$. This method of embedding cycle lengths was noted by Breuer [2].

Table I (from [3]) shows that this is not the only way to obtain submaximal cycles. Notice that $n = 17$ has a submaximal cycle of length 85 (the maximal cycle length is 255). Since 17 is prime, the submaximal cycle is not produced from a divisor. This fact leads to the following open problems:

Open Problem 1 *How and when do submaximal cycles occur when not produced by divisors?*

Open Problem 2 *How fast do the number of cycles with different lengths grow asymptotically for large n?*

Clearly, the embedding by divisors implies the growth is bounded below by the function $n - \phi(n)$ where ϕ is the Euler totient function. An answer to Question 1 will likely give one to Question 2.

MAXIMAL CYCLE LENGTHS AS A FUNCTION OF *n*

It is desirable to obtain the maximal cycle length c as a function of n. Ehrlich obtained divisibility conditions in Ref. [6] which provide some direction to this goal. Let n be odd

TABLE I Period lengths under iterations of D_n[3]

Vector length	*Number of cycles of different lengths*	*Cycle lengths*	*Vector length*	*Number of cycles of different lengths*	*Cycle lengths*
$n = 3$	2	1,3	$n = 22$	3	1,341,682
$n = 4$	1	1	$n = 23$	2	1,2047
$n = 5$	2	1,15	$n = 24$	5	1,3,6,12,24
$n = 6$	3	1,3,6	$n = 25$	3	1,15,25575
$n = 7$	2	1,7	$n = 26$	3	1,819,1638
$n = 8$	1	1	$n = 27$	4	1,3,63,13797
$n = 9$	3	1,3,63	$n = 28$	4	1,7,14,28
$n = 10$	3	1,15,30	$n = 29$	2	475107
$n = 11$	2	1,341	$n = 30$	7	1,3,5,6,10,15,30
$n = 12$	4	1,3,6,12	$n = 31$	2	1,31
$n = 13$	2	1,819	$n = 32$	1	1
$n = 14$	3	1,7,14	n = 33	4	1,3,341,1023
$n = 15$	4	1,3,5,15	$n = 34$	5	1,85,170,255,510
$n = 16$	1	1	$n = 35$	6	1,7,15,105,819,4095
$n = 17$	3	1,85,255	$n = 36$	7	1,3,6,12,63,126,252
$n = 18$	5	1,3,6,63,126	$n = 37$	2	1,3233097
$n = 19$	2	1,9709	$n = 38$	3	1,9709,19418
$n = 20$	4	1,15,30,60	$n = 39$	6	1,3,455,819,1365,4095
$n = 21$	5	1,3,7,21,63	$n = 40$	5	1,15,30,60,120

and define $c_1 = 2^j - 1$ where j is the order of 2 modulo n. If $n|2^l + 1$ for some l, then let $m = \min\{l : n|2^l + 1\}$ and define $c_2 = n(2^m - 1)$. Note that the existence of c_1 is always guaranteed by Euler's Theorem. Ehrlich proved that $c|c_1$ and $c|c_2$ if c_2 exists.

Although it seemed in most cases $c = c_2$ when c_2 existed and c_1 otherwise, Ehrlich provided four examples to show that c does not necessarily have to equal c_1 or c_2; namely $n = 37, 95, 101$ and 111. Obviously, the maximal period in these cases was a proper common divisor of c_1 and c_2 and is also a multiple of n. These results lead to the following open questions related to D_n:

Open Problem 3 *For what values of n will the length of the maximal cycle c not be equal to c_1 or c_2?*

Open Problem 4 *If $c \neq c_1, c_2$, then by Ehrlich's result, c must be a proper common divisor of c_1, c_2. What is the exact form of c in this case?*

GENERAL DUCCI MAPS

Besides the map $\tilde{D}$, Chamberland [4] has considered similar maps with different "weightings". Most of the work in Ref. [4] concerns the weighting $(-1, \underline{2}, -1)$ which corresponds to the map

$$f(x_1, x_2, \ldots, x_n) = (|2x_1 - x_2 - x_n|, |2x_2 - x_3 - x_1|, \ldots, |2x_n - x_1 - x_{n-1}|)$$

Similar to the case for the map $\tilde{D}$, when n is not a power of two, there exists a string whose forward iterates do not converge to the zero string. In a rather complicated proof it was shown for the case $n = 2^2$ all initial strings converge to the zero string. However, counter-examples were developed showing that this does not hold for $n = 2^3$ and $n = 2^4$. Specifically, we have

$$f^{(24)}(1, 2, 3, 0, 1, 0, 1, 2) = 2^8(1, 2, 3, 0, 1, 0, 1, 2)$$
$$f^{(240)}(2, 1, 1, 1, 0, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2, 1) = 2^{40}(2, 1, 1, 1, 0, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2, 1)$$

and hence the forward iterates of these two strings diverge.

Moreover, it was also found that when considering the map over the reals, the 2^2-string

$$S := \left(1, \frac{1+\sqrt{5}}{2}, 2+\sqrt{5}, \frac{1+\sqrt{5}}{2}\right)$$

iterates to $(\sqrt{5} - 1)S$, and hence diverges. This leads to the interesting phenomena that while all rational 2^2-strings iterate to the zero string, an arbitrarily close real 2^2-string diverges. Chamberland also gives a concrete theorem on the set of divergent 3-strings [4].

One may consider other weightings, such as $(\underline{w_1}, w_2, \ldots, w_p)$, acting on strings of length at least p, defined as

$$f(x_1, x_2, \ldots, x_n) = \big(|w_1x_1 + w_2x_2 + \ldots + w_px_p|,$$
$$|w_1x_2 + w_2x_3 + \ldots + w_px_{p+1}|, \ldots, |w_1x_n + w_2x_1 + \ldots + w_px_{p-1}|\big)$$

The underlined term in the weighting may be moved; it simply indicates the location of the weight's components applied to each term in the string. The weighting $(\underline{1}, -1)$ is *bounded* because any string's largest term (in magnitude) will not increase in size. Another example of a bounded weighting is $(1, 0, \underline{0}, -1)$. The dynamics of integer strings under bounded weightings with integer weights must eventually cycle.

Many questions surrounding general weightings are wide open:

OPEN PROBLEM 5 *What are the dynamics of bounded integer weightings besides* $(\underline{1}, -1)$*?*

OPEN PROBLEM 6 *Are there unbounded weightings with a sum of zero which have only the zero-string as a cycle?*

OPEN PROBLEM 7 *What are the dynamics of weightings with rational terms? real terms?*

OPEN PROBLEM 8 *What are the dynamics of weightings with infinitely many terms?*

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