AUTOMORPHISMS OF THE COMPLEX NUMBERS

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One of the best known bits of mathematical folklore is that there are many automorphisms of the field of complex numbers, i.e., that the complex numbers can be permuted in many ways (besides the familiar complex conjugation) that preserve addition and multiplication. As evidence that it is folklore we point out that it appears without proof in a popular projective geometry text [6] as well as an undergraduate algebra text [5]. This expository paper is devoted to a proof of this bit of folklore. The average mathematician is vaguely aware that the "wild" automorphisms of the complex number system probably require for their construction the axiom of choice or some equivalent assumption about sets. In our existence proof for wild automorphisms we illustrate a typical application of Zorn's lemma; moreover, we present evidence (Theorem 4) that wild automorphisms are so wild that an assumption such as Zorn's lemma or the axiom of choice seems to be essential.

1. Subfields of the field of complex numbers. Any field considered in this paper will be a subfield of the complex numbers, **C**, i.e., will be a subset of **C** containing 0, 1 and containing $a+b$, ab, $a-b$, and (if $b\neq 0$) a/b whenever it contains a and b. Familiar examples of subfields are **Q**, the field of rational numbers; **R**, the field of real numbers; and **C** itself. It is easy to show that the intersection of any collection of subfields of **C** is itself a subfield of **C**, and, in particular, that the intersection of all subfields is **Q**.

DEFINITIONS. Let **F** be a subfield of **C** and let $\alpha, \beta, \cdots, \lambda$ be complex numbers. The intersection of all subfields of **C** containing **F** and $\alpha, \beta, \cdots, \lambda$ is denoted by $\mathbf{F}(\alpha, \beta, \cdots, \lambda)$ and is called the *extension field* of **F** *generated* by $\alpha, \beta, \cdots, \lambda$. The numbers $\alpha, \beta, \cdots, \lambda$ are called *generators*. If **G** is a subfield of **C** containing **F** such that $\mathbf{G}=\mathbf{F}(\alpha, \beta, \cdots, \lambda)$ for some finite set of generators, then we say that **G** is a *finitely generated extension* of **F**. If only one generator is required then we say that **G** is a *simple extension* of **F**.

A complex number, α, is called *algebraic* or *transcendental* over **F** according as it does or does not satisfy at least one polynomial equation with coefficients in **F**. If α is algebraic over **F** then the (unique!) monic polynomial, p, of least degree with coefficients in **F** such that $p(\alpha)=0$ is called the *minimal polynomial* of α over **F**.

The structure of a simple extension, $\mathbf{F}(\alpha)$, of **F** depends on the "algebraic relationship" between α and **F**. If there is none, i.e., if α is transcendental over **F**, then distinct rational forms, $p(x)/q(x)$ (p, q polynomials with coefficients in **F**) yield distinct complex numbers, $p(\alpha)/q(\alpha)$, all in $\mathbf{F}(\alpha)$. However if α is algebraic over **F**, and if m is the degree of its minimal polynomial over **F**, then one can show that $q(\alpha)\neq 0$ implies $p(\alpha)/q(\alpha)=a_0+a_1\alpha+\cdots+a_{m-1}\alpha^{m-1}$ for exactly one set of $a_i\in\mathbf{F}$. In any case $\mathbf{F}(\alpha)=\{p(\alpha)/q(\alpha)\mid p, q \text{ polynomials with coefficients in } \mathbf{F} \text{ and } q(\alpha)\neq 0\}$. For details see [2], Chapter 14, Theorems 1 and 4.

DEFINITIONS. A subfield, $\mathbf{F}$, of $\mathbf{C}$ is said to be *algebraically closed* if every complex number algebraic over $\mathbf{F}$ is in $\mathbf{F}$, or, equivalently, if every polynomial with coefficients in $\mathbf{F}$ can be factored into linear factors with coefficients in $\mathbf{F}$. We shall denote by $\mathbf{F}^a$ the set of all complex numbers which are algebraic over $\mathbf{F}$. $\mathbf{F}^a$ is called the *algebraic closure* of $\mathbf{F}$ in $\mathbf{C}$.

THEOREM 1. *Let* $\mathbf{F}$ *be a subfield of* $\mathbf{C}$. $\mathbf{F}^a$ *is a subfield of* $\mathbf{C}$ *that is algebraically closed.*

There are two "standard" proofs of this theorem. In the first proof one shows that any rational combination of numbers algebraic over $\mathbf{F}$ is itself algebraic over $\mathbf{F}$. To see that this is not easy the reader should try to prove that $\alpha+\beta$ and $\alpha\beta$ are algebraic over $\mathbf{F}$ whenever α and β are. The second proof depends on the fact that a field is also a vector space over any of its subfields. The two main lemmas for this proof are a multiplicative property of dimensions for extension fields and the fact that α is algebraic over $\mathbf{F}$ if and only if $\mathbf{F}(\alpha)$ is a *finite dimensional* vector space over $\mathbf{F}$. Both proofs are in [3].

2. Isomorphisms between fields. Roughly speaking an isomorphism between fields is a one to one correspondence between the elements of the two fields which "preserves" algebraic operations. Since we shall be concerned with isomorphisms between subfields of the complex number system and because we plan to apply Zorn's lemma to sets of isomorphisms we choose the following specialized definition of isomorphism.

DEFINITIONS. An *isomorphism* (between subfields of $\mathbf{C}$) is a set, ϕ, of ordered pairs of complex numbers such that: 1. If $\langle a, x\rangle$ and $\langle b, y\rangle$ are in ϕ, i.e., if $\phi(a)=x$ and $\phi(b)=y$, then $a=b$ if and only if $x=y$. (In other words, ϕ is a function and is one to one.) 2. If $\langle a, x\rangle$ and $\langle b, y\rangle$ belong to ϕ then so do $\langle a+b, x+y\rangle$, $\langle ab, xy\rangle$, $\langle a-b, x-y\rangle$, and (if b and y are nonzero) $\langle a/b, x/y\rangle$. (ϕ preserves algebraic operations.) 3. $\langle 0, 0\rangle$ and $\langle 1, 1\rangle$ are both in ϕ. (This assures that ϕ is not trivial in the sense of being empty or containing only $\langle 0, 0\rangle$.)

As is customary we call the set of all first components of ordered pairs in an isomorphism ϕ the *domain* of ϕ, and the set of all second components the *range* of ϕ. Also, as usual, we write $\phi(a)=x$ and say ϕ *sends* a to x if and only if $\langle a, x\rangle \in\phi$. It is easy to show that the domain and range of an isomorphism are subfields of $\mathbf{C}$. If the domain and range are the same field, $\mathbf{F}$, then we say that ϕ is an *automorphism* of $\mathbf{F}$. Clearly the identity map on a subfield $\mathbf{F}$, $I_F=\{\langle x, x\rangle \mid x \in\mathbf{F}\}$, is an automorphism of $\mathbf{F}$. This is called the *trivial* automorphism of $\mathbf{F}$. All other automorphisms of $\mathbf{F}$ are called *nontrivial.*

Let ϕ and σ be two isomorphisms. We say that ϕ *extends* σ if σ is a subset of ϕ. If, in addition, the domain of ϕ is $\mathbf{F}$ then we say that ϕ *extends* σ *to* $\mathbf{F}$.

Caution! Our use of the word isomorphism is very restricted in that we allow only complex numbers in the domain and range. The usual definition of "isomorphism" (between arbitrary fields) is more complicated in that not only are more general "numbers" allowed but also the operations in the two fields involved may be different. The reader who is familiar with a different definition

of isomorphism should prove that if all fields involved are subfields of **C** then the definition above is equivalent to his "standard" definition.

Examples of isomorphisms. The most familiar example of an isomorphism is complex conjugation, $\{\langle a+bi, a-bi\rangle \mid a, b \in \mathbf{R}\}$, which is a nontrivial automorphism of **C**. Slightly more complicated examples are $\sigma = \{\langle a+c\sqrt{7}, a-c\sqrt{7}\rangle \mid a, c \in \mathbf{Q}\}$ and $\psi = \{\langle a+b\sqrt[4]{7}+c\sqrt{7}+d\sqrt[4]{343},\ a+ib\sqrt[4]{7}-c\sqrt{7}-id\sqrt[4]{343}\rangle \mid a,\ b,\ c,\ d \in \mathbf{Q}\}$. The reader should verify that σ is an automorphism of $\mathbf{Q}(\sqrt{7})$, that ψ extends σ to $\mathbf{Q}(\sqrt[4]{7})$, and that the range of ψ is $\mathbf{Q}(i\sqrt[4]{7})$.

THEOREM 2. *Any isomorphism between subfields of* **C** *extends* $I_{\mathbf{Q}}$, *the identity map on* **Q**.

Proof. Let ϕ be an isomorphism and let $\mathbf{F} = \{a \mid \phi(a) = a\} = \{a \mid \langle a, a\rangle \in \phi\}$. It is easy to show that **F** is a subfield of **C**. Since **Q** is contained in any subfield, ϕ must extend $I_{\mathbf{Q}}$.

This result asserts that the field of rational numbers is an "algebraically rigid" structure. This is not too surprising since 0 and 1 can be characterized algebraically, (0 as the only solution of $x+x=x$ and 1 as the only nonzero solution of $xx=x$) and all other rational numbers are built up by rational operations from 0 and 1. Thus if a function is to preserve algebraic properties it should leave the rational numbers undisturbed. A similar (but more surprising) result is valid for the field of real numbers.

THEOREM 3. *The only isomorphisms between subfields of* **C** *whose domains include* **R** *and which map* **R** *into* **R** *are* $I_{\mathbf{R}}$, $I_{\mathbf{C}}$, *and complex conjugation.*

Proof. Let ϕ be such an isomorphism, i.e., assume $\mathbf{R} \subseteq$ domain ϕ and $x \in \mathbf{R}$ implies $\phi(x) \in \mathbf{R}$. We first show that ϕ preserves order in **R**. If $x<y$, then there is a real number w such that $w \neq 0$ and $y-x=w^2$. But then $\phi(y)-\phi(x) = [\phi(w)]^2$ with $\phi(w) \in \mathbf{R}$ and $\phi(w) \neq 0$. Hence $\phi(y)-\phi(x)$ is positive, i.e., $\phi(x) < \phi(y)$. Now assume $a \in \mathbf{R}$, but that $a \neq \phi(a)$. Choose a rational number, q, between a and $\phi(a)$. Since $\phi(q) = q$ by Theorem 2, the order between a and q is reversed by ϕ and we have a contradiction. Hence $a \in \mathbf{R}$ implies $\phi(a) = a$, i.e., $I_{\mathbf{R}} \subseteq \phi$.

If $\phi \neq I_{\mathbf{R}}$, then the domain of ϕ is a subfield of **C** containing **R** as a proper subset. In any such subfield we can find a complex number, $a+bi$, with $b \neq 0$ as well as all real numbers. But, since $x+yi = x+y([(a+bi)-a]/b)$, this implies that the subfield is **C** itself. Thus the domain of ϕ is **C**. Consider $\phi(i)$. Since $i^2 = -1$, $[\phi(i)]^2 = \phi(-1) = -1$. The only roots of $x^2 = -1$ are $\pm i$; hence $\phi(i) = \pm i$. If $\phi(i) = i$, then $\phi = I_{\mathbf{C}}$, and if $\phi(i) = -i$, then ϕ is complex conjugation.

Theorem 2 implies that **Q** has no nontrivial automorphism, and Theorem 3 implies the same for **R**. Theorem 3 also implies that a nontrivial automorphism of a subfield of **R** cannot be extended to an automorphism of **R**. For example, the automorphism σ defined just before Theorem 2 cannot be extended to an automorphism of **R**.

We shall call any automorphism of **C** which is not $I_{\mathbf{C}}$ nor complex conjugation a *wild automorphism* of **C**. That these automorphisms are really "wild" is shown by the following theorem.

THEOREM 4. *If ϕ is a wild automorphism of* **C** *then ϕ is a discontinuous mapping of the complex plane onto itself; in fact, ϕ leaves a dense subset of the real line pointwise fixed but maps the real line onto a dense subset of the plane.*

Proof. By Theorem 2, ϕ leaves **Q** (a dense subset of the real line!) pointwise fixed. By Theorem 3 we can choose $b \in \mathbf{R}$ such that $\phi(b) \notin \mathbf{R}$. Every neighborhood of b contains a rational number (which is left fixed by ϕ) and the number b (which is moved by ϕ); hence ϕ is discontinuous.

For every pair of rational numbers, q and r, $\phi(rb+q) = \phi(r)\phi(b) + \phi(q) = r\phi(b) + q$. Thus for a fixed r, $\{\phi(rb+q) \mid q \in \mathbf{Q}\}$ is a set of images of real numbers which is a dense subset of the horizontal line through $r\phi(b)$. As r varies this horizontal line moves up and down; moreover the various $r\phi(b)$ form a dense subset of the (nonhorizontal) line through 0 and $\phi(b)$. Thus the set $\{\phi(rb+q) \mid r, q \in \mathbf{Q}\}$ is a dense subset of the plane. This set is contained in $\phi(\mathbf{R})$; hence $\phi(\mathbf{R})$ is also a dense subset of **C**.

3. Isomorphisms and simple extensions. As a first step in extending an automorphism to an automorphism of **C** we need to know how to extend it to a "slightly larger" subfield. Since the proofs are no harder we discuss the more general topic of extending an isomorphism to a simple extension of its domain. There are two cases to consider according as the generator of the simple extension is algebraic or transcendental over the original field.

THEOREM 5A. *Let ϕ be an isomorphism with domain* **F** *and range* **F'**. *If α is algebraic over* **F** *then there is an isomorphism extending ϕ to $\mathbf{F}(\alpha)$ and sending α to β if and only if β is a root of the polynomial obtained by applying ϕ to the coefficients of the minimal polynomial of α over* **F**.

THEOREM 5B. *Let ϕ be an isomorphism with domain* **F** *and range* **F'**. *If α is transcendental over* **F**, *then there is an isomorphism extending ϕ to $\mathbf{F}(\alpha)$ and sending α to β if and only if β is transcendental over* **F'**.

An outline of the proof. In either case it is easy to show that $\sigma = \{\langle p(\alpha)/q(\alpha), p'(\beta)/q'(\beta)\rangle \mid p, q$ are polynomials with coefficients in **F**, $q(\alpha) \neq 0$, p', q' obtained by applying ϕ to the coefficients of p and $q\}$ is the only possible isomorphism extending ϕ to $\mathbf{F}(\alpha)$ and sending α to β. It is tedious, but not difficult, to show that σ is an isomorphism if and only if α and β are related as stated in Theorems 5A or 5B. For more details of this proof see [2], Chapter 14, Theorem 1 and Chapter 15, Lemma 1.

A special case of Theorem 5A comprises part of the proof of Theorem 3. In that proof we showed that the only extensions of I_R to $\mathbf{R}(i) = \mathbf{C}$ send i to $\pm i$.

If we combine Theorems 5A and B we find that any isomorphism with domain **F** and range **F'** can be extended to $\mathbf{F}(\alpha)$ *unless* α is transcendental over **F** and there are no complex numbers transcendental over **F'**. We shall show at the end of the paper that this "unless" clause is an essential qualification.

Examples. Let ψ and σ be the isomorphisms defined just before Theorem 2. By Theorem 5A the only extensions of I_Q to $\mathbf{Q}(\sqrt{7})$ are σ and the identity map on $\mathbf{Q}(\sqrt{7})$ since $\sqrt{7}$ and $-\sqrt{7}$ are the only two roots in **C** of the polynomial

x^2-7. The minimal polynomial of $\sqrt[4]{7}$ over $\mathbf{Q}(\sqrt{7})$ is $x^2-\sqrt{7}$ which is sent by σ to $x^2+\sqrt{7}$. The only two roots of $x^2+\sqrt{7}$ are $\pm i\sqrt[4]{7}$; hence an extension of σ to $\mathbf{Q}(\sqrt[4]{7})$ must send $\sqrt[4]{7}$ to one of these two numbers. Thus there are only two possible extensions of σ to $\mathbf{Q}(\sqrt[4]{7})$, one of which is ψ.

There are uncountably many complex numbers which are transcendental over the range of ψ, $\mathbf{Q}(i\sqrt[4]{7})$. Thus by Theorem 5B there are uncountably many ways of extending ψ to $\mathbf{Q}(\sqrt[4]{7}, \pi)$. A few of these possibilities send π to $1/\pi$, $1-\pi$, $\pi+\sqrt{57}$, or $e/17$.

These examples should convince the reader that there are many isomorphisms between finitely generated extensions of $\mathbf{Q}$. Since many of these are clearly automorphisms differing radically in their action from I_C or complex conjugation, it will follow from our main result (any automorphism can be extended to an automorphism of $\mathbf{C}$) that there are many wild automorphisms of $\mathbf{C}$.

Using ordinary induction and Theorems 5A and B we could extend any automorphism of a field to a finitely generated extension of that field. Unfortunately $\mathbf{C}$ is not a finitely (or even countably) generated extension of $\mathbf{Q}$ so ordinary induction will not suffice to prove that *any* automorphism of a subfield of $\mathbf{C}$ can be extended to $\mathbf{C}$. We therefore pause to discuss a tool to handle the "transfinite" aspect of our induction.

4. Zorn's lemma. A nonempty collection, $\mathfrak{C}$, of sets is called a *chain* of sets if for any two sets A, B in $\mathfrak{C}$, either $A\subseteq B$ or $B\subseteq A$. A family, $\mathfrak{F}$, of sets is said to have the *chain property* if $\mathfrak{F}$ contains the union of every chain of sets taken from $\mathfrak{F}$. Since the union of any finite chain of sets is simply the largest set in that chain, it is clear that any finite family of sets has the chain property. Two more examples of families with the chain property are $\mathfrak{F}_1$ = the set of all subsets of a given set A, and $\mathfrak{F}_2=\{B \mid B\subseteq\mathbf{R}$ and B contains no integers$\}$. Two families without the chain property are $\mathfrak{G}_1=\{A \mid A$ is a finite subset of $\mathbf{R}\}$ and $\mathfrak{G}_2=\{\mathbf{F} \mid \mathbf{F}$ is a subfield of $\mathbf{C}$ and a finitely generated extension of $\mathbf{Q}\}$.

ZORN'S LEMMA. *If $\mathfrak{F}$ is a nonempty family of subsets of a given set B and $\mathfrak{F}$ has the chain property, then there is at least one set, M, in $\mathfrak{F}$ such that $A\in\mathfrak{F}$ and $M\subseteq A$ implies $M=A$.*

A set with the property specified for M is called a *maximal* set in $\mathfrak{F}$. It is quite possible for a family of sets to have many maximal elements. Think of them as located at the tips of branches rather than at the top of the heap. Zorn's lemma only requires that under certain conditions there must be at least one maximal element. Several other properties of sets, notably the axiom of choice, are equivalent to Zorn's lemma. For a discussion of these equivalences (including proofs) see [4] or [7]. Returning to the examples of families with the chain property we note that the maximal set in $\mathfrak{F}_1$ and $\mathfrak{F}_2$ is unique. The reader should have no difficulty constructing a finite family of sets in which there is more than one maximal set. Neither of the families $\mathfrak{G}_1$ or $\mathfrak{G}_2$ has a maximal element. The family $\mathfrak{H}=\{A \mid$ Either A is a finite subset of $\mathbf{Q}$ or $A=\mathbf{Q}\}$ is an example of a family of sets which has a maximal member but does not satisfy the chain property; hence the converse of Zorn's lemma is not true.

5. Extending automorphisms to C. We now show that any automorphism, ϕ, can be extended to **C** by applying Zorn's lemma to the family of automorphisms extending ϕ. It is awkward to do this directly since the only isomorphisms extending ϕ to a simple extension of its domain may not be automorphisms. (Consider our examples σ and ψ!) To avoid this difficulty we first prove the following theorem.

THEOREM 6. *If ϕ is an isomorphism with domain* **F** *and range* **G**, *then ϕ can be extended to an isomorphism with domain* $\mathbf{F}^a$ *and range* $\mathbf{G}^a$.

Proof. Let $\mathfrak{F} = \{\theta \mid \theta$ is an isomorphism extending ϕ to a subfield of $\mathbf{F}^a\}$. We shall show that $\mathfrak{F}$ satisfies the three hypotheses of Zorn's lemma. $\mathfrak{F}$ is nonempty since ϕ extends itself to **F**. Isomorphisms are sets of ordered pairs; hence all members of $\mathfrak{F}$ are subsets of $\mathbf{C}\times\mathbf{C}$. Let $\mathfrak{C}$ be a chain taken from $\mathfrak{F}$ and let σ be the union of all θ in $\mathfrak{C}$. σ is clearly a set of ordered pairs of complex numbers. $\mathfrak{C}$, as a chain, is nonempty; hence it contains at least one isomorphism and thus $\langle 0, 0\rangle$ and $\langle 1, 1\rangle$ are in σ. Let $\langle a, b\rangle$ and $\langle x, y\rangle$ be in σ. Then $\langle a, b\rangle \in \theta_1$ and $\langle x, y\rangle \in \theta_2$ for some $\theta_1, \theta_2 \in \mathfrak{C}$. Since $\mathfrak{C}$ is a chain either $\theta_1 \subseteq \theta_2$ or $\theta_1 \supseteq \theta_2$ and thus the two ordered pairs are both in the larger of θ_1 and θ_2. From this it follows easily that σ is a one to one function which preserves algebraic operations. The isomorphism σ is in the family $\mathfrak{F}$ since it clearly extends ϕ and its domain, the union of subfields of $\mathbf{F}^a$, is contained in $\mathbf{F}^a$. We apply Zorn's lemma and let ψ be a maximal member of $\mathfrak{F}$. We must show that the domain and range of ψ are $\mathbf{F}^a$ and $\mathbf{G}^a$.

If the domain of ψ is not all of $\mathbf{F}^a$, then there is at least one element α in $\mathbf{F}^a$ but not in the domain of ψ. Since α is algebraic over **F** and $\mathbf{G}^a$ is algebraically closed there is at least one β in $\mathbf{G}^a$ which is a root of the ψ transform of the minimal polynomial of α over **F**. Thus by Theorem 5A there is at least one way of extending ψ to a larger isomorphism still in $\mathfrak{F}$. This is a contradiction and thus $\mathbf{F}^a$ is the domain of ψ.

Since $\mathbf{F}^a$ is algebraically closed and ψ is an isomorphism, the range of ψ is an algebraically closed subfield of $\mathbf{G}^a$ which contains **G**. But the only such subfield of $\mathbf{G}^a$ is $\mathbf{G}^a$ itself; hence $\mathbf{G}^a$ is the range of ψ and the proof is complete.

THEOREM 7. *Any automorphism of a subfield of* **C** *can be extended to an automorphism of* **C**.

Proof. Let ϕ be an automorphism of a subfield of **C**, and let $\mathfrak{F} = \{\theta \mid \theta$ is an automorphism extending ϕ to some subfield of $\mathbf{C}\}$. The proof that $\mathfrak{F}$ satisfies the three hypotheses of Zorn's lemma is virtually the same as in the proof of Theorem 6, the only change necessary is to show that domain σ = range σ instead of domain $\sigma \subseteq \mathbf{F}^a$. We leave this to the reader. Applying Zorn's lemma let ψ be a maximal member of $\mathfrak{F}$. We must show domain $\psi = \mathbf{C}$. If not, then there is a complex number, α, not in domain $\psi = \mathbf{F}$. If α is algebraic over **F** then, by Theorem 6, we could extend ψ to an automorphism of $\mathbf{F}^a$ contradicting the maximality of ψ in $\mathfrak{F}$. If α is transcendental over **F**, then by Theorem 5B we could extend ψ to an automorphism of $\mathbf{F}(\alpha)$, sending α to α for example, since α is also transcendental over range $\psi = \mathbf{F}$. This again contradicts the maximality of ψ, so there can be no complex numbers outside of domain ψ and the proof is complete.

6. Concluding remarks.

1. Although it is doubtful that anyone will give a complete recipe for an automorphism of **C** aside from I_C or complex conjugation, we see from the Theorem above that any automorphism that can be constructed in a finitely generated extension of **Q** can be extended to **C**. Thus, for example, there are automorphisms of **C** which interchange π and e, send $\sqrt[4]{3}$ to $i\sqrt[4]{3}$, and leave $\sqrt{7}$ fixed.

2. It is not true that any isomorphism between subfields of **C** can be extended to an automorphism of **C**. In particular there *are* isomorphisms with domain **C** whose range is properly contained in **C**. For example, choose $\alpha_1, \alpha_2, \alpha_3, \cdots$, a countable set of complex numbers that are algebraically independent over **Q**. There is an isomorphism, ϕ, of $\mathbf{Q}(\alpha_1, \alpha_2, \cdots)$ into itself such that $\phi(\alpha_i) = \alpha_{i+1}$. Applying Zorn's lemma to $\mathfrak{F} = \{\theta \mid \theta$ is an isomorphism extending ϕ, range $\theta \subseteq$ domain θ, and α_1 transcendental over range $\theta\}$ leads to a maximal isomorphism, ψ, whose domain is all of **C** but such that α_1 is not in the range. Note that ψ^{-1} is an example of an isomorphism defined on a subfield, **F**, of **C** which cannot be extended to $\mathbf{F}(\alpha_1)$.

3. As the final comment I mention an additional bit of mathematical folklore. In [1] it is claimed, without proof or reference to the proof, that the cardinality of the set of automorphisms of **C** is $2^{2^{\aleph_0}}$. I have heard this from other sources and am convinced that it is true although I do not know where the proof may be found.

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ON SOLUTIONS OF CERTAIN RICCATI DIFFERENTIAL EQUATIONS

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In search of exact solutions of the general Riccati differential equation

$$y' = f + gy + hy^2, \tag{1}$$

where the differentiation is with respect to x and f, g, h are functions of x, it is customary to find conditions on the coefficients f, g and h such that equation (1) may be transformed into another first order equation where the variables