Dynamic Nonlinear Networks: State-of-the-Art

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Abstract—This paper surveys the state-of-the-art of the *qualitative aspects* of nonlinear *RLC* networks. The class of networks being surveyed may contain *multi-terminal* and *multi-port* RESISTORS, INDUCTORS, AND CAPACITORS, as well as dc and time-dependent voltage and current sources.

The concepts of *impasse points* and *local solvability* are introduced and shown to be of fundamental importance in modeling a physical network. Simple criteria are given which guarantee the existence of a global state equation.

General theorems are presented for identifying or testing whether a dynamic nonlinear network possesses one or more of the following *basic qualitative properties:*

1. No finite-forward-escape-time solutions.
2. Local asymptotic stability of equilibrium points and observability of operating points.
3. Eventual uniform-boundedness of solutions.
4. Complete stability and global asymptotic stability.
5. Existence of a dc or periodic steady-state solution.
6. Unique steady-state response and spectrum conservation.

The hypotheses of most of these theorems are couched in *graph-* and *circuit-theoretic* terms so they can be easily checked, often by *inspection*.

Special efforts are made to state the concepts and results in a form that can be easily understood and used by the *nonspecialist*. Moreover, each concept and property is profusely illustrated with carefully conceived examples, and intuitive explanations so as to make this paper both motivating and somewhat self-contained. Extensive references are provided to facilitate researchers interested in conducting future research on the many unsolved problems in dynamic nonlinear networks.

I. Introduction

MOST MODERN electronic devices [1]–[6] and electrical power system components [7]–[10] are *nonlinear*. In this paper, we assume that they are modeled using a basic set of circuit elements consisting of 2-terminal, multi-terminal, and multi-port RESISTORS, INDUCTORS, and CAPACITORS,[1] as well as independent voltage and current sources.

A network $\mathfrak{N}$ made of an arbitrary interconnection of these basic circuit elements is called a *dynamic lumped RLC network*. For simplicity, we assume $\mathfrak{N}$ contains only a finite number of elements and that all resistors, inductors, and capacitors are time-invariant.

Every *properly modeled* dynamic lumped network has a well-defined *state equation*

$$\dot{x}=f(x,t), \qquad t \geqslant t_0 \tag{1.1}$$

for all $x \in \mathfrak{D}$, where $\mathfrak{D}$ denotes a nonempty subset of the n-dimensional Euclidean space $\mathbb{R}^n$, and where $f(x,t)$ is a continuous function of x in $\mathfrak{D}$. Assuming that (1.1) has a unique *solution*

$$x=\hat{x}(t), \qquad t \geqslant t_0 \tag{1.2}$$

with an initial state $x(t_0) \triangleq x_0 \in \mathfrak{D}$, efficient numerical methods are available for computing $\hat{x}(t)$ [12].

Unlike linear networks, however, computing the solution of (1.1) corresponding to some initial state does not constitute a meaningful analysis in the nonlinear case. This is because a given nonlinear network can exhibit many *qualitatively different* solutions—some could be extremely complex [13]–[14] and bizarre [16]–[18]—depending only on the choice of the initial state.

For example, under fault situations, power systems normally operating in 60-Hz sinusoidal regime have been observed to switch to an abnormally high-current nonsinusoidal regime [10], [19]. If not quenched quickly, this abnormal operating mode could cause serious damage to the system—the expensive power transformer is often the first to burn-out.

Other examples abound in electronic circuits especially at microwave frequencies where undesirable subharmonic oscillations are frequently observed and pose a perennial challenge to the microwave circuit designer for quenching, if not suppressing, such undesirable operating modes. Likewise, in computer networks, noise and transient disturbances could cause a memory circuit to switch inadvertently to a different equilibrium point, thereby translating the shift into an incorrect information signal.

Hence, unless one carries out a qualitative analysis first to determine the different operating modes and regimes in a network $\mathfrak{N}$, one cannot evaluate, let alone predict, the performance of $\mathfrak{N}$ by numerical simulation.

The objective of this paper is to present a survey of the *state-of-the-art* concerning the *qualitative behavior of dynamic nonlinear networks*.[2] Since an extensive recent survey of this subject (including precise statements of the main results and a comprehensive bibliography) is available in

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[1] An $(n+1)$-terminal or n-port circuit element N with terminal (or port) vector v, current i, charge q, and flux ϕ is said to be an $(n+1)$-terminal (or n-port) RESISTOR, INDUCTOR, or CAPACITOR if N is described by a *constitutive relation* $f_R(v,i;t)=0$, $f_L(\phi,i;t)=0$, or $f_C(q,v;t)=0$, respectively [11]. An element is said to be *time-invariant* if t is explicitly absent in the element's constitutive relation.

[2] This paper represents an expanded version of a recent presentation at an *NSF Workshop on Nonlinear Circuits and Systems* held in Houston, TX, Jan. 4–5, 1980.

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[20], the emphasis of this paper will be on interpreting the circuit-theoretic meanings of the many results surveyed in [20] and assessing their significance.

Although the proofs of many of the results to be presented in this survey require rather sophisticated modern mathematical tools, special attempts are made to state these results in simple and intuitively plausible circuit-theoretic terms so that they can be understood and used by the nonspecialist. Consequently, stronger than necessary hypotheses will usually be invoked. Special attention will be given to the interpretation of these hypotheses so that the reader will appreciate why they are needed. Some of these hypotheses will be couched in *graph-theoretic* terms so that they can be checked by inspection. Whenever possible, all hypotheses will be expressed at the "circuit element" level so that often there will be no need to write down any equation explicitly. We strongly believe that results of this nature are most useful in practice.

The following is a *table of contents* of the topics presented in this paper:

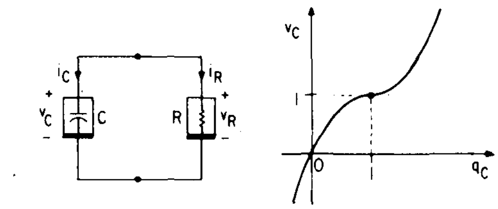


Fig. 1. A 2-element *RC* network $\mathfrak{N}$ (*Example* 1.).

II. A GLIMPSE AT SOME MOTIVATING EXAMPLES

This section is mainly tutorial. It contains a large number of simple but nontrivial—some rather subtle—examples which show why certain "good" or "bad" things can occur in nonlinear networks. They also provide a lead-in motivation for the subsequent sections.

2.1. Examples Concerning State Equation Formulation

Example 1. Consider the nonlinear *RC* network $\mathfrak{N}$ shown in Fig. 1(a). Let the element constitutive relations be described as follows:

$$\textit{Resistor } R\colon \quad i_R = h(v_R), \qquad v_R \in \mathbb{R} \tag{2.1}$$

$$\textit{Capacitor } C\colon \quad v_C = g(q_C), \qquad q_C \in \mathbb{R}. \tag{2.2}$$

If we choose the *capacitor charge* q_C as state variable, the state equation is given by

$$\boxed{\dot{q}_C = -h(g(q_C)) \triangleq f(q_C), \qquad q_C \in \mathbb{R}.} \tag{2.3}$$

Now suppose $g(\cdot)$ in (2.2) is *bijective* so that its inverse

$$q_C = \hat{q}_C(v_C), \qquad v_C \in \mathbb{R} \tag{2.4}$$

exists. Then applying KCL and the chain rule, we obtain

$$C(v_C)\dot{v}_C = -h(v_C), \qquad v_C \in \mathbb{R} \tag{2.5}$$

where $C(v_C) \triangleq d\hat{q}_C(v_C)/dv_C$ is the incremental capacitance. Assuming that $C(v_C) \neq 0$ for all v_C, then (2.5) can be rewritten as

$$\boxed{\dot{v}_C = -C^{-1}(v_C)h(v_C) \triangleq f(v_C), \qquad v_C \in \mathbb{R}.} \tag{2.6}$$

Hence, (2.6) is also a well-defined state equation with the *capacitor voltage* v_C as the state variable, provided

$$\boxed{C(v_C) \neq 0 \text{ is defined for all } v_C \in \mathbb{R}.} \tag{2.7}$$

To illustrate the importance of (2.7), let R be a 1-Ω linear resistor and let C be described by [21]:

$$v_C = g(q_C) \triangleq (q_C - 1)^3 + 1 \tag{2.8}$$

[3]Since much of this survey is based on results presented in [25], [31], we take this opportunity to include a brief errata.

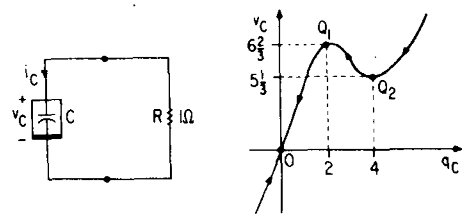


Fig. 2. The q_C–v_C curve for charge-controlled capacitor C of Example 2. Here q_C is a global coordinate for this curve. The capacitor voltage v_C does not qualify as a global coordinate. Nor does it qualify as a "local" coordinate at Q_1 or Q_2.

as shown in Fig. 1(b). Then (2.3) assumes the form

$$\dot{q}_C = -(q_C - 1)^3 - 1 \triangleq f(q_C), \qquad q_C \in \mathbb{R}. \qquad (2.9)$$

Note that $g(\cdot)$ in (2.8) is bijective and has a *continuous* inverse function

$$q_C = \hat{q}_C(v_C) = 1 + (v_C - 1)^{1/3}. \qquad (2.10)$$

Its incremental capacitance is given by

$$C(v_C) = \frac{1}{3}(v_C - 1)^{-2/3} \neq 0, \qquad \text{for all } v_C \in \mathbb{R}. \quad (2.11)$$

Substituting (2.11) into (2.6), we obtain

$$\dot{v}_C = -3(v_C - 1)^{2/3} v_C \triangleq f(v_C), \qquad v_C \in \mathbb{R}. \quad (2.12)$$

Note that $f(v_C)$ is a continuously differentiable function which vanishes at $v_C = 1$. Hence $v_C = 1$ is a well-defined *equilibrium point* of (2.12) [22].

Observe, however, that $v_C = v_R = 1$ implies $i_C = -i_R = -1 \neq 0$. Hence we have arrived at a *paradoxical* situation where the capacitor current is *not zero* at equilibrium!

To resolve this paradox, note that $C(v_C)$ in (2.11) is *undefined* at $v_C = 1$; namely, $C(1) = \infty$. Consequently, (2.7) is violated. It follows, therefore, that the state equation which correctly describes $\mathfrak{N}$ is not (2.12) but rather (2.13):

$$\dot{v}_C = -3(v_C - 1)^{2/3} v_C \triangleq f(v_C), \qquad v_C \in \mathbb{R},\ v_C \neq 1. \qquad (2.13)$$

The almost trivial distinction between (2.12) and (2.13) turns out to be crucially important in this example. Indeed, (2.12) can be interpreted as the state equation describing an infinite number of distinct networks (all sharing the same topology as Fig. 1) all of which differ from that defined by (2.8); namely; redefine R and C in Fig. 1(a) as follows:

$$\textit{Resistor:} \quad i_R = 3C_0(v_R - 1)^{2/3} v_R \qquad (2.14)$$

$$\textit{Capacitor:} \quad q_C = C_0 v_C \qquad (2.15)$$

where C_0 is any real number. Observe that at the equilibrium point $v_C = v_R = 1$, (2.14) implies $i_C = -i_R = 0$, as it should.

This example shows that while each network has a *unique* state equation (if it exists) with respect to a given state variable, each state equation represents infinitely many distinct networks.

The above example also demonstrates that even if a capacitor {resp., inductor} constitutive relation is bijective, it would still be preferable to choose the *capacitor charge* {resp., *inductor flux*} as a state variable. Such a choice not only obviates the need for checking additional conditions (such as (2.7)), but it offers certain additional numerical advantages [12, pp. 43–45, 431–432] over the capacitor voltage {resp., inductor current}.

Example 2. Let R in Fig. 2(a) be a 1-Ω linear resistor and let C be a *charge-controlled capacitor*[4] described by

$$v_C = \hat{v}_C(q_C) \triangleq \frac{1}{3}q_C^3 - 3q_C^2 + 8q_C, \qquad q_C \in \mathbb{R} \ (2.16)$$

as shown in Fig. 2(b). Then state equation (2.3) reduces to

$$\dot{q}_C = -\left[\frac{1}{3}q_C^3 - 3q_C^2 + 8q_C\right] \triangleq f(q_C), \qquad q_C \in \mathbb{R}. \qquad (2.17)$$

Note that C is *not* voltage-controlled. In particular, its incremental capacitance $C(v_C)$ is *multi-valued* over the interval $[5\frac{1}{3}, 6\frac{2}{3}]$. If we choose v_C as state variable as in (2.6), then the state equation is defined only over 2 disjoint intervals $(-\infty, 5\frac{1}{3})$ and $(6\frac{2}{3}, \infty)$:

$$\dot{v}_C = -\frac{v_C}{C(v_C)}, \qquad -\infty < v_C < 5\frac{1}{3} \text{ and } 6\frac{2}{3} < v_C < \infty. \qquad (2.18)$$

Since $\dot{q}_C = -v_C$ implies $\dot{q}_C > 0$ whenever $v_C < 0$, and $\dot{q}_C < 0$ whenever $v_C > 0$, it follows that all solutions must tend toward the equilibrium point $q_C = 0$, as depicted by the "dynamic route" in Fig. 2(b) [22]. Hence, $\mathfrak{N}$ has a well-defined solution $q_C(t)$ satisfying (2.17), in spite of the fact that the state equation (2.18) does not exist globally (i.e., in $\mathbb{R}$).

Since $v_C(t) = \hat{v}_C(q_C(t))$ is well defined, one might argue that (2.18) is also well defined so long as one picks the appropriate branch of the curve when numerically solving the state equation: the initial point specifies which branch applies at any time. This is in fact the way circuit simulation programs are designed, where the computation is carried out in a *small neighborhood* of an initial point [12]. Mathematically, v_C is called a *local coordinate* about the initial point $v_C(t_0)$. This argument is valid at all points in the curve in Fig. 2 *except* at Q_1 and Q_2. Note that given any neighborhood of either point, no matter how small, the $q_C - v_C$ curve remains a multivalued function of v_C. This difficulty is reflected in (2.18) in the value of $C(v_C)$ which tends to ∞ at precisely these 2 points. We conclude that v_C does not qualify even as a *local coordinate* at Q_1 and Q_2.

Example 3. Let R in Fig. 3(a) be a 1-Ω linear resistor and let C be a *voltage-controlled capacitor* described by the

[4] An $(n+1)$-terminal (or n-port) element described by $y = f(x)$, $x \in \mathbb{R}^n$ is said to be x-controlled iff f is a continuous function for all $x \in \mathbb{R}^n$. In other words, "x-controlled" always means "x" is the independent variable of a function defined in the *entire* space $\mathbb{R}^n$.

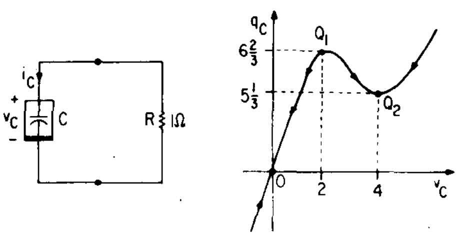


Fig. 3. The q_C–v_C curve for voltage-controlled capacitor C of *Example 3*. Here v_C is a global coordinate for this curve. The capacitor charge q_C does not qualify as a global coordinate. Nor does it qualify as a local coordinate at Q_1 or Q_2.

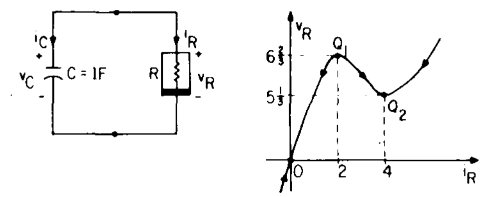


Fig. 4. The i_R–v_R curve for current-controlled resistor R of *Example 4*. Here i_R is a global coordinate for this curve. The resistor voltage v_R does not qualify as a global coordinate. Nor does it qualify as a local coordinate at Q_1 or Q_2.

"dual" of (2.16):

$$q_C = \hat{q}_C(v_C) \triangleq \frac{1}{3}v_C^3 - 3v_C^2 + 8v_C, \qquad v_C \in \mathbb{R} \quad (2.19)$$

as shown in Fig. 3(b). Note that q_C does not qualify as a global coordinate for the q_C–v_C curve, but v_C does. In fact, q_C does not even qualify as a local coordinate at Q_1 and Q_2 in Fig. 3(b). Hence, the state equation (2.3) does not exist globally.

Since v_C qualifies as a global coordinate for the q_C–v_C curve, it is natural to write the state equation in the form of (2.6). Note however that since $C(v_C)=0$ at Q_1 and Q_2, the state equation

$$\dot{v}_C = -\frac{v_C}{v_C^2 - 6v_C + 8} \triangleq f(v_C), \qquad v_C \neq 2,4 \quad (2.20)$$

is undefined at $v_C = 2$ and $v_C = 4$.

Unlike Fig. 2, however, the dynamic route (note that $\dot{q}_C < 0$ for all points with $v_C > 0$ is *qualitatively* quite different in Fig. 3. Here, the solution starting from a point on either side near Q_1 diverges from it {resp. Q_2 converges toward it}. One might dismiss this observation as no more than a simple example of an unstable equilibrium point of (2.19). Observe, however, that Q_1 and Q_2 are *not* equilibrium points since $\dot{v}_C$ does not tend to zero as $v_C \to 2$ or 4. In fact, $\dot{v}_C \to \infty$ as $v_C \to 4$ and $\dot{v}_C \to -\infty$ as $v_C \to 2$! This means that the solution starting at any point near Q_2 at $t=0$ would arrive at Q_2 at some *finite* time $T>0$. Likewise, the solution starting at any point near Q_1 at $t=0$ would return to Q_1 at some *finite* time $T<0$. Note that upon reaching either Q_2 or Q_1 in Fig. 3(b), the solution cannot be continued in forward or backward time, respectively. We call Q_1 and Q_2 *impasse points*[5] for this reason.

One could object to choosing either Q_1 or Q_2 as initial conditions on the ground that $f(v_C)$ in (2.20) is undefined at these points. However, the issue here is that starting in a small neighborhood of Q_2, any computer simulation would get us to Q_2 in *finite time* T, after which the computer will be unable to continue the integration routine since the solution ceases to exist.[6] Since all *physical* networks must have solution for *all* times $t \geq t_0$, any network which exhibits such *impasse points* is nonphysical and should be *remodeled*.

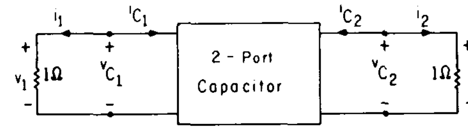


Fig. 5. Network for *Example 5*.

Example 4. Let C in Fig. 4(a) be a 1-F linear capacitor and let R be a *current-controlled resistor* described by

$$v_R = \hat{v}_R(i_R) \triangleq \frac{1}{3}i_R^3 - 3i_R^2 + 8i_R, \qquad i_R \in \mathbb{R} \quad (2.21)$$

as shown in Fig. 4(b). Since $\hat{v}(i_R)$ is not bijective, $h(v_R)$ in (2.1) is multivalued over the interval $5\frac{1}{3} \leq v_R \leq 6\frac{2}{3}$. Since both (2.3) and (2.6) are expressed in terms of $h(\cdot)$, *neither state equation exists globally* in this case. Note that this situation differs from those of Examples 2 and 3 where at least one state equation exists everywhere except for some isolated points.

To give a physical interpretation of this situation, observe that $\dot{v}_R = \dot{v}_C = i_C = -i_R$ implies $\dot{v}_R > 0$ whenever $i_R < 0$, and $\dot{v}_R < 0$ whenever $i_R > 0$. This gives rise to the dynamic route shown in Fig. 4(b). Again, we reach an impasse situation when the solution reaches Q_2 in some "finite" forward time $T < \infty$, or Q_1 in some "finite" backward time $T > -\infty$. (Note that Q_1 and Q_2 are *not* equilibrium points of either state equation (2.3) or (2.6), even if they are defined at Q_1 and Q_2.)

Hence, Q_1 and Q_2 are impasse points and we conclude that Fig. 4(a) represents a *nonphysical* circuit. To remodel this circuit, we need only insert a small series *linear inductance* representing the line inductance. It is easy to see that the resulting second-order circuit has a state equation defined globally in $\mathbb{R}^2$ [22].

To uncover the source of the difficulty in Fig. 4(b), note that since $q_C = v_C = v_R$, neither q_C nor v_C qualifies as a global coordinate, or as a local coordinate at Q_1 and Q_2.

Example 5. Consider the network shown in Fig. 5, where the two-port capacitor C is described by

$$q_{C_1} = \hat{q}_1(v_{C_1}, v_{C_2}) \triangleq e^{v_{C_1}} \cos v_{C_2} \quad (2.22a)$$

$$q_{C_2} = \hat{q}_2(v_{C_1}, v_{C_2}) \triangleq e^{v_{C_1}} \sin v_{C_2} \quad (2.22b)$$

[5]We call a point x^* of an autonomous system $\dot{x}=f(x)$ an impasse point, if the solution ceases to exist after it reaches x^* in *finite* forward or backward time. A more precise definition is given in *Definition 3* of Section 3.4.

[6]No existing circuit simulation program is capable of detecting this nonphysical situation. Clearly, whatever solution it generates after $t=T$ is plain garbage, which could be highly misleading to the unsuspecting user.

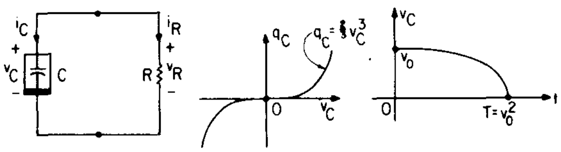


Fig. 6. A network having no solution for $t > v_0^2$. However, the solution exists and is bounded for $0 < t < v_0^2$.

where $v_C \in \mathbb{R}^2$. Note that $q_C = \hat{q}(v_C)$ is *not* bijective because there exist at least 2 distinct points, $(v_{C_1}, v_{C_2}) = (0,0)$ and $(0, 2\pi)$, which map into the same point $(q_{C_1}, q_{C_2}) = (1,0)$. In other words, C is voltage-controlled but *not* charge-controlled—the same situation encountered earlier in Fig. 3(b) of Example 3. Since q_C does not qualify as a *global* coordinate, the state equation in terms of q_C does not exist globally in $\mathbb{R}^2$. In *Example 3*, we have seen that this situation led to the presence of impasse points so that the state equation (2.20) (in terms of v_C) is undefined at these points.

To show that this *nonphysical* situation does not arise in *this example*—in spite of the other striking similarities—note that the *incremental capacitance matrix* $C(v_C)$ associated with (2.22) is *nonsingular* for all $v_C \in \mathbb{R}^2$. Hence, using capacitor voltages as state variables, the following state equation for Fig. 5 exists for all $v_C \in \mathbb{R}^2$.

$$\begin{bmatrix} \dot{v}_{C_1} \\ \dot{v}_{C_2} \end{bmatrix} = -\begin{bmatrix} e^{v_{C_1}} \cos v_{C_2} & -e^{v_{C_1}} \sin v_{C_2} \\ e^{v_{C_1}} \sin v_{C_2} & e^{v_{C_1}} \cos v_{C_2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} v_{C_1} \\ v_{C_2} \end{bmatrix}. \tag{2.23}$$

The subtle difference between *Examples 3* and *5* is that whereas q does not qualify as a *local coordinate* for the q_C–v_C curve defined by (2.19) *for some* $q_C \in \mathbb{R}$, q_C does qualify as a local coordinate for (2.22) for all $q_C \in \mathbb{R}^2$. To show this, note that since $\det C(v_C) = e^{2v_{C_1}} \neq 0$, the function $q_C = \hat{q}(v_C)$ in (2.22) is *locally* one-to-one.[7] Hence, for any point $v_C^* \in \mathbb{R}^2$, there is an open neighborhood $N(v_C^*)$ about v_C^* such that q_C defines a single-valued function for all $q_C \in N(v_C^*)$.

2.2. Examples Concerning Existence and Uniqueness of Solutions

Example 6. Let R in Fig. 6(a) be a 1-Ω linear resistor and let C be described by $q_C = \frac{2}{3} v_C^3$, as shown in Fig. 6(b). The state equation is

$$\dot{v}_C = -\frac{1}{2v_C}, \qquad v_C \neq 0. \tag{2.24}$$

The solution with *initial* capacitor voltage $v_C(0) = v_0$ exists analytically:

$$v_C(t) = \sqrt{v_0^2 - t}\,, \qquad t \geq 0. \tag{2.25}$$

Note that this solution does not exist for $t \geq v_0^2$ (see Fig. 6(c). In fact, if we choose $v_C(0) = 0$, then no solution exists at all. This should not be surprising because $v_C = 0$ is in fact an impasse point. Note that both v_C and q_C qualify as a global and a local coordinate in this case, but $v_C(q_C)$ is *not differentiable* at $q_C = 0$.

[7]Note that $q_C = \hat{q}(v_C)$ is *not* locally one-to-one at Q_1 and Q_2 in *Example 3.*

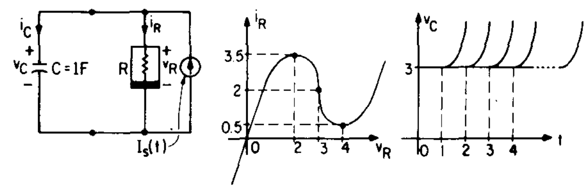


Fig. 7. A network with infinitely many solutions all of which having the same initial voltage $v_C(0) = 3$.

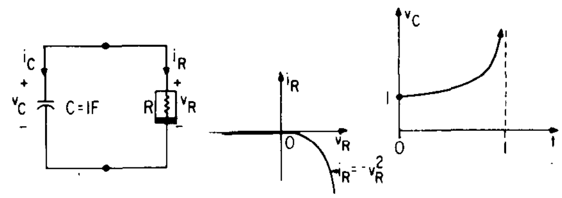


Fig. 8. A *nonphysical* network exhibiting a finite-forward-escape-time solution.

Example 7. Let C in Fig. 7(a) be a 1-F linear capacitor and let R be described by the i_R–v_R curve shown in Fig. 7(b). Let this curve be represented analytically in a small neighborhood of $v_R = 3$ as follows [23]:

$$i_R = -\frac{3}{2}(v_R - 3)^{1/3} + 2, \qquad 2 \leq v_R \leq 4. \tag{2.26}$$

The state equation is

$$\dot{v}_C = \frac{3}{2}(v_C - 3)^{1/3} - 2 + I_s(t), \qquad 2 \leq v_C \leq 4. \tag{2.27}$$

The solution with initial capacitor voltage $v_C(0) = 3$ and $I_s(t) = 2$ exists analytically:

$$\begin{aligned} v_C(t) &= 3, & 0 \leq t \leq T \\ &= 3 + (t - T)^{3/2}, & T < t \leq T + 1 \end{aligned} \tag{2.28}$$

where T is an arbitrary real number. Hence, this network has *infinitely many* solutions. Note that $\hat{i}_R(v_R)$ is not differentiable at $v_R = 3$.

Example 8. Let C in Fig. 8(a) be a 1-F linear capacitor and let R be described by the i_R–v_R curve shown in Fig. 8(b). The state equation is

$$\begin{aligned} \dot{v}_C &= v_C^2, & v_C \geq 0 \\ &= 0, & v_C < 0. \end{aligned} \tag{2.29a}$$

The solution with initial voltage $v_C(0) = 1$ exists *analytically*:

$$v_C(t) = \frac{1}{1-t}, \qquad t \geq 0. \tag{2.29b}$$

Note from Fig. 8(c) that this solution does *not* exist at $t = 1$: $v_C(t)$ blows up at $t = 1$. Such a network is said to exhibit a *finite-forward-escape time* and is clearly *nonphysical*.

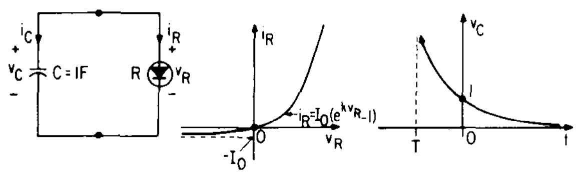


Fig. 9. A network exhibiting a finite-backward-escape-time solution.

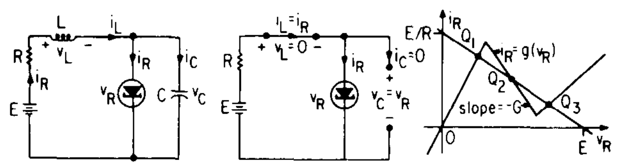


Fig. 10. A *dynamic* network and its associated *resistive* network obtained by open-circuiting the capacitor and short-circuiting the inductor.

Example 9. Let C in Fig. 9(a) be a 1-F linear capacitor and let R be a *p-n* junction diode described by the i_R–v_R curve shown in Fig. 9(b). The state equation is

$$\dot{v}_C = -I_0[e^{kv_C} - 1]. \tag{2.30}$$

The solution with initial voltage $v_C(0)=1$ *exists analytically*:

$$v_C(t) = \ln\left\{\frac{e^{f(t)}}{e^{f(t)}-1}\right\}^{1/k} \tag{2.31a}$$

where

$$f(t) \triangleq (kI_0)t + \ln\left\{\frac{1}{1-e^{-k}}\right\}. \tag{2.31b}$$

Note from Fig. 9(c) that this solution does *not* exist at $t = T \triangleq -(1/kI_0)\ln[1/1-e^{-k}]$. Such a network is said to exhibit a *finite-backward-escape time*.

2.3 Examples Concerning Equilibrium Points and Operating Points

Example 10. Choosing v_C and i_L as state variables, the state equation for the typical tunnel-diode network shown in Fig. 10(a) is given by

$$\dot{v}_C = -\frac{1}{C}[i_L - g(v_C)] \triangleq f_C(v_C, i_L) \tag{2.32a}$$

$$\dot{i}_L = -\frac{1}{L}[e - Ri_L - v_C] \triangleq f_L(v_C, i_L). \tag{2.32b}$$

Any solution (v_C^*, i_L^*) of the *algebraic equation* obtained by setting the right-hand side of (2.32) to zero is called an *equilibrium point* of the state equation (2.32) [22]. Since $v_C = v_R$ and $i_L = i_R$, for this example, there is a one-to-one correspondence between the equilibrium points of (2.32) and the *operating points*[8] of the tunnel diode in the associated resistive network obtained by open-circuiting the capacitor and short-circuiting the inductor as shown in Fig. 10(b).

The operating points in this case are simply obtained by the load-line construction shown in Fig. 10(c). Note that there are 3 *isolated* operating points so long as $E \neq RG$. On the other hand, there are infinitely many *nonisolated* equilibrium points if $E = RG$.

Since any *physical* network can have only one solution at any time, a purely *resistive* network, such as Fig. 10(b), having *multiple* operating points is ill-posed unless it is associated with a *dynamic* network, thereby allowing one to determine which operating point is actually observable in practice.

Whether an operating point of a *resistive network* can be physically observed depends on the *stability* of the corresponding equilibrium point of the associated *dynamic* network. It is important, therefore, that simple criteria be developed for testing whether an equilibrium point is observable in practice.

[8]Each solution of a Resistive network N is called an *operating point* of N. The corresponding voltages and currents associated with an internal Resistor R_j is called an *operating point of R_j*.

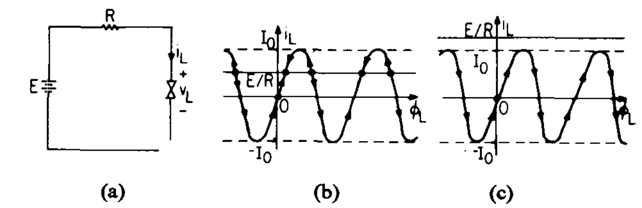


Fig. 11. (a) Josephson junction circuit: The constitutive relation of the nonlinear inductor is given by $i_L = I_0 \sin k\phi_L$. (b) Although the resistive circuit has a *unique* operating point, there are infinitely many equilibrium points if $E < RI_0$. (c) There are no equilibrium points if $E > RI_0$.

Example 11. The state equation for the Josephson junction network shown in Fig. 11(a) is given by

$$\dot{\phi}_L = E - RI_0 \sin k\phi_L \tag{2.33}$$

where I_0 and k are device constants. Note that we pick ϕ_L as the state variable because $i_L = I_0 \sin k\phi_L$ is not bijective. The equilibrium points of (2.33) can be obtained in 2 steps: 1) Find the operating point ($i_L = E/R$) of the *resistive* network obtained by short-circuiting the inductor ($v_L = \dot{\phi}_L = 0$). 2) Each intersection of the ϕ_L–i_L curve with the horizontal line $i_L = E/R$ represents an equilibrium point of (2.33).

If $E < RI_0$, (2.33) has *infinitely* many isolated equilibrium points as shown in Fig. 11(b). Since $\dot{\phi}_L > 0$ whenever $i_L < E/R$, and $\dot{\phi}_L < 0$ whenever $i_L > E/R$, the dynamic route is as shown in Fig. 11(b). Note the equilibrium points alternate from stable to unstable points.

If $E > RI_0$, (2.33) has *no* equilibrium points. In this case, the dynamic route in Fig. 11(c) shows $\phi_L(t)$ will increase indefinitely (from left to right along the curve).

This example demonstrates 3 important observations:

1. Equilibrium and operating points are generally distinct concepts: Here, we have a *unique* operating point but infinitely many equilibrium points if $E \leqslant RI_0$ and no equilibrium point if $E > RI_0$.
2. To avoid ambiguity, it is necessary to specify the *state variable* before finding equilibrium points.

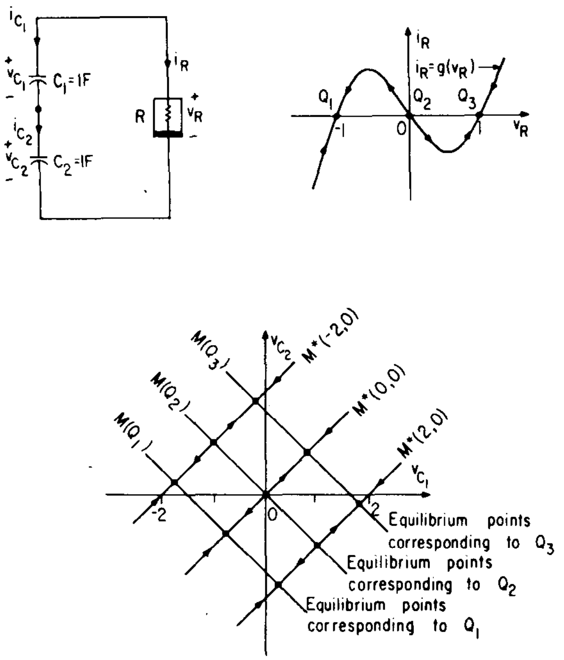


Fig. 12. A network containing a cut set of capacitors.

3. A realistic physical inductor circuit model can support a solution $\phi_L(t)\to\infty$ at $t\to\infty$ so long as its associated current $i_L(t)$ remains bounded for all times.

Example 12. The state equation for the network in Fig. 12(a) is given by

$$\dot{v}_{C_1} = -g(v_{C_1}+v_{C_2}) \triangleq f_1(v_{C_1}, v_{C_2}) \qquad (2.34a)$$

$$\dot{v}_{C_2} = -g(v_{C_1}+v_{C_2}) \triangleq f_2(v_{C_1}, v_{C_2}). \qquad (2.34b)$$

The nonlinear resistor has 3 isolated operating points when $i_R=0$ (open-circuiting both capacitors C_1 and C_2), as shown in Fig. 12(b); namely, $v_R=-1,0,1$. Since $v_{C_1}+v_{C_2}=v_R$, it follows that each operating point gives rise to a whole *line* of *nonisolated* equilibrium points, as shown in Fig. 12(c).

The qualitative behavior of the *nonisolated* equilibrium points in Fig. 12(c) differs drastically from those in Fig. 10(c) when $E=RG$: a slight perturbation of the i_R–v_R curve in Fig. 12(c) changes the equilibrium point locations only slightly, whereas a slight perturbation of R from the value $R=E/G$ in Fig. 10(c) would change the *nonisolated* equilibrium points into one, two, or three *isolated* equilibrium points. In other words, the network in Fig. 12(a) is *structurally stable* whereas that in Fig. 10(a) is *not* when $E=RG$.

Nonisolated equilibrium points are generally of no practical interest if they are structurally unstable. In this example, however, they represent a nonpathological circuit property. In fact, the equilibrium points in Fig. 12(c) can be considered to be isolated in the sense that once the initial voltages $v_{C_1}(0)$ and $v_{C_2}(0)$ are specified, the trajectories are constrained to move along the straight line $v_{C_1}+v_{C_2}=v_{C_1}(0)+v_{C_2}(0)=v_0$. This property follows from KCL and the observation that $q_{C_1}=v_{C_1}$ and $q_{C_2}=v_{C_2}$:

$$\begin{aligned} i_{C_1}(t)+i_{C_2}(t)=0 \Rightarrow q_{C_1}(t)+q_{C_2}(t) &= q_{C_1}(0)+q_{C_2}(0) \\ &= v_{C_1}(0)+v_{C_2}(0) \triangleq v_0. \end{aligned}$$

Hence, each initial condition specifies a 45°-trajectory line whose intersections with the 3 lines of equilibrium points in Fig. 12(c) give rise to 3 "isolated" equilibrium points. To determine which one will actually be attained, we translate the dynamic route in Fig. 12(b) (note that $\dot{v}_{C_1}>0$ and $\dot{v}_{C_2}>0$ whenever $i_R<0$; and $\dot{v}_{C_1}<0$ and $\dot{v}_{C_2}<0$ whenever $i_R>0$) into the trajectory lines in Fig. 12(c) and observe that each equilibrium point corresponding to Q_1 or Q_3 is *asymptotically stable*, whereas that corresponding to Q_2 is *unstable*.

Each trajectory line in Fig. 12(c) is called a *one-dimensional invariant submanifold* because any solution starting from a point on such a line must remain on that line.

Example 13. The state equation for the network shown in Fig. 13(a) is given by

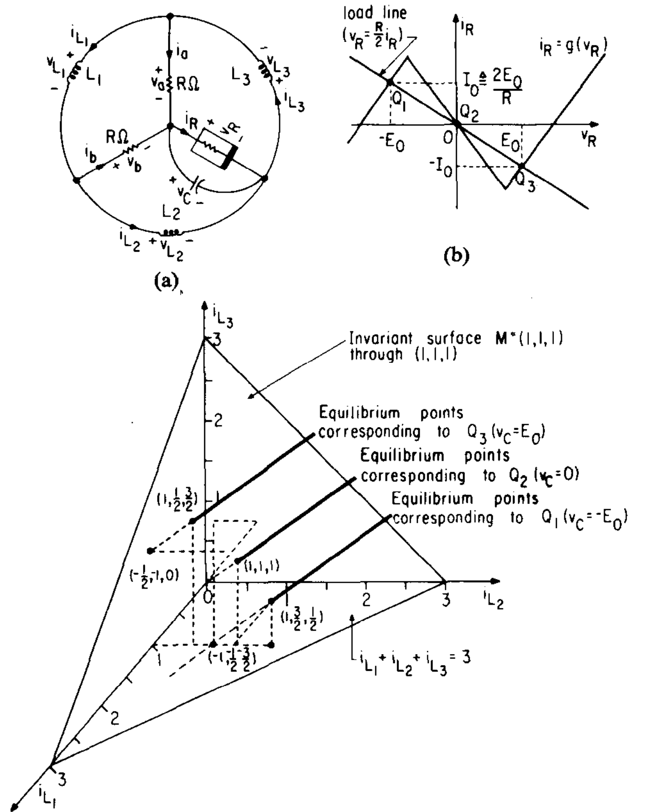


Fig. 13. (a) A network containing a loop of identical inductors $L_1=L_2=L_3=L$. (b) The load-line intersects the i_R–v_R curve at 3 operating points Q_1, Q_2, and Q_3. (c) The set of equilibrium points plotted on the $i_{L_1}-i_{L_2}-i_{L_3}$ space (drawn with $I_0=1$). Each line of equilibrium points is drawn by identifying 2 points on the line: (0,0,0) and (1,1,1) for Q_2; $(-1,-\frac{1}{2},-\frac{3}{2})$ and $(1,\frac{3}{2},\frac{1}{2})$ for Q_1; $(-\frac{1}{2},-1,0)$ and $(1,\frac{1}{2},\frac{3}{2})$ for Q_3. The 4th coordinate v_C is a constant along each line. Each initial current $(i_{L_1}(0), i_{L_2}(0), i_{L_3}(0))$ identifies a 2-dimensional invariant submanifold $i_{L_1}+i_{L_2}+i_{L_3}=i_0$ (drawn with $i_0=3$).

$$\dot{v}_{C_1} = -\frac{1}{C}\left[g(v_C)+i_{L_2}-i_{L_3}\right] \triangleq f_1(v_C, i_{L_1}, i_{L_2}, i_{L_3}) \tag{2.35a}$$

$$\dot{i}_{L_1} = -\frac{R}{L}\left[2i_{L_1}-i_{L_2}-i_{L_3}\right] \triangleq f_2(v_C, i_{L_1}, i_{L_2}, i_{L_3}) \tag{2.35b}$$

$$\dot{i}_{L_2} = -\frac{R}{L}\left[-i_{L_1}+i_{L_2}-\frac{v_C}{R}\right] \triangleq f_3(v_C, i_{L_1}, i_{L_2}, i_{L_3}) \tag{2.35c}$$

$$\dot{i}_{L_3} = -\frac{R}{L}\left[-i_{L_1}+i_{L_3}+\frac{v_C}{R}\right] \triangleq f_4(v_C, i_{L_1}, i_{L_2}, i_{L_3}) \tag{2.35d}$$

where $g(v_C)$ denotes the i_R–v_R curve shown in Fig. 13(b).

To determine the equilibrium points, let us first open-circuit the capacitor, short-circuit the 3 inductors and find the operating points of the nonlinear resistor as shown in Fig. 13(b): Q_1: $v_R=-E_0$, $i_R=I_0$; Q_2: $v_R=0$, $i_R=0$; Q_3: $v_R=E_0$, $i_R=-I_0$. Each operating point uniquely determines the voltage and current of the 2 R-Ω linear resistors ($v_a=v_b=-v_R$, $i_a=i_b=i_R/2$). The equilibrium points can now be determined by finding $(v_C, i_{L_1}, i_{L_2}, i_{L_3})$ corresponding to Q_1, Q_2, and Q_3 by applying KCL and KVL:

$$Q_1: v_C=-E_0, \quad i_{L_1}=i_{L_2}-\frac{I_0}{2}, \quad i_{L_3}=i_{L_2}-I_0 \tag{2.36a}$$

$$Q_2: v_C=0, \quad i_{L_1}=i_{L_2}=i_{L_3} \tag{2.36b}$$

$$Q_3: v_C=E_0, \quad i_{L_1}=i_{L_2}+\frac{I_0}{2}, \quad i_{L_3}=i_{L_2}+I_0. \tag{2.36c}$$

Note that for each operating point Q_j, there correspond a unique capacitor voltage v_C, but a *continuum* of inductor currents $(i_{L_1}, i_{L_2}, i_{L_3})$. If we plot only the equilibrium inductor currents in the $i_{L_1}-i_{L_2}-i_{L_3}$ space, we would obtain 3 infinite lines of *nonisolated* equilibrium points as shown in Fig. 13(c), where the 4th coordinate v_C is also indicated.

Integrating $v_{L_1}(t)+v_{L_2}(t)+v_{L_3}(t)=0$ from 0 to t, we obtain

$$\begin{aligned}\phi_{L_1}(t)+\phi_{L_2}(t)+\phi_{L_3}(t) &= \phi_{L_1}(0)+\phi_{L_2}(0)+\phi_{L_3}(0)\\ &= Li_{L_1}(0)+Li_{L_2}(0)+Li_{L_3}(0)\\ &\triangleq Li_0.\end{aligned}$$

Hence, once the initial inductor currents are specified, the solution $(i_{L_1}(t), i_{L_2}(t), i_{L_3}(t))$ is constrained to lie along the plane $i_{L_1}+i_{L_2}+i_{L_3}=i_0$ as shown in Fig. 13(c) (drawn with $i_0=3$). Since any trajectory starting from a point on this plane must remain on this plane, we have once again an *invariant* submanifold (a two-dimensional surface in this case). Since each line of equilibrium points intersects this plane at exactly one point, the equilibrium points are actually *isolated* once the initial inductor currents are specified. Hence, the *qualitative* behavior of trajectories in the vicinity of each equilibrium point can be analyzed as if the equilibrium points are isolated.

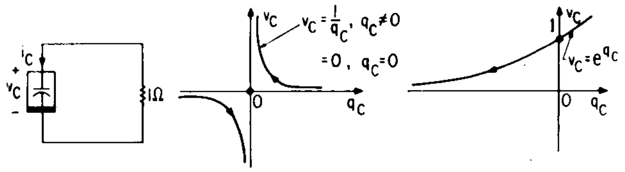


Fig. 14. A network having an *unbounded* capacitor *voltage* solution when the q_C–v_C curve is as shown in Fig. 14(b), and *unbounded* capacitor *charge* solution when the q_C–v_C curve is as shown in Fig. 14(c).

For example, using the assumed initial value $i_{L_1}(0)+i_{L_2}(0)+i_{L_3}(0) \triangleq i_0=3$ in Fig. 13(c), we substitute $i_{L_3}=-i_{L_1}-i_{L_2}+3$ into (2.35(a), (b), (c)) and obtain a reduced third-order state equation

$$\dot{v}_{C_1} = -\frac{1}{C}\left[g(v_C)+i_{L_1}+2i_{L_2}-3\right] \triangleq \hat{f}_1(v_C, i_{L_1}, i_{L_2}) \tag{2.37a}$$

$$\dot{i}_{L_1} = -\frac{R}{L}\left[3i_{L_1}-3\right] \triangleq \hat{f}_2(v_C, i_{L_1}, i_{L_2}) \tag{2.37b}$$

$$\dot{i}_{L_2} = -\frac{R}{L}\left[-i_{L_1}+i_{L_2}+\frac{v_C}{R}\right] \triangleq \hat{f}_3(v_C, i_{L_1}, i_{L_2}). \tag{2.37c}$$

We can interpret (2.37) as the state equation of a third-order dynamic network obtained by replacing the third inductor in Fig. 13(a) by a *current-controlled current source* described by $i_{L_3}=-i_{L_1}-i_{L_2}+3$.

Observe that the *state variable* i_{L_3} has been eliminated and (2.37) gives rise to 3 *isolated* equilibrium points: Q_1: $(-E_0, 1, \frac{3}{2})$, Q_2: $(0,1,1)$, Q_3: $(E_0, 1, \frac{1}{2})$, Using the assumed value $I_0 \triangleq 2E_0/R$ in Fig. 3.18(c), we calculate the eigenvalues of the Jacobian matrix at each equilibrium point and find Q_1 and Q_3 to be *asymptotically stable*, and Q_2 to be *unstable*.

2.4. Examples Concerning Boundedness of Solutions

Example 14. Let the capacitor in Fig. 14(a) be described by the constitutive relation shown in Fig. 14(b). The associated state equation is given by

$$\begin{aligned}\dot{q}_C &= -\frac{1}{q_C}, \qquad q_C \neq 0\\ &= 0, \qquad q_C=0.\end{aligned} \tag{2.38}$$

The dynamic route in Fig. 14(b) shows that $|q_C(t)|<\infty$ for all t. However, its associated voltage

$$v_C(t)=\frac{1}{\sqrt{1-2t}} \to \infty \qquad \text{as } t \to \frac{1}{2}. \tag{2.39}$$

Hence, a bounded capacitor "charge" solution does not imply a bounded capacitor "voltage" solution.

Example 15. Let the capacitor in Fig. 14(a) be described by the constitutive relation shown in Fig. 14(c).

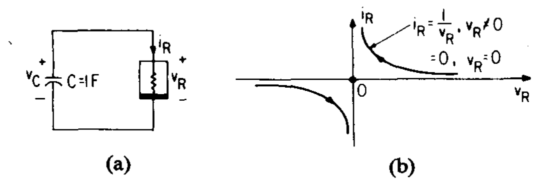


Fig. 15. A passive network having a bounded capacitor charge and voltage solution but an *unbounded* resistor current solution.

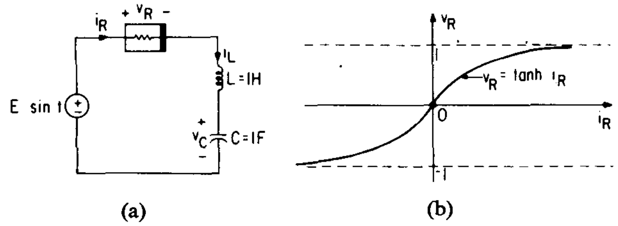


Fig. 16. A network made of strictly passive and strictly locally passive elements driven by a sinusoidal voltage source can give an unbounded zero-state solution.

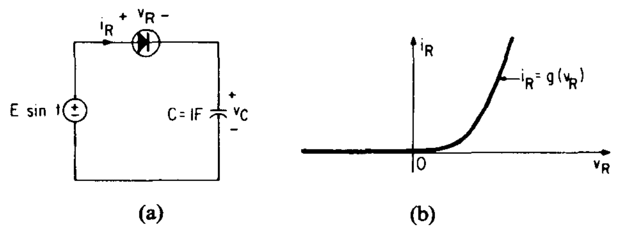


Fig. 17. A network (driven by a periodic source) having infinitely many distinct periodic solutions depending only on the initial voltage $v_C(0) < E$.

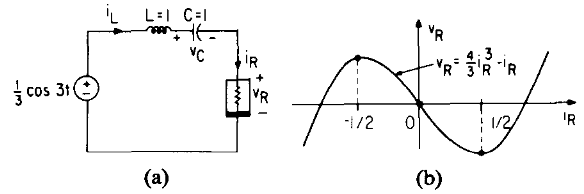


Fig. 18. A network having at least 3 subharmonic solutions.

The associated state equation is given by

$$\dot{q}_C = -e^{q_C}. \tag{2.40}$$

The dynamic route in Fig. 14(c) shows that all solutions $v_C(t) \to 0$ and $q_C(t) = \ln(1/t) \to -\infty$ as $t \to \infty$. Hence, a bounded capacitor "voltage" solution does not imply a bounded capacitor "charge" solution.

Example 16. The state equation for the network shown in Fig. 15(a) is given by

$$\dot{v}_C = -\frac{1}{v_C}, \qquad v_C \neq 0$$
$$= 0, \qquad v_C = 0. \tag{2.41}$$

The capacitor charge and voltage solution $q_C(t) = v_C(t) = \sqrt{1-2t}$ is clearly bounded. However, the resistor current solution

$$i_R(t) = \frac{1}{\sqrt{1-2t}} \to \infty \qquad \text{as } t \to \frac{1}{2}. \tag{2.42}$$

Since both elements in Fig. 15(a) are passive,[9] this example shows that passivity alone is not sufficient to guarantee that all solutions are bounded.

Example 17. The state equation for the network in Fig. 16(a) is given by

$$\dot{v}_C = i_L \tag{2.43a}$$

$$\dot{i}_L = -\tanh i_L - v_C + E\sin t. \tag{2.43b}$$

Note that the resistor, inductor, and capacitor are both *strictly passive* and *strictly locally passive*.[10] It is easy to show that if $E = 0$, the origin is a *globally asymptotically stable* equilibrium point in the sense that all solutions tend to the origin at $t \to \infty$, regardless of initial conditions.

However, if $E > 4/\pi$, the zero-state solution $i_L(t) \to \infty$ as $t \to \infty$ [24].

This example shows that it is not possible to guarantee boundedness of solutions even if all elements are both *strictly passive* and *strictly locally passive*.

2.5. Examples Concerning Steady-State Solutions

Example 18. The state equation for the network shown in Fig. 17(a) is given by

$$\dot{v}_C = g(E\sin t - v_C). \tag{2.44}$$

Since $g(v_R) \geq 0$, this network cannot have a *nonconstant periodic* solution even though it is driven by a periodic input signal.

Moreover for any *initial* voltage $v_C(0) > E$, (2.44) admits the *constant* solution

$$v_C(t) = v_C(0), \qquad t \geq 0. \tag{2.45}$$

This example shows that it is possible for a nonlinear network to exhibit more than one—infinitely many for this example—distinct periodic solutions, even if all elements (except the source) are passive and locally passive.

Example 19. The state equation for the network shown in Fig. 18(a) is given by

$$\dot{v}_C = i_L \tag{2.46a}$$

$$\dot{i}_L = i_L - \frac{4}{3} i_L^3 - v_C + \cos 3t. \tag{2.46b}$$

Depending on the initial condition, (2.46) has at least 3 distinct periodic solutions at $1/3$ of the input frequency:

$$v_C(t) = \sin\left(t + \frac{2k\pi}{3}\right), \qquad k = 0,1,2 \tag{2.47a}$$

$$i_L(t) = \cos\left(t + \frac{2k\pi}{3}\right), \qquad k = 0,1,2. \tag{2.47b}$$

[9] A multi-terminal (or multi-port) Resistor R is said to be passive {resp.; *strictly passive*} *iff* $i_R^T v_R \geq 0$ {resp.; $i_R^T v_R > 0$, except at origin} *for all* (i_R, v_R) satisfying the constitutive relation of R [20].

Hence, a 2-terminal Resistor R is passive $\Leftrightarrow$ its v_R–i_R curve lies only in the first and the third quadrants. R is *strictly passive* $\Leftrightarrow$ it is passive and its v_R–i_R curve does not touch the v_R- and i_R-axis except the origin.

[10] See *Definition 5* in Section 4.1.

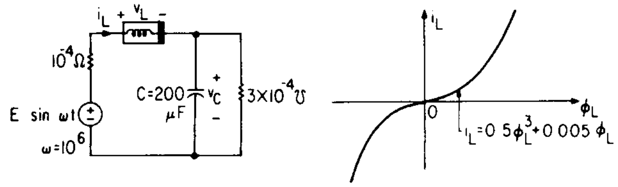


Fig. 19. A network having 2 distinct nonconstant periodic solutions.

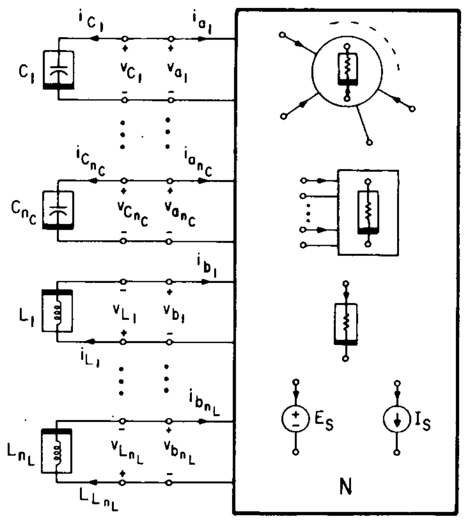


Fig. 20. Any *RLC* network $\mathfrak{N}$ can be represented by an n-port resistor N terminated by n_C (possibly coupled) capacitors and n_L (possibly coupled) inductors, where $n = n_C + n_L$.

Since the "output" frequency is smaller than the "input" frequency, we say the network has at least 3 distinct *subharmonic* solutions.

Example 20. To show that even networks containing only strictly locally passive elements can give rise to more than one steady-state regime, the network shown in Fig. 19 has been simulated on a computer [25]. The simulated results give at least 2 distinct steady-state solutions: a "periodic" waveform of the same frequency and an "almost subharmonic" waveform.

III. Global State Equation Formulation and Local Solvability

Any *RLC* network can be represented as shown in Fig. 20, where all capacitors and inductors are connected externally to a *resistive* n-port N. Since the capacitors {resp., inductors} may be coupled to each other, multi-terminal and/or multi-port capacitors and inductors are included in this representation. Two-terminal, multi-terminal, and/or multi-port resistors, as well as independent voltage and current sources are also included in N. For example, diodes, transistors (described by the Ebers–Moll equation, controlled sources, gyrators, and ideal transformers are resistors (because their constitutive relations are described by *algebraic* equations involving only currents and voltages) and are therefore included in N.

To simplify notation, we assume all elements, except possibly the independent sources, are time-invariant and are described as in Table I:

Table I. Constitutive Relations of the Elements in Fig. 20.

1. *Resistive n-port N*
   Hybrid Representation:
$$i_a = h_a(v_a, i_b; u_s(t)) \tag{3.1}$$
$$v_b = h_b(v_a, i_b; u_s(t)) \tag{3.2}$$
   where $u_s(t) \triangleq [E_s(t), I_s(t)]^T$ denotes the independent sources.
2. *Capacitors*
   Voltage-controlled Representation:
$$q_C = \hat{q}_C(v_C) \tag{3.3a}$$
   charge-controlled Representation:
$$v_C = \hat{v}_C(q_C) \tag{3.3b}$$
3. *Inductors*
   Current-controlled Representation:
$$\phi_L = \hat{\phi}_L(i_L) \tag{3.4a}$$
   flux-controlled Representation:
$$i_L = \hat{i}_L(\phi_L). \tag{3.4b}$$

3.1. Two Common Formulations

The examples in Section 2.1 show that a network is *ill-posed* if there exist *impasse* points because solutions starting from such points do not exist, while solutions arriving at such points at t_0 cannot be continued in forward time $t > t_0$, or in backward time $t < t_0$. Such networks are *nonphysical* and can not have a *globally* defined state equation.

In this paper, we exclude ill-posed networks (except Section 3.4) and consider only those networks having a globally defined state equation

$$\dot{x} = f(x, t) \tag{3.5}$$

where f: $\mathbb{R}^{n+1} \to \mathbb{R}^n$ is a C^k-function of x, $k \geq 0$.[11]

Given (3.5), we can define *infinitely many equivalent* state equations

$$\dot{y} = \left.\frac{\partial \hat{y}(x)}{\partial x} f(x, t)\right|_{x = \hat{x}(y)} \triangleq \hat{f}(y, t) \tag{3.6}$$

where $y = \hat{y}(x)$ and its inverse function $x = \hat{x}(y)$ are any C^1*-bijective* functions. In other words, if $\mathfrak{N}$ has a *global* state equation in terms of some state variable x, then we find infinitely many other variables which also qualify as state variables.

To simplify our notation, we will consider only the two most common choices;[12] namely, $x = (v_C, i_L)$ or $z = (q_C, \phi_L)$

[11] $f(x, t)$ is said to be a C^k-function of x *iff* at any time t, all of its kth-order derivatives exist [26] for all $x \in \mathbf{R}^n$, and each *partial derivative* of order k is a continuous function. For example, f is a C^0-function iff it is *continuous*, and a C^1-function iff it is *continuously differentiable*.

[12] For some special classes of networks, the *Lagrangian* or *Hamiltonian* formulation may be more appropriate [27]–[29].

(1) v_C–i_L *Formulation*:

Assume all capacitors are voltage-controlled and the incremental Capacitance matrix

$$C(v_C) \triangleq \frac{\partial \hat{q}_C(v_C)}{\partial v_C} \quad (3.7a)$$

is nonsingular for all $v_C \in \mathbb{R}^{n_C}$.

Assume all inductors are current-controlled and the incremental Inductance matrix

$$L(i_L) \triangleq \frac{\partial \hat{\phi}_L(i_L)}{\partial i_L} \quad (3.7b)$$

is nonsingular for all $i_L \in \mathbb{R}^{n_L}$.

Substituting $i_C = \dot{q}_C = C(v_C)\dot{v}_C = -i_a$ into (3.1) and $v_L = \dot{\phi}_L = L(i_L)\dot{i}_L = -v_b$ into (3.2), we obtain the following state equation in terms of the state variables v_C and i_L:

$$\boxed{\begin{aligned} \dot{v}_C &= -C^{-1}(v_C)h_a(v_a, i_b; u_s(t)) \\ \dot{i}_L &= -L^{-1}(i_L)h_b(v_a, i_b; u_s(t)). \end{aligned}} \quad (3.8)$$

(2) q_C–ϕ_L *Formulation*:

Substituting (3.3b) for v_a and (3.4b) for i_b in (3.1)–(3.2), and noting that $\dot{q}_C = -i_a$ and $\dot{\phi}_L = -v_b$, we obtain the following state equation in terms of the state variables q_C and ϕ_L:

$$\boxed{\begin{aligned} \dot{q}_C &= -h_a\big(\hat{v}_C(q_C), \hat{i}_L(\phi_L); u_s(t)\big) \\ \dot{\phi}_L &= -h_b\big(\hat{v}_C(q_C), \hat{i}_L(\phi_L); u_s(t)\big). \end{aligned}} \quad (3.9)$$

A network $\mathcal{N}$ described by either (3.8) or (3.9) is said to be *autonomous* iff $u_s(t) = k$ is a constant vector; i.e., $\mathcal{N}$ contains only *dc* sources. Otherwise, $\mathcal{N}$ is said to be *nonautonomous*.

If we define

$$x \triangleq \begin{bmatrix} v_a \\ i_b \end{bmatrix} = \begin{bmatrix} v_C \\ i_L \end{bmatrix}, \quad y \triangleq \begin{bmatrix} i_a \\ v_b \end{bmatrix}, \quad z \triangleq \begin{bmatrix} q_C \\ \phi_L \end{bmatrix} \quad (3.10)$$

then (3.1)–(3.4) and (3.7) can be written as follows:

Resistor Function:

$$y = h(x; u_s(t)) \triangleq \begin{bmatrix} h_a(v_a, i_b; u_s(t)) \\ h_b(v_a, i_b; u_s(t)) \end{bmatrix} \quad (3.11)$$

Capacitor–Inductor Function:

$$x = g(z) \triangleq \begin{bmatrix} \hat{v}_C(q_C) \\ \hat{i}_L(\phi_L) \end{bmatrix} \quad (3.12)$$

Capacitance–Inductance Matrix:

$$D(x) \triangleq \begin{bmatrix} C(v_C) & 0 \\ 0 & L(i_L) \end{bmatrix}. \quad (3.13)$$

Using these abbreviated notations, the state equations (3.8) and (3.9) for autonomous (where we suppressed the constant source vector) and *nonautonomous* networks assume the following compact forms [25], [30], [31][13]

[13]The reader is cautioned that our notations differ from those in [25], [30], [31]. Specifically, $x, y, z, h(x)$, and $g(z)$ in this paper correspond to $x_p, -y_p, z_p, g_p(x_p)$, and $h_p(z_p)$ in [25], [30], [31].

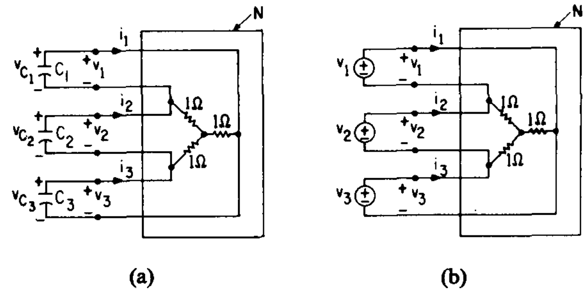


Fig. 21. (a) A 3-capacitor autonomous *RC* network $\mathcal{N}$. (b) The resistive 3-port N associated with $\mathcal{N}$ driven by 3 voltage sources v_1, v_2, and v_3 corresponding to the 3 capacitor voltages v_{C_1}, v_{C_2}, and v_{C_3}.

(1) $v_C - i_L$ Formulation: $x \triangleq (v_C, i_L)$

Autonomous Networks:

$$\dot{x} = -D^{-1}(x)h(x) \quad (3.14a)$$

Nonautonomous Networks:

$$\dot{x} = -D^{-1}(x)h(x; u_s(t)) \quad (3.14b)$$

where $h(\cdot)$ and $D^{-1}(x)$ are C^0-functions of $x \in \mathbb{R}^n$.

(2) $q_C - \phi_L$ Formulation: $z \triangleq (q_C, \phi_L)$

Autonomous Networks:

$$\dot{z} = -h(g(z)) \quad (3.15a)$$

Nonautonomous Networks:

$$\dot{z} = -h(g(z); u_s(t)) \quad (3.15b)$$

where $h(\cdot)$ and $g(\cdot)$ are C^0-functions of $x \in \mathbb{R}^n$.

3.2. On the Existence of the Resistor Function

The preceding state equation formulation is deceptively simple because it assumes the *resistor function* $h(x; u_s(t))$ defined in (3.1) is given *a priori*. For each network, this function must be derived before the state equation can be written. The following examples show that this is not only a nontrivial task, but it could happen that such a function does not exist.

Example 1. Consider the 3-capacitor autonomous *RC* network $\mathcal{N}$ shown in Fig. 21(a). Since $x = (v_{C_1}, v_{C_2}, v_{C_3})$, the resistor function $h(x)$ for the associated resistive 3-port N is simply its conductance representation $i = Gv$. To derive this function, we drive N with 3 voltage sources v_1, v_2, v_3, as shown in Fig. 21(b), and determine the resulting port currents i_1, i_2, and i_3. Observe, however, that since the voltage sources form a loop, they can not be chosen as *independent* variables for N. Hence $h(x)$ *does not exist.*

Example 2. Consider the 3-inductor autonomous *RL* network $\mathcal{N}$ shown in Fig. 22(a). Since $x = (i_{L_1}, i_{L_2}, i_{L_3})$, the resistor function $h(x)$ for the associated resistive 3-port N is simply its *resistance representation* $v = Ri$. To derive this function, we drive N with 3 current sources i_1, i_2, and i_3, as shown in Fig. 22(b), and determine the resulting port voltages v_1, v_2, and v_3. Observe, however, that since the current sources form a cut set, they can not be chosen as *independent* variables for N. Hence $h(x)$ *does not exist.*

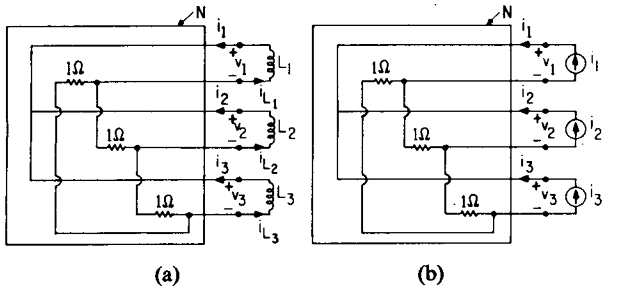


Fig. 22. (a) A 3-inductor autonomous network $\mathfrak{N}$. (b) The resistive 3-port N associated with $\mathfrak{N}$ driven by 3 current sources i_1, i_2, and i_3 corresponding to the 3 inductor currents i_{L_1}, i_{L_2}, and i_{L_3}.

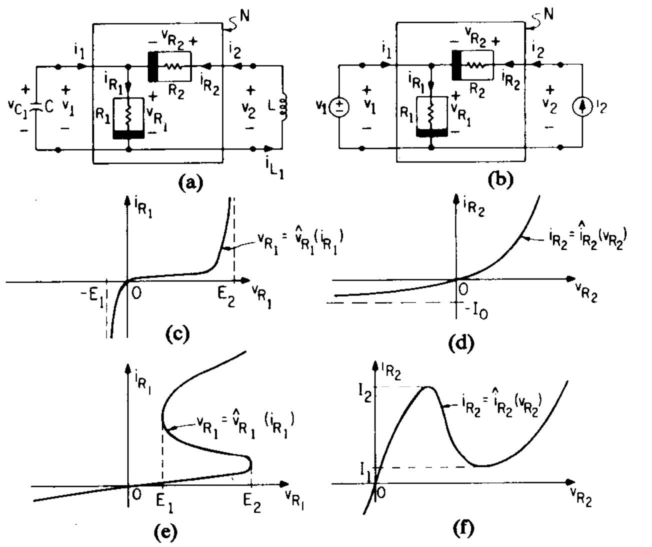


Fig. 23. (a) An autonomous RC network $\mathfrak{N}$. (b) The resistive 2-port N associated with $\mathfrak{N}$ driven by a voltage source v_1 (corresponding to capacitor C) and a current source i_2 (corresponding to inductor L). (c) A typical "Zener diode" v_{R_1}–i_{R_1} curve. (d) A typical "p-n junction diode" v_{R_2}–i_{R_2} curve. (e) A typical "neon bulb" v_{R_1}–i_{R_1} curve. (f) A typical "tunnel diode" v_{R_2}–i_{R_2} curve.

Example 3. Consider the autonomous RLC network $\mathfrak{N}$ shown in Fig. 23(a). Since $x=(v_{C_1}, i_{L_1})$, the resistor function $h(x)$ for the associated resistive 2-port N is obtained by driving port 1 with a voltage source v_1 and port 2 with a current source i_2, as shown in Fig. 23(b), and then solving for $i_1=h_1(v_1,i_2)$ and $v_2=h_2(v_1,i_2)$. Since $i_1=i_{R_1}-i_2$ and $v_2=v_{R_2}+v_1$, it is necessary to express i_{R_1} and v_{R_2} in terms of the 2 *independent* variables v_1 and i_2. Let us consider 2 illustrative cases:

Case 1. Assume R_1 is a *Zener diode* described by the i_{R_1}–v_{R_1} curve shown in Fig. 23(c) and R_2 is a *p-n junction diode* described by the i_{R_2}–v_{R_2} curve shown in Fig. 23(d). Note that R_1 is current-controlled but not voltage-controlled because i_{R_1} is undefined for $v_{R_1}\geqslant E_2$ and $v_{R_1}\leqslant -E_1$. Similarly, R_2 is voltage-controlled but not current-controlled because v_{R_2} is undefined for $i_{R_2}\leqslant -I_0$. Hence, even though both curves are strictly increasing, the resistor function

$$i_1=h_1(v_1,i_2)\triangleq \hat{i}_{R_1}(v_1)-i_2, \qquad -E_1<v_1<E_2 \tag{3.16a}$$

$$i_2=h_2(v_1,i_2)\triangleq \hat{v}_{R_2}(i_2)+v_2, \qquad i_2>-I_0 \tag{3.16b}$$

is *not* defined for all $(v_1,i_2)\in\mathbb{R}^2$. Hence $\boldsymbol{h}(\boldsymbol{x})$ *does not exist* in the region $E_2\leqslant v_1\leqslant -E_1$ and $i_2\leqslant -I_0$.

Case 2. Assume R_1 is a *neon bulb* described by the $i_{R_1}\to v_{R_1}$ curve shown in Fig. 23(e) and R_2 is a *tunnel diode* described by the i_{R_2}–v_{R_2} curve shown in Fig. 23(f). Since R_1 is a *multi-valued* function of v_{R_1} over $E_1\leqslant v_{R_1}\leqslant E_2$ and R_2 is a *multivalued* function of i_{R_2} over $I_1\leqslant i_{R_2}\leqslant I_2$, the resistor function $\boldsymbol{h}(\boldsymbol{x})$ *does not exist* in the region $E_1\leqslant v_1\leqslant E_2$ and $I_1\leqslant i_2\leqslant I_2$.

The preceding examples give rise to several important observations concerning the existence of $\boldsymbol{h}(\boldsymbol{x};\,\boldsymbol{u}_s(t))$.

Observation 1. The resistor function $h(x;\,u_s(t))$ does *not* exist whenever $\mathfrak{N}$ contains a *loop* made exclusively of capacitors and/or voltage sources (henceforth called *C–E_s loop*), or a *cut set* made exclusively of inductors and/or current sources (henceforth called *L–I_s cut set*).

The nonexistence of $h(x;\,u_s(t))$ under *Observation 1* does not necessarily imply that $\mathfrak{N}$ does not have a state equation. In fact, by choosing only 2 out of 3 state variables, both circuits in Figs. 21 and 22 have a well-defined state equation.

Systematic methods for deriving state equations for nonlinear networks containing C–E_s loops and L–I_s cut sets are given in [12, ch. 10]. Unfortunately, these methods often involve an excessive amount of algebra, thereby making it all but impossible to derive the "nonlinear" state equation in an explicit analytical form. This is why most results and theorems concerning dynamic nonlinear networks are stated only for networks, containing neither C–E_s loops nor L–I_s cut sets. From a practical point of view, this represents a severe restriction because most realistic semiconductor device models contain capacitor loops (representing stray capacitances) and inductor cut sets (representing parasitic inductances) [32], especially for high-frequency operations.

Fortunately, the following result shows that the above restriction is unnecessary.

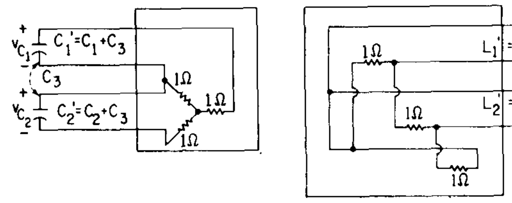
Theorem 1. C–E_s Loop and L–I_s Cut Set Transformation [30], [33][14]

(a) Every C–E_s loop in $\mathfrak{N}$ may be eliminated by open-circuiting any one capacitor in the loop, and by modifying the constitutive relations of the remaining elements in the loop, without altering the solutions of $\mathfrak{N}$.

(b) Every L–I_s cut set in $\mathfrak{N}$ may be eliminated by *short-circuiting* any one inductor in the cut set, and by modifying the constitutive relations of the remaining elements in the cut set, without altering the solution of $\mathfrak{N}$.

(c) The *modified* constitutive relations of the capacitors and the inductors can be derived *explicitly* from the original constitutive relations [12], [30], [33]. Moreover, most *circuit-theoretic* properties (e.g., *reciprocity (Definition 4), passivity, (Definition 5) strict local passivity (Definition 5), strong local passivity (Definition 1)*)

[14]A special case of Theorem 1 which holds only for *linear* networks is given in [12, pp. 434–436].



(a) (b)

Fig. 24. (a) Equivalent network obtained by open-circuiting capacitor C_3 from Fig. 21(a), thereby eliminating the capacitor loop. (b) Equivalent network obtained by short-circuiting inductor L_3 from Fig. 22(a), thereby eliminating the inductor cut set.

possessed by the original elements are inherited by the transformed elements.

Applying *Theorem 1* to the networks shown in Figs. 21(a) and 22(a), we obtain the equivalent networks shown in Fig. 24(a) and (b). In the modified network $\mathfrak{N}'$ of Fig. 24(a), the value C_3 of the open-circuited capacitor is added to C_1 and C_2 while a "mutual capacitance" equal to C_3 is introduced between C_1' and C_2'. Note that $\mathfrak{N}'$ no longer contains a C–E_s loop. In the modified network $\mathfrak{N}'$ of Fig. 24(b), the value L_3 of the short-circuited inductor is added to L_1 and L_2 while a "mutual inductance" equal to L_3 is introduced between L_1' and L_2'. Note that $\mathfrak{N}'$ no longer contains an L–I_s cut set.

From the *qualitative analysis* point of view, we seldom need to actually implement the above transformations. It is property (c) which we will find most useful: it allows us to generalize trivially any result or theorem (which previously exclude C–E_s loops and L–I_s cut sets) whose hypotheses invoke one or more of the "preserved" circuit-theoretic or structural properties to networks that allow both C–E_s loops and L–I_s cut sets.

Observation 1 shows that the state equations formulated in Table II of Section II implicitly assume that the networks $\mathfrak{N}$ contains neither C–E_s loop nor L–I_s cut sets. *Theorem 1(c)* shows that this assumption involves little loss of generality.

Observation 2. The resistor function $\boldsymbol{h}(\boldsymbol{x};\ \boldsymbol{u}_s(t))$ does *not* exist whenever a current-controlled (but not voltage-controlled) resistor is in parallel with a capacitor, or whenever a voltage-controlled (but not current-controlled) resistor is in series with an inductor.

Observe that if we interchange the 2 resistors in Fig. 23(a), the resistor function $\boldsymbol{h}(\boldsymbol{x})$ becomes well defined (for both sets of curves in Fig. 23). This is because the *independent* variable defining each resistor coincides with an *independent* variable defining N; namely, $v_{R_1}=v_1$ and $i_{R_2}=i_2$.

Except for contrived cases, $\boldsymbol{h}(\boldsymbol{x};\ \boldsymbol{u}_s(t))$ usually does *not* exist whenever N contains one or more voltage-controlled (but not current-controlled) resistors which are not in parallel with capacitors, or current-controlled (but not voltage-controlled) resistors which are not in series with inductors. Moreover, the nonexistence of $\boldsymbol{h}(\boldsymbol{x};\ \boldsymbol{u}_s(t))$ in this case implies the nonexistence of the state equation.

Observation 3. The problem of determining whether the resistor function $\boldsymbol{h}(\boldsymbol{x};\ \boldsymbol{u}_s(t))$ exists for the dynamic network $\mathfrak{N}$ in Fig. 20 is equivalent to the problem of determining whether the associated *resistive* network (obtained by connecting n_C voltage sources $\boldsymbol{E}_s$ across the capacitor ports and n_L current sources $\boldsymbol{I}_s$ across the inductor ports of the resistive n-port N) has a *unique* solution (or operating point) *for all* $\boldsymbol{E}_s \in \mathbb{R}^{n_C}$ and $\boldsymbol{I}_s \in \mathbb{R}^{n_L}$.

If we examine the curves in Figs. 23(c)–(f) carefully, we will find points on each curve whose slope either tends to zero or infinity. Note that the resistor function $\boldsymbol{h}(\boldsymbol{x})$ would exist if the slope on each point of each curve is bounded from above and below by 2 positive finite constants; namely, $\underline{\gamma} \leqslant f'(x) \leqslant \bar{\gamma}$. This property turns out to be a rather basic condition often invoked in theorems for nonlinear network containing both 2-terminal and multi-terminal elements. Hence, let us formally define this property for vector-valued functions:

Definition 1. Strong-Local Passivity

A multi-terminal (or multi-port) *Resistor*, *Inductor*, or *Capacitor*[15] described by a constitutive relation [11] $\boldsymbol{y}=\boldsymbol{h}(\boldsymbol{x})$, where $\boldsymbol{h}$: $\mathbb{R}^\alpha \to \mathbb{R}^\alpha$ is a continuous function, is said to be *Strongly Locally Passive*[16] iff there exist finite constants $\bar{\gamma} > \underline{\gamma} > 0$ such that for all $\boldsymbol{x}'$ and $\boldsymbol{x}'' \in \mathbb{R}^\alpha$, we have[17]

$$\underline{\gamma}\|\boldsymbol{x}'-\boldsymbol{x}''\|^2 \leqslant (\boldsymbol{x}'-\boldsymbol{x}'')^T[\boldsymbol{h}(\boldsymbol{x}')-\boldsymbol{h}(\boldsymbol{x}'')] \leqslant \bar{\gamma}\|\boldsymbol{x}'-\boldsymbol{x}''\|^2 \tag{3.17}$$

Theorem 2. Existence of Resistor Function

The Resistor function $\boldsymbol{h}(\boldsymbol{x};\ \boldsymbol{u}_s(t))$ associated with N of Fig. 20 exists for any *bounded* source vector $\boldsymbol{u}_s(t)$ if the following 5 conditions are satisfied:

1. There is no loop {resp., no cut set} formed exclusively by capacitors, inductors and/or independent voltage sources {resp.; current sources}.
2. Each *voltage-controlled* (but not current-controlled) 2-terminal resistor is in parallel with a *capacitor*. Each *current-controlled* (but not voltage-controlled) 2-terminal resistor is in series with an *inductor*.
3. Each remaining 2-terminal resistor is *strongly locally passive*, or else it is either in parallel with a capacitor or in series with an inductor.
4. Each multi-terminal (or multi-port) Resistor R_α is *strongly locally passive*, or else each *voltage-controlled* terminal pair (or port)[18] of R_α is in parallel with a

[15]The inductors and capacitors are assumed to be Reciprocal [20] (see Definition 4) in this definition.

[16]Also called *strongly uniformly increasing* in [30]–[31]. In spite of the superficial similarity between the terminologies, "strong local" passivity and passivity, they are *entirely independent* concepts. As a useful mnemonic aid, whenever the word "local" appears, the curve (in the case of 2-terminal elements) must *at least be monotone increasing*. On the other hand, when the term passivity appears without "local," the curve need only lie in the first and third quadrants. However, the *slope* may be *positive* or *negative* at a point on the curve.

[17]Throughout this paper, $\|\cdot\|$, denotes the Euclidean norm; i.e., $\|\boldsymbol{x}\| = [x_1^2+x_2^2+\cdots+x_\alpha^2]^{1/2}$.

[18]A *terminal pair* (or port) of a multi-terminal (or multi-port) resistor R_α is said to be voltage-controlled {resp., current-controlled} *iff* its associated voltage {resp., current} qualifies as an *independent* variable in a well-defined representation $\boldsymbol{y}_\alpha=\boldsymbol{h}(\boldsymbol{x}_\alpha)$ of R_α. For example, if R_α is described by $i_1=h_1(v_1,i_2) \triangleq v_1^2+i_2^2$ and $v_2=h_2(v_1,i_2) \triangleq \sin(v_1+i_2)$, then port 1 is voltage-controlled (but not current-controlled) while port 2 is current-controlled (but not voltage-controlled). Note that v_1 and i_2 are *multi-valued* functions of i_1 and v_2 and, therefore, do *not* qualify as independent variables.

capacitor, and each *current-controlled* terminal pair (or port) of R_α is in series with an *inductor*.

5. The constitutive relations of all $(k+1)$-terminal or k-port resistors are described by *continuous* functions in $\mathbb{R}^k$, $k \geqslant 1$.

The proof of *Theorem 2* depends on several results from [30], [34] and is given in *Appendix A.1.*

Note that conditions 1–5 are mostly topological in nature and hence can be easily checked by inspection. The only condition that needs to be checked for *strong local passivity* can be determined by inspection of the *i–v* curve in the case of 2-terminal resistors, or by the following algebraic test in the case of multi-terminal (or multi-port) resistors:

Strong Local Passivity criterion [30][19]

Let the constitutive relation $\boldsymbol{h}(\boldsymbol{x}_\alpha)$ in *Definition 1* be a C^1-function and let $\boldsymbol{H}(\boldsymbol{x}_\alpha)$ be its associated incremental *hybrid* (Jacobian) matrix.

A multi-terminal (or multi-port) *Resistor*, *Inductor*, or *Capacitor* is *strongly locally passive*$\Leftrightarrow$here exist 2 positive constants $\underline{\lambda}$ and $\overline{\lambda}$ such that the 2 matrices $\boldsymbol{H}(\boldsymbol{x}_\alpha)-\underline{\lambda}\mathbf{1}$ and $\overline{\lambda}\mathbf{1}-\boldsymbol{H}(\boldsymbol{x}_\alpha)$ are both *positive semi-definite* for all $\boldsymbol{x}_\alpha \in \mathbb{R}^\alpha$ ($\mathbf{1}$ denotes the identity matrix).

Remarks. 1. It follows from the above criterion that *locally active* elements, such as transistors, cannot be strongly locally passive. However, realistic models of such elements usually contain a capacitor in parallel with each voltage-controlled terminal pair (or port) and an inductor in series with each current-controlled terminal pair (or port). Consequently, *Theorem 2* is in fact applicable to a very large class of properly modeled dynamic networks.

2. Conditions 1–3 of *Theorem 2* are *necessary* for the existence of the resistor function $\boldsymbol{h}(\boldsymbol{x};\, \boldsymbol{u}_s(t))$.

3. Conditions 4 and 5 of *Theorem 2* are "almost" necessary for the existence of $\boldsymbol{h}(\boldsymbol{x};\, \boldsymbol{u}_s(t))$ in the following sense: if port j of R_α is voltage-controlled but not current-controlled, then it is usually necessary that port j be in parallel with a capacitor. Likewise, if port j of R_α is current-controlled but not voltage-controlled, then it is usually necessary that port j be in series with an inductor. However, exceptions can occur in some artificial and contrived examples.

4. For the special class of transistor-diode networks which may not satisfy condition 5, the existence of $\boldsymbol{h}(\boldsymbol{x};\, \boldsymbol{u}_s(t))$ can be checked by necessary and sufficient conditions given in [9]. In this case, $\boldsymbol{h}(\boldsymbol{x};\, \boldsymbol{u}_s(t))$ for a given network may exist for certain numerical values of element parameters (say the value of some resistances) but not exist for certain other values.

In contrast to this element parameter dependency,[20] *Theorem 2* depends only on the network topology and on the strong local passivity of certain elements.

3.3. *On the Existence of a Global State Equation*

We are now ready to address the basic question posed earlier: When does an *RLC* network have a global C^0-state equation?

Theorem 3. Global State Equation Existence Criteria

(a) v_C–i_L *Formulation*

The state equation (3.14) exists for all $\boldsymbol{x} \triangleq (\boldsymbol{v}_C, \boldsymbol{i}_L) \in \mathbb{R}^n$ if the following conditions are satisfied:

1. The topological condition 1 of *Theorem 2* is satisfied.

2. The *resistors* satisfy conditions 2–5 of *Theorem 2*.

3. The *capacitors* can be described by a C^1-voltage-controlled representation (3.3a) whose incremental capacitance matrix $\boldsymbol{C}(\boldsymbol{v}_C)$ is nonsingular for all $\boldsymbol{q}_C \in \mathbb{R}^{n_C}$.

4. The *inductors* can be described by a C^1-*current-controlled* representation (3.4a) whose incremental inductance matrix $\boldsymbol{L}(\boldsymbol{i}_L)$ is nonsingular for all $\boldsymbol{\phi}_L \in \mathbb{R}^{n_L}$.

(b) q_C–ϕ_L *Formulation*

The state equation (3.15) exists for all $\boldsymbol{z} \triangleq (\boldsymbol{q}_C, \boldsymbol{\phi}_L) \in \mathbb{R}^n$ if the following conditions are satisfied:

1. The topological condition 1 of *Theorem 2* is satisfied.

2. The *resistors* satisfy conditions 2–5 of *Theorem 2*.

3. The *capacitors* can be described by a C^0-*charge-controlled* representation (3.3b).

4. The *inductors* can be described by a C^0-flux-controlled representation (3.4b).

The proof of *Theorem 3* follows directly from (3.14), (3.15), and *Theorem 2.*

Remarks. 1. In the more general mixed formulation where some capacitors {resp.; inductors} are C^1-*voltage-controlled* {resp.; C^1-*current-controlled*} while others are C^0-*charge-controlled* {resp.; C^0-flux-controlled}, *Theorem 3* is still applicable (*mutatis-mutandis*) by combining the conditions (a) and (b) in an obvious way.

2. Although not all conditions in *Theorem 3* are necessary for the existence of a global state equation, they are "almost" necessary in the sense that except in contrived cases, most networks which violate one or more conditions of *Theorem 3* do not have a global state equation regardless of the choice of state variables.

3. The non-existence of a global state equation for $\mathfrak{N}$ usually, but not always, leads to the existence of *impasse* points, as demonstrated in *Examples 3 and 4* of Section 2.1. In order to uncover the precise condition which gives rise to impasse points, let us review Examples 1–5 of Section 2.1.

3.4. *On Local Solvability*

Observe that impasse points occur in *Examples 3 and 4* because it is impossible to write a state equation even if

[19] For a time-varying constitutive relation $\boldsymbol{h}(\boldsymbol{x}_\alpha, t)$, simply apply this criterion at each time t.

[20] It is important to distinguish between conditions which depend on element parameters from those which do not. We will refer to the latter as *graph-* or *circuit-theoretic* conditions. One of the most significant and challenging problems in nonlinear network theory is to derive *qualitative properties* involving only graph- and circuit-theoretic conditions if at all possible.

the state space is restricted to an arbitrarily small neighborhood of Q_1 and Q_2. This observation motivates our next definition:

Let $z \triangleq (q_C, \phi_L)$ and $x \triangleq (v_C, \phi_L)$. A point Q in $\mathbb{R}^n \times \mathbb{R}^n$ with coordinates $(z_a, x_a) = (q_{C_Q}, \phi_{L_Q}, v_{C_Q}, i_{L_Q})$ is called a *capacitor–inductor operating point* of the *RLC* network $\mathfrak{N}$ in Fig. 20 iff (q_{C_Q}, v_{C_Q}) and (ϕ_{L_Q}, i_{L_Q}) satisfy the *constitutive relation*[21] of the capacitors and inductors, respectively.

Definition 2. Local Solvability[22]

The *RLC* network $\mathfrak{N}$ in Fig. 20 is said to be *locally solvable* iff given *any* capacitor–inductor operating point Q: (z_Q, x_Q) and *any* time t_0, there exists an ϵ-neighborhood of Q and t_0 such that $\mathfrak{N}$ has a C^1 *local state equation*

$$\dot{z} = f_Q(z, t), \qquad \text{for all } \|z - z_Q\| < \epsilon \text{ and } |t - t_0| < \epsilon \tag{3.18}$$

where $f_Q(\cdot)$ is a C^1 function which generally depends on both Q and t_0, and $\epsilon > 0$.

Theorem 4. Local Solvability Criteria

The *RLC* network in Fig. 20 is *locally solvable* $\Leftrightarrow$ given *any* capacitor–inductor operating point Q: $(q_{C_Q}, \phi_{L_Q}, v_{C_Q}, i_{L_Q})$ and *any* initial time t_0, the constitutive relations in Table I have the following C^1 *local*[23] representation in an ϵ-neighborhood of Q and t_0:

1. The *resistive n-port N* has a C^1 *local hybrid* representation $h_Q(\cdot)$ about the *resistor* operating point $(v_a, i_b) = (v_{C_Q}, i_{L_Q})$; namely,

$$\boxed{\begin{aligned} i_a &= h_{a_Q}(v_a, i_b; u_s(t)) \\ v_b &= h_{b_Q}(v_a, i_b; u_s(t)) \end{aligned}} \quad \text{for all } \|v_a - v_{C_Q}\| < \epsilon,$$
$$\|i_b - i_{L_Q}\| < \epsilon \text{ and } |t - t_0| < \epsilon. \tag{3.19}$$

2. Each *capacitor* has a C^1 *local charge-controlled* representation about the *capacitor* operating point (q_{C_Q}, v_{C_Q}):

$$v_C = \hat{v}_{C_Q}(q_C), \qquad \|q_C - q_{C_Q}\| < \epsilon. \tag{3.20}$$

3. Each *inductor* has a C^1 *local flux-controlled* representation about the *inductor* operating point (ϕ_{L_Q}, i_{L_Q}):

$$i_L = \hat{i}_{L_Q}(\phi_L), \qquad \|\phi_L - \phi_{L_Q}\| < \epsilon. \tag{3.21}$$

Proof: Sufficiency is obvious. Necessity follows, *mutatis-mutandis*, from the $q_C - \phi_L$ formulation given in Section 3.1. □

Corollary. $\mathfrak{N}$ is *locally solvable* $\Leftrightarrow$(3.18) can be written in the form

$$\begin{aligned} \dot{z} &= -h_Q(g_Q(z); u_s(t)) \\ &\triangleq f_Q(z, t), \qquad \|z - z_Q\| < \epsilon, \ |t - t_0| < \epsilon \end{aligned} \tag{3.22}$$

where $h_Q(\cdot) \triangleq (h_{a_Q}(\cdot), h_{b_Q}(\cdot))$ and $g_Q(\cdot) \triangleq (\hat{v}_{C_Q}(\cdot), \hat{i}_{L_Q}(\cdot))$ are C^1 *functions* about Q and t_0. □

In the usual case where $h_Q(\cdot)$ and $g_Q(\cdot)$ do not depend on Q and t_0, then (3.22) becomes the *global* state equation (3.15b).

The preceding definition on local solvability was stated in terms of *one particular* choice of state variables—out of infinitely many possibilities— namely, the *capacitor charge* q_C and *inductor flux* ϕ_L. Our next theorem shows that this is the correct choice.

Theorem 5. Capacitor Charge and Inductor Flux are Basic State Variables

If $\mathfrak{N}$ is *not* locally solvable, then there exists no other state variable which can give rise to a C^1 local state equation about *each* capacitor–inductor operating point Q.

Proof: The proof requires a coordinate-free geometric approach [237]–[39]. A more abstract differential-geometric version of this theorem is stated and proved in [39]. □

Observe that *Theorem 5* does *not* hold if local solvability is defined in terms of capacitor voltage v_C and inductor current i_L. Indeed, *Example 2* (Fig. 2) does not have a local state equation with respect to v_C at Q_1 and Q_2. Yet the network has a C^1 global state equation (2.17) in terms of q_C. Likewise, *Example 11* (Fig. 11) does not have a local state equation with respect i_L at infinitely many isolated points ($k\phi_L = \pm(m+1/2)\pi$, $m = 0, \pm 1, \pm 2, \cdots$). Yet the network has a C^1 global state equation (2.33) in terms of ϕ_L.

Our main motivation for defining *local solvability* was to derive the weakest condition which guarantees that there are no "impasse" points. To prove this formally, we must define impasse points precisely:

Definition 3. Impasse Point[24]

A *capacitor–inductor operating point* Q: $(q_{C_Q}, \phi_{L_Q}, v_{C_Q}, i_{L_Q})$ is called an *impasse point* of the *RLC* network $\mathfrak{N}$ *iff* there are *no* C^1 functions $(q_C(t), \phi_L(t), v_C(t), i_L(t))$ with $(q_C(t_0), \phi_L(t_0), v_C(t_0), i_L(t_0)) = (q_{C_Q}, \phi_{L_Q}, v_{C_Q}, i_{L_Q})$ which satisfy KCL, KVL, and the elements constitutive relations over $(t_0 - \epsilon) < t < (t_0 + \epsilon)$ for arbitrarily small $\epsilon > 0$. □

It follows from *Definition 3* that $\mathfrak{N}$ can not have a solution passing through an *impasse point* over any finite time interval. Our next theorem shows that the existence of an *impasse point* is a *local* phenomenon and, therefore, can be avoided by weaker "local" conditions which do not require that $\mathfrak{N}$ have a global state equation.

[21] The constitutive relations of the capacitors and inductors are defined by $f_C(q_C, v_C) = 0$ and $f_L(\phi_L, q_L) = 0$, respectively. Hence they need not be x-controlled or z-controlled. This is why it is generally necessary to specify *both* z and x in order to identify an operating point Q unambiguously. For example, $v_C = 6$ gives rise to 3 distinct points in Fig. 2(b) while $q_C = 6$ gives rise to 3 distinct points in Fig. 3(b).

[22] This definition is a *nonautonomous version* of *Definition 1* in [36] but with the state variables chosen to be *capacitor charge* q_C and *inductor flux* ϕ_L.

[23] By *local*, we mean the functions $h_Q(\cdot)$, $\hat{v}_{C_Q}(\cdot)$, and $\hat{i}_{L_Q}(\cdot)$ in (3.19), (3.20), and (3.21) generally depend on both the operating point Q and the initial time t_0.

[24] Note that this definition is stated directly in terms of the elements' constitutive relation and therefore does *not* require that $\mathfrak{N}$ have a local or global state equation.

Theorem 6. No-Impasse-Point Criterion
If the *RLC* network $\mathfrak{N}$ in Fig. 20 is *locally solvable*, then $\mathfrak{N}$ has no impasse points.

Proof: If $\mathfrak{N}$ is locally solvable, then given *any* $z_Q=(q_{C_Q},\phi_{L_Q})\in\mathbb{R}^n$ and *any* t_0, $\mathfrak{N}$ can be described by a local state equation (3.18) where $f_Q(z,t)$ is a C^1-function about (z_Q,t_0). Hence there exists a unique solution $z(t)=(q_C(t),\phi_L(t))$ such that $z(t_0)=z_Q$ for $|t-t_0|<\epsilon$, where $\epsilon>0$ [12]. It follows from (3.20) and (3.21) that $\mathfrak{N}$ has no impasse points. □

To illustrate the significance of *Theorems 4, 5, and 6*, consider the following examples from Section 2.1 again:

Example 1 (Fig. 1). $\mathfrak{N}$ is *locally solvable*$\Leftrightarrow\mathfrak{N}$ has *no impasse points* inspite of the fact that its state equation (2.13) with respect to v_C does not exist at $v_C=1$.

Example 2 (Fig. 2). $\mathfrak{N}$ is *locally solvable*$\Leftrightarrow\mathfrak{N}$ has *no impasse points* inspite of the fact that its state equation (2.18) with respect to v_C does not exist over the interval $5\frac{1}{3}\leqslant v_C\leqslant 6\frac{2}{3}$.

Example 3 (Fig. 3). $\mathfrak{N}$ is *not* locally solvable because (3.20) fails (C is not locally charge-controlled at Q_1 and Q_2). Here, we find 2 impasse points at Q_1 and Q_2.

Example 4 (Fig. 4). $\mathfrak{N}$ is *not* locally solvable because (3.19) fails (R is not locally voltage-controlled at Q_1 and Q_2). Here, we find 2 impasse points at Q_1 and Q_2.

Example 5 (Fig. 5). $\mathfrak{N}$ is *locally solvable*$\Leftrightarrow\mathfrak{N}$ has *no impasse points* inspite of the fact that its state equation with respect to q_C does *not* exist *globally*.

Example 6 (Fig. 6). $\mathfrak{N}$ is *not* locally solvable because $v_C=\hat{v}_C(q_C)=(\frac{3}{2}q_C)^{1/3}$ is *not* C^1 at $q_C=0$. Here, we find $q_C=0$ is an impasse point.

Remark. Although *Theorem 6* provides only a *sufficient* condition for no impasse points, *Examples 3, 4, and 6* strongly suggest that any network which is *not locally solvable* is potentially *ill-posed* and should be remodeled.

In the following sections, we make the *standing assumption* that all networks under consideration can be described by either a C^1 global state equation (3.14) or (3.15) in order to simplify the statement of the theorems. In particular, we assume (unless otherwise stated) that the *resistor* function $\boldsymbol{h}(\cdot)$ in (3.11) and the *capacitor–inductor function* $\boldsymbol{g}(\cdot)$ in (3.12) are C^1 *functions* in $\mathbb{R}^n$, where $n=n_C+n_L$.

IV. Qualitative Property 1: No Finite-Forward-Escape-Time Solutions

The absence of impasse points does *not* guarantee that the solution exists for all time t. Neither does it guarantee that the associated *initial-value problem*

$$\dot{x}=f(x,t),\quad x(t_0)=x_0 \tag{4.1}$$

has a *unique* solution: *Example 6* does not have a solution for $t>T$ and *Example 7* does *not* have a unique solution.

Both of these nonphysical situations can *not* occur if $f(\cdot)$ satisfies a *local Lipschitz* condition with respect to x in a neighborhood of x_0 [12]. In particular, if $\mathfrak{N}$ is locally solvable, then $\mathfrak{N}$ has a unique solution through each capacitor–inductor operating point Q at any initial time t_0.

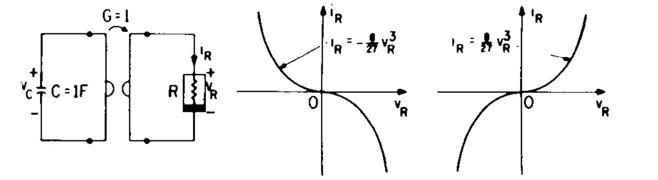


Fig. 25. A gyrator network which does not have a local C^1 state equation at $q_C=v_L=0$.

It is not possible to state a *general* uniqueness theorem which involves only conditions that can be easily checked at the element level. This is because even if all elements are described by a C^1 constitutive relation, the resulting state equation need not be C^1.

For example, consider the simple gyrator network in Fig. 25(a), where the resistor is described by the C^1 function shown in Fig. 25(b). All elements are clearly C^1 but the state equation[25]

$$\dot{q}_C=\tfrac{3}{2}q_C^{1/3} \tag{4.2}$$

is *not* C^1 at $q_C=0$. Indeed (4.2) has *infinitely* many solutions

$$q_C(t)=0,\qquad 0\leqslant t\leqslant k$$
$$=(t-k)^{3/2},\qquad t>k \tag{4.3}$$

where k is *any real* number.

If we replace the i_R–v_R curve by that shown in Fig. 25(c), the state equation

$$\dot{q}_C=-\tfrac{3}{2}q_C^{1/3} \tag{4.4}$$

is still *not* C^1 at $q_C=0$. However, in this case, (4.4) has a *unique* solution corresponding to any initial condition.

The above example suggests that sufficiently strong circuit-theoretic properties must be imposed at the element level to guarantee local uniqueness. Hence, let us define some of these properties first.[26]

4.1. Basic Circuit-Theoretic Properties

Definition 4. Reciprocity

(a) A multi-terminal (or multi-port) *Resistor* described by a C^1 function $\boldsymbol{h}(\cdot)$ in (3.1)–(3.2) is *reciprocal* iff its associated *incremental hybrid matrix*

$$\boldsymbol{H}(\boldsymbol{v}_a,\boldsymbol{i}_b;\boldsymbol{u}_s(t))=\begin{bmatrix}\partial\boldsymbol{h}_a/\partial\boldsymbol{v}_a & \partial\boldsymbol{h}_a/\partial\boldsymbol{i}_b\\ \partial\boldsymbol{h}_b/\partial\boldsymbol{v}_a & \partial\boldsymbol{h}_b/\partial\boldsymbol{i}_b\end{bmatrix}\triangleq\begin{bmatrix}\boldsymbol{H}_{aa} & \boldsymbol{H}_{ab}\\ \boldsymbol{H}_{ba} & \boldsymbol{H}_{bb}\end{bmatrix} \tag{4.5}$$

satisfies the following property *for all* $(\boldsymbol{v}_a,\boldsymbol{i}_b)\in\mathbb{R}^n$ and for all $t\in\mathbb{R}$:

1. $\boldsymbol{H}_{aa}$ and $\boldsymbol{H}_{bb}$ are *symmetric*. (4.6)
2. $\boldsymbol{H}_{ab}=-\boldsymbol{H}_{ba}^T$. (4.7)

[25] Recall the effect of the gyrator is to reflect the i_R–v_R curve along the 45°-line through the origin [22].

[26] The following definitions are stated in terms of the representations given in Table I. They are *equivalent* to the generalized *coordinate-independent* definitions given in [20], [30], [38].

(b) A multi-terminal (or multi-port) *Capacitor* described by a C^1 function in (3.3a) or (3.3b) is *reciprocal iff* its associated *incremental capacitance matrix* $\boldsymbol{C}(\boldsymbol{v}_C) \triangleq \partial \hat{\boldsymbol{q}}_C)/\partial \boldsymbol{v}_C$ or its *incremental elastance matrix* $\boldsymbol{S}(\boldsymbol{q}_C) \triangleq \partial \hat{\boldsymbol{v}}_C(\boldsymbol{q}_C)/\partial \boldsymbol{q}_C$ are *symmetric* for all $\boldsymbol{v}_C \in \mathbb{R}^{n_C}$ and $\boldsymbol{q}_C \in \mathbb{R}^{n_C}$, respectively.

(c) A multi-terminal (or multi-port) *Inductor* described by a C^1 function in (3.4a) or (3.4b) is *reciprocal iff* its associated *incremental inductance matrix* $\boldsymbol{L}(\boldsymbol{i}_L) \triangleq \partial \hat{\boldsymbol{\phi}}_L(\boldsymbol{i}_L)/\partial \boldsymbol{i}_L$ or its *incremental reciprocal inductance matrix* $\boldsymbol{\Gamma}(\boldsymbol{\phi}_L) \triangleq \partial \hat{\boldsymbol{i}}_L(\boldsymbol{\phi}_L)/\partial \boldsymbol{i}_L$ are *symmetric* for all $\boldsymbol{i}_L \in \mathbb{R}^{n_L}$ and $\boldsymbol{\phi}_L \in \mathbb{R}^{n_L}$, respectively.

The following properties concerning reciprocity are proved in [43], [44].[27]

Theorem 7. Properties Involving Reciprocity.

1. *Closure Property*: A multi-terminal (or multi-port) N made of arbitrary interconnection of *reciprocal* resistors {resp.; capacitors, inductors} results in a *reciprocal* resistor {resp.; capacitor, inductor} provided N can be described by a C^1 constitutive relation.
2. Every 2-terminal Resistor, Capacitor, inductor characterized by a C^1 function is reciprocal.
3. Every *nonreciprocal* capacitor or inductor is *active*.
4. A C^1 charge-controlled or voltage-controlled *capacitor* is lossless$\Leftrightarrow$it is reciprocal.
5. A C^1 flux-controlled or current-controlled *inductor* is *lossless*$\Leftrightarrow$it is reciprocal.

It follows from *Properties* 1 and 2 that the resistive n-port N in Fig. 20 is reciprocal if it contains only 2-terminal resistors and independent sources. However, if N contains even one nonreciprocal resistor, such as a transistor or gyrator, N will generally become *nonreciprocal* except in contrived cases. This means that any hypothesis which requires that N be reciprocal would virtually exclude transistors or other nonreciprocal elements.

Properties 3 and 4 show that all realistically modeled capacitors and inductors must be *reciprocal*. Therefore, there is little loss of generality in assuming that all capacitors and inductors in Fig. 20 are Reciprocal.

Definition 5. Local Passivity and Strict Local Passivity

A multi-terminal (or multi-port) *Resistor*, *Inductor*, or *Capacitor* described by a constitutive relation $\boldsymbol{y}_\alpha = \boldsymbol{h}(\boldsymbol{x}_\alpha)$, where $\boldsymbol{h}$: $\mathbb{R}^\alpha \to \mathbb{R}^\alpha$ is a C^1 function, is said to be *locally passive* {resp.; *strictly locally passive*} *at a point* $\boldsymbol{x}^* \in \mathbb{R}^\alpha$ *iff* its *incremental matrix*

$$\boldsymbol{H}(\boldsymbol{x}_\alpha) \triangleq \frac{\partial \boldsymbol{h}(\boldsymbol{x}_\alpha)}{\partial \boldsymbol{x}_\alpha} \tag{4.8}$$

is *positive-semi-definite* {resp.; *positive definite*} at $\boldsymbol{x}_\alpha = \boldsymbol{x}_\alpha^*$.

The element is said to be *locally passive* {resp.; *strictly locally passive*} iff $\boldsymbol{H}(\boldsymbol{x}_\alpha)$ is positive-semi-definite {resp.; positive definite}[28] *for all* $\boldsymbol{x}_\alpha \in \mathbb{R}^\alpha$.

The element is said to be *locally active iff* it is *not* locally passive.

Remarks.

1. A 2-terminal Resistor, Inductor, and Capacitor is *locally passive* {resp.; *strictly locally passive*} at $x_\alpha^* \Leftrightarrow$the *slope* of the i_R–v_R, ϕ_L–i_L, and q_C–v_C curve at x_α^* is *nonnegative* {resp.; *positive*}.

2. *Strong local passivity* (*Definition* 1)$\Rightarrow$*strict local passivity*$\Rightarrow$*local passivity*. The converse is of course false. For example, the resistor described by the i_R–v_R curve in Fig. 17(b) is locally passive but not strictly locally passive because the slope is zero for $v_R \leqslant 0$. That shown in Fig. 16(b) is strictly locally passive but not strongly locally passive because the slope is *not* bounded from below by $\gamma > 0$.

3. For 2-terminal elements, the only difference among these 3 circuit-theoretic properties is that the characteristic curve of a locally passive element may contain isolated points having a *zero* slope, or even an entire horizontal segment (which is not allowed in the other two properties); that of a strictly locally passive resistor may saturate and therefore need not tend to ∞ as $x_\alpha \to \infty$. In contrast, the curve for a strongly locally passive element must tend to $\pm\infty$ as $v_R \to \pm\infty$.

4. For the time-varying case where $y = h(x;\ t)$, simply apply *Definition 5* at each instant of time. In particular, note that a dc or time-dependent voltage source or current source is *locally passive* although it is globally active.

Theorem 8. Properties Involving Local Passivity and Strict Local Passivity

1. *Closure Property*: The resistive n-port N in Fig. 20 is *locally passive* if all elements inside N are *locally passive*.
2. *Almost Closure Property*:[29] The resistive n-port N in Fig. 20 is *strictly locally passive* if:

(a) all elements inside N are *strictly locally passive*.

(b) there is no loop {resp.; *cut set*} formed exclusively by capacitors, inductors, and/or voltage {resp.; current} sources.

Proof: See proofs for *Theorem 1* and *Theorem 7* in [30].

4.2. Local Existence and Uniqueness of Solutions

Theorem 9. Local Uniqueness Theorem

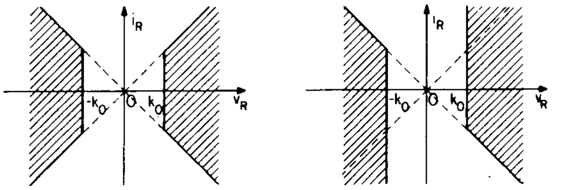
An *RLC* network $\mathfrak{N}$ having a continuous local state equation about any initial point $(\boldsymbol{v}_C(t_0), \boldsymbol{i}_L(t_0))$ has a unique solution over some time interval $(t_0 - \epsilon) < t < (t_0 + \epsilon)$, $\epsilon > 0$, if the following conditions are satisfied:

1. All Resistors in $\mathfrak{N}$ are *locally passive*.

[27] Unless otherwise stated, all Resistors, Inductors, and Capacitors in this paper are assumed to be *time-invariant*.

[28] An $n \times n$ real (not necessarily symmetric) matrix $\boldsymbol{A}$ is *positive semi-definite* {resp.; *positive definite*} iff $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \geqslant 0$ {resp.; >0} for all real n-vectors $\boldsymbol{x} \neq \boldsymbol{0}$. It can be shown that $\boldsymbol{A}$ is positive-semi-definite {resp.; positive definite} if and only if $\boldsymbol{A}+\boldsymbol{A}^T$ is positive semi-definite {resp.; positive definite}.

[29] Arbitrary interconnection of strictly locally passive *elements* do not always give rise to a strictly locally passive element! See [30] for counter examples.



(a) (b)

Fig. 26. (a) If the v_R–i_R curve lies within the shaded region for all $|v_R| > k_0$, the network in Figs. 8 and 9 have no finite-*forward and backward* escape-time solutions. (b) If the v_R–i_R curve lies within the expanded shaded region for all $|v_R| > k_0$ the network in Figs. 8 and 9 have no finite-*forward*-escape-time solutions.

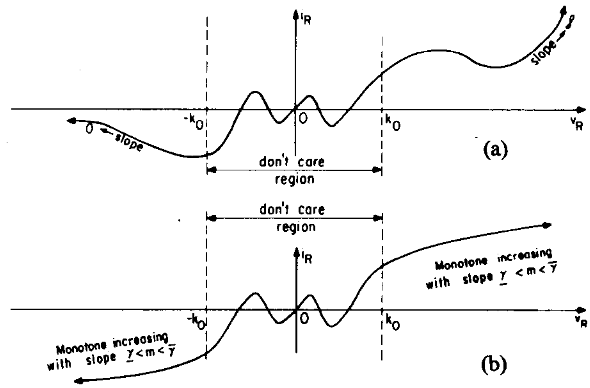


Fig. 27. (a) The v_R–i_R curve of an *eventually passive* 2-terminal resistor. (b) The v_R–i_R curve of an *eventually strongly locally passive* 2-terminal resistor.

2. All Capacitors and Inductors in $\mathfrak{N}$ are *linear*, *Reciprocal*, and *strictly locally passive*.

Proof: See [23]. □

Applying *Theorem 9* to the gyrator network in Fig. 25(a) with the i_R–v_R curve in Fig. 25(c), we find all conditions are satisfied and conclude that the network has a locally unique solution.

The hypotheses of *Theorem 9* are extremely strong and hence *Theorem 9* is applicable to only a very small class of networks. For a more general local uniqueness theorem, see [23].

4.3. No Finite-Forward-Escape-Time Solutions

Example 8 shows a network having a unique local solution can have a finite-*forward*-escape-time solution. *Example 9* shows a network having a locally unique solution can have a finite-*backward*-escape-time solution.

One of the most general theorems for guaranteeing no finite *forward and backward* escape time-solutions is due to Wintner [31]. Unfortunately, Wintner's theorem has little practical value because its conditions are so strong as to exclude any *polynomial* nonlinearity of degree greater than one. For example, *Wintner's Theorem* says that the networks in Figs. 8 and 9 have no finite escape-time-solutions if the i_R–v_R curve of the nonlinear resistor lies within the shaded region shown in Fig. 26(a). Note that the i_R–v_R curve in Figs. 8(b) and 9(b) eventually lie *outside* the shaded region shown in Fig. 26(a), and indeed the former has a finite-forward-escape-time solution while the latter has a finite-backward-escape-time-solution.

These examples (Example 8 has a second-degree polynomial) shows that *Wintner's Theorem* is in fact quite sharp and that the conditions can not be significantly weakened. This suggests that most networks having polynomial nonlinearities of degree greater than one have either a *finite* forward or backward escape-time-solution.

From the circuit point of view, however, the property of having no *finite-backward*-escape-time-solutions is a luxury that is seldom needed. After all, one is usually interested on the solution after some initial time t_0 (e.g., the time a switch is closed) and hence only finite-forward-escape-time-solutions represent a *nonphysical* situation that must be avoided [11], [40].

It turns out that a *much weaker* theorem can be derived which guarantees that there will be no finite-forward-escape-time-solutions. See *Theorem B-2* of [31]. The conditions in *Theorem B-2* are so weak that most networks of practical interest would be allowed. For example, this theorem says that the network in Fig. 8 has no finite-*forward*-escape-time-solution if the i_R–v_R curve lies in the *expanded* shaded region in Fig. 26(b). Observe that the p-n junction diode i_R–v_R curve in Fig. 9(b) meets this requirement and hence the network in Fig. 9(a) has no finite-*forward*-escape-time solutions. Note that since the shaded region in Fig. 26(b) includes points in the 2nd and 4th quadrants, even *active* resistors are allowed.

Roughly speaking, *Theorem B-2* of [31] guarantees that $\mathfrak{N}$ in Fig. 20 will have no finite *forward*-escape-time solutions if the resistive n-port N is "no more active" than an *active linear n*-port resistor.

To derive conditions at the element level, it is convenient to assume that *all* voltage and current sources have been combined with internal resistors (as in Appendix A1) so that each "composite" resistor $\hat{R}_\alpha$ inside the resistive n-port N is described by

$$\hat{y}_{R_\alpha} = h_{R_\alpha}\big(\hat{x}_{R_\alpha} + b_\alpha(t)\big) + c_\alpha(t) \triangleq h_{R_\alpha}\big(\hat{x}_{R_\alpha}, t\big). \quad (4.9)$$

Definition 6. Eventual Passivity and Eventual Strict Passivity

A multi-terminal (or multi-port) Resistor having a constitutive relation (4.9) is said to be *eventually passive* in the sense that at any time t,

$$x_{R_\alpha}^T \hat{h}_{R_\alpha}\big(\hat{x}_{R_\alpha}, t\big) \geq 0, \qquad \text{for all } \|\hat{x}_{R_\alpha}\| > k_0 \quad (4.10)$$

where k_0 is any finite number. It is said to be *eventually strictly passive* iff the inequality sign in (4.10) is replaced by a *strict* inequality.

Definition 7. Eventual Strong Local Passivity

A multi-terminal (or multi-port) *Resistor*, *Inductor*, or *Capacitor* is said to be *eventually strongly locally passive* iff (3.17) holds for all x' and x'' satisfying $\|x'\| > k_0$ and $\|x''\| > k_0$, where k_0 is any finite number.

To illustrate the difference between *Definitions 6 and 7*, Fig. 27(a) shows the i_R–v_R curve of an *active* but *eventually passive* 2-terminal resistor, whereas Fig. 27(b) shows the i_R–v_R curve of an *active* but *eventually strongly locally passive* as well as *eventually passive* 2-terminal resistor.

Note that the slope in Fig. 27(a) tends to ∞ as $v_R \to \infty$, and to 0 as $v_R \to -\infty$. Hence this resistor is *not* eventually strongly locally passive. In contrast to this the slopes in Fig. 27(b) are bounded from below by some positive number γ and from above by some positive number $\bar{\gamma}$. It can be shown that eventual-strong-local passivity implies eventual passivity.

Theorem 10. No-Finite Forward-Escape-Time-Criteria
An *RLC* network $\mathfrak{N}$ described by either state equation 93.14) or (3.15) has *no finite-forward-escape time solutions* if the following 3 conditions are satisfied:

1. There are no loop and no cut set made exclusively of capacitors and/or inductors. (Assume all voltage and current sources have been combined with the internal resistors forming "composite" resistors described by (4.9).)
2. All capacitors and inductors are *Reciprocal* and eventually strongly locally passive.[30]
3. All "composite" resistors are *eventually passive*.

Proof: Follows directly from *Theorem A* and *Theorem 3* from [31]. □

Remark. Conditions 1–3 are stronger than necessary and can be further relaxed. For example, the "0" on the right-hand side of inequality (4.10) can be replaced by $-k$, where k is any *finite* number; etc. Even then, the present hypotheses are already quite weak and are easily satisfied by most practical networks. For example, the networks in Figs. 9, 10, 12, 13, 16, 17, 18, and 19 all satisfy the hypotheses and hence have no finite-forward-escape-time solutions. The same is true of most networks containing capacitors, inductors (except Josephson junction devices) resistors, diodes, transistors, op amp's, etc.

V. Qualitative Property 2: Local Asymptotic Stability of Equilibrium Points and Observability of Operating Points

Examples 10 and *11* from Section 2.3 show how to give physical meaning to a resistive network $\mathfrak{N}_R$ having multiple operating points (i.e., solutions): we must remodel $\mathfrak{N}_R$ by a dynamic network $\mathfrak{N}$ which reduces to $\mathfrak{N}_R$ when all capacitors are replaced by open-circuits and all inductors by short-circuits. Each operating point of $\mathfrak{N}_R$ corresponds to one or more equilibrium points of $\mathfrak{N}$: In *Example 10 (Fig.10)*, there is a one-to-one correspondence. In *Example 11 (Fig. 11)*, $\mathfrak{N}_R$ has only *one* operating point, but $\mathfrak{N}$ has infinitely many *isolated* equilibrium points. In *Example 12 (Fig. 12)*, $\mathfrak{N}_R$ has 3 isolated operating points,[31] but each gives rise to a line of *nonisolated* equilibrium points. Likewise, in *Example 13* (Fig. 13), $\mathfrak{N}_R$ has 3 isolated operating points, each giving rise to a line of equilibrium points.

Whether an operating point Q of a resistive network $\mathfrak{N}_R$ is *observable at a given time* or not depends on whether the *initial* (just prior to measurement) capacitor voltages and inductor currents of the associated dynamic network $\mathfrak{N}$ is located within the *basin* of a corresponding *asymptotically stable* equilibrium point [41]. For example, the operating point Q_2 in Fig. 10(c) is *not observable* if $R > 1/G$ because its corresponding equilibrium point is unstable [22]. On the other hand, the other 2 operating points Q_1 and Q_3 are observable because *Theorem 9* below implies that their corresponding equilibrium points are asymptotically *stable*. The basin of Q_1 {resp.; Q_3} consists of all points to the *left* {resp.; *right*} of an imaginary curve passing through Q_2, as can be easily verified from the *phase-portrait* [22] of Fig. 10(a).

The preceding discussion shows that equilibrium points are of basic importance in understanding the qualitative behaviors of nonlinear networks. Our objective in this section is to present some useful theorems on this subject.

5.1. Mathematical Characterization of Equilibrium Points

Definition 8. Equilibrium Point[32]
Let $\mathfrak{N}$ be an *autonomous RLC* network described by either state equation (3.14a) or (3.15a).

(a) A point $x^* \in \mathbb{R}^n$ is said to be an *equilibrium point* of (3.14a) *iff*

$$D^{-1}(x^*)h(x^*) = 0. \tag{5.1}$$

(b) A point $z^* \in \mathbb{R}^n$ is said to be a *equilibrium point* of (3.15a) *iff*

$$h(g(z^*)) = 0. \tag{5.2}$$

We call x^* and z^* equilibrium points because $\dot{x}(t) \equiv 0$ at $x = x^*$ and $z(t) \equiv 0$ at $z = z^*$. Observe that $x^* = g(z^*)$ in view of (3.12).

Since $D(x^*)$ is *nonsingular*, by assumption, it follows from (3.8)–(3.10) that $x = x^*$ and $z = z^*$ if and only if $i_a = 0$, and $v_b = 0$. Hence, we have:

Observation 1. The capacitor currents and inductor voltages must vanish at an equilibrium point of (3.14a) or (3.15a).

Observation 2. The equilibrium points $x^* = (v_C^*, i_L^*)$ of (3.14a) can be found by open-circuiting all capacitors and short-circuiting all inductors, and then finding the open-circuit voltages v_a and the short-circuit currents i_b of the associated Resistive n-port N. The equilibrium points $z^* = (q_C^*, \phi_L^*)$ is found by solving q_C^* from $v_C^* = \hat{v}_C(q_C^*)$ and ϕ_L^* from $i_L^* = \hat{i}_L(\phi_L^*)$.

Theorem 11. Local Stability Criteria
Consider the state equation

$$\dot{x} = f(x) \tag{5.3}$$

[30]To apply the strong-local-passivity criterion in Sec. 3.2 for checking eventual strong local passivity, we only need to check the conditions for all x satisfying $\|x_\alpha\| > k_0$.

[31]When obtaining the resistive network $\mathfrak{N}_R$ associated with a dynamic network $\mathfrak{N}$, we *delete* any *cut set* made of open circuits and *contract* any *loop* made of short circuits.

For example, $\mathfrak{N}_R$ associated with Fig. 12(a) consists of only the open-circuited resistor. The 2 open circuits associated with C_1 and C_2 are *deleted* because they form a cut set. This means that *all voltages across capacitors forming a cut set do not appear in* $\mathfrak{N}_R$.

Likewise, $\mathfrak{N}_R$ associated with Fig. 13(a) consists of 3 resistors in parallel. The 3 short circuits associated with the 3 inductors are *contracted* (by coalescing the 3 nodes) because they form a loop. This means that *all currents in the inductors forming a loop do not appear in* $\mathfrak{N}_R$.

[32]Usually called *singular points* in the mathematical literature.

where $f(\cdot)$: $\mathbb{R}^n \to \mathbb{R}^n$ is a C^1 *function*. Then the *local* qualitative behavior of any equilibrium point x^* of (5.3) is determined by its associated *linearized* system:

$$\dot{x} = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x^*} \triangleq Ax. \tag{5.4}$$

In particular, we have

(a) x^* is *asymptotically stable* if all eigenvalues of A has negative real parts.

(b) x^* is *unstable* if at least one eigenvalue of A has a positive real part.

Proof: This is a special case of a theorem proved in [42].

Remarks.

1. Since *Theorem 11* pertains only to a small neighborhood of x^*, it is obviously applicable if the *RLC* network $\mathfrak{N}$ is *locally solvable* at x^*.

2. The C^1 requirement on $f(\cdot)$ can be further relaxed [42]. In particular, if we can write $f(x) = Ax + g(x)$ where $g(0) = 0$, then *Theorem 11* remains valid if $g(\cdot)$ is continuous and $g(x) = 0(\|x\|)^2$ as $\|x\| \to 0$.

3. If A has an eigenvalue with a *zero* real part, *higher order* partial derivatives of $f(\cdot)$ at x^* are necessary to determine the local behavior of x^*.

Let us now derive some useful criteria at the element level.

5.2. Circuit-Theoretic Characterization of Equilibrium Points

Theorem 12. Local asymptotic stability criteria

Let $\mathfrak{N}$ be a *locally solvable RLC* network. An equilibrium point of $\mathfrak{N}$ (in terms of any choice of state variable) is *asymptotically stable* if the following conditions are satisfied:

1. There is no *loop* {resp.; *cut set*} formed exclusively by the ports and/or voltage {resp.; current} sources.
2. Each Resistor, Inductor, and Capacitor is *strictly locally passive* at its respective operating point corresponding to the equilibrium point under consideration.
3. All Capacitors and Inductors are *reciprocal.*

Proof: Since $\mathfrak{N}$ is locally solvable, it follows from (3.22) that it has a C^1 local state equation

$$\dot{z} = -h_Q(g_Q(z)) \triangleq f_Q(z) \tag{5.5}$$

about any equilibrium point z^*. The associated Jacobian matrix is

$$A = \left.\frac{\partial f_Q(z)}{\partial z}\right|_{z=z^*} = \left.\frac{\partial h_Q(x)}{\partial x}\right|_{x=g(z^*)} \cdot \left.\frac{\partial g_Q(z)}{\partial z}\right|_{z=z^*}$$
$$\triangleq -HD^{-1}. \tag{5.6}$$

Since N is *strictly* locally passive in view of *Theorem 8(b)*, both H and D^{-1} are *positive definite* matrices. Moreover, D^{-1} is symmetric.[33] It follows from *Lemma 1* in [45, p. 529] that all eigenvalues of $D^{-1}H^T$ have positive real parts. Since $A = -(D^{-1}H^T)^T$, it follows that all eigenvalues of A have negative real parts. Hence z^* is asymptotically stable in view of *Theorem 11(a)*. Since any other state variable is related to Z *bijectively*, the qualitative behavior is preserved. □

[33] The incremental hybrid matrix H is generally *not* symmetric even if N is Reciprocal in view of (4.7).

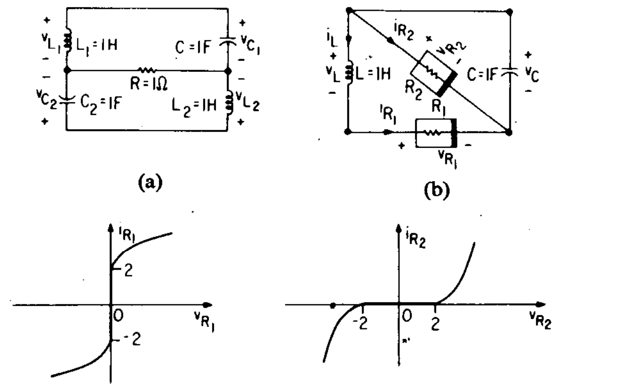


Fig. 28. (a) An oscillatory network which satisfies all hypotheses except condition 1 of *Theorem 12*. (b) An oscillatory network which satisfies all hypotheses except condition 2 of *Theorem 12*, when R_1 and R_2 are described as in (c) and (d).

Applying *Theorem 12* to the network in Fig. 10(a), we conclude *by inspection* of Fig. 10(c) that both Q_1 and Q_3 are asymptotically stable. Similarly, for the network in Fig. 11(a), we conclude *by inspection* of Fig. 11(b) that all equilibrium points having a *positive* slope are asymptotically stable.

To demonstrate the conditions 1 and 2 in *Theorem 12* are generally necessary, consider the 2 networks shown in Figs. 28(a) and (b). Since the linear balanced bridge network in Fig. 28(a) oscillates with *any* nonzero initial condition, the equilibrium point $(v_C, i_L) = (0,0)$ is *not* asymptotically stable, although it is *stable-in-the sense of Lyapunov* [22, 42]. Note that this network violates *only* condition 1.

Likewise, by restricting the initial condition to $|v_C(0)| < 2$ and $|i_L(0)| < 2$, the network in Fig. 28(b) with R_1 and R_2 described in Fig. 28(c) and (d) behaves like a parallel *LC* oscillatory tank circuit. Hence the equilibrium point $(v_C, i_L) = (0,0)$ is also *not* asymptotically stable. Note that this network violates *only* condition 2. It is easily verified that the eigenvalues for these 2 networks are *purely imaginary* and hence neither *Theorem 11* nor *Theorem 12* is applicable.

Observe next the networks analyzed earlier in *Example 12* (Fig. 12) and *Example 13* (Fig. 13). Both violate *only* condition 1 of *Theorem 12*: C_1 and C_2 form a *cut set* in Fig. 12(a) while L_1, L_2, and L_3 form a *loop* in Fig. 13(a). It is easily seen from (2.34) and (2.35) that the associated Jacobian matrix has a *zero* eigenvalue because their component equations are *not* linearly independent. Hence, *Theorem 11* is also not applicable. For these 2 networks, however, unlike those of Fig. 28, our previous analysis shows that there is no oscillation in a neighborhood of each equilibrium point, and hence an operating point is actually *observable* if its associated equilibrium point on the invariant submanifold (which is fixed by the initial condition) is asymptotically stable.

In other words, even though *Lyapunov's* definition would have classified the nonisolated equilibrium points as *not* asymptotically stable (note that 2 arbitrarily close initial conditions falling on a line of equilibrium points remain stationary), this classification would be misleading for these networks because it would suggest that the associated operating points are *not* observable. This conclusion is of course contrary to the actual phenomenon that is taking place in Figs. 12(c) and 13(c).

Observe that all trajectories in Fig. 12(c) are restricted to lie on a *line* (one-dimensional submanifold of $\mathbb{R}^2$) $M^*(v_{C_1}(0), v_{C_2}(0))$ once the initial condition $(v_{C_1}(0), v_{C_2}(0))$ is given. Likewise, all trajectories[34] in Fig. 13(c) are restricted to lie on a *plane* (two-dimensional submanifold of $\mathbb{R}^4$) $M^*(i_{L_1}(0), i_{L_2}(0), i_{L_3}(0))$ once the initial condition $(i_{L_1}(0), i_{L_2}(0), i_{L_3}(0))$ is specified. It follows from these observations that it is physically more meaningful to classify the local stability of equilibrium points in Figs. 12 and 13 directly on the invariant submanifold M^*. Mathematically, this is equivalent to eliminating one of the state variables in *Examples* 12 and 13 and then analyzing a *reduced-order* state equation, as we have done in (2.37). The *zero* eigenvalue will no longer be present and hence *Theorem 11* can be applied to deduce the asymptotic stability or instability of an equilibrium point Q_j. This translates directly into a physically meaningful conclusion as to whether the associated operating point is observable or not.

It turns out that the phenomena displayed in *Examples* 12 and 13 are quite typical and can be completely characterized by the following theorems:

Theorem 13. Isolated Equilibrium Point Criterion

Let $\mathfrak{N}$ be an autonomous *RLC* network described by state equation (3.14a) with (v_C, i_L) as the state variables. Assume the associated resistive n-port network $\mathfrak{N}_R$ obtained by open-circuiting all capacitors and short-circuiting all inductors (recall footnote 31) has only *isolated operating points*.

Then (3.14a) has only *isolated equilibrium points*$\Leftrightarrow\mathfrak{N}$ contains *no capacitor-only cut sets* and *no inductor-only loops*.

Proof: See [46].

Remarks.

1. *Theorem 13* does not hold if (q_C, ϕ_L) is chosen as the state variables. For example, consider the network shown in Fig. 29. The resistive network $\mathfrak{N}_R$ has a single operating point $i_L = E/R$. Observe, however, any $\phi_L \in \{[1,2], [3,4], [6,7]\}$ is an equilibrium point.

2. We can avoid the problem in Fig. 24(b) by requiring the ϕ_L–i_L curve to be *transversal* (*not tangent* in this case) with respect to the resistor load line. It turns out that this observation can be generalized with the help of *transversality theory* from Differential Topology. The reader is referred to [46] for a self-contained introduction to this powerful tool. We will close this section with an *informal* statement of this generalization:

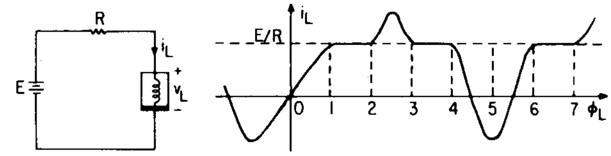


Fig. 29. A network having an *isolated* dc operating point but *nonisolated* equilibrium points over the intervals [1,2], [3,4], [6,7].

Theorem 14. Implications of Capacitor-Only Cut Sets and Inductor-Only Loops

Let $\mathfrak{N}$ be an autonomous *RLC* network described by (3.15a) with $z = (q_C, \phi_L)$ as the state variables. Assume the associated Resistive network $\mathfrak{N}_R$ has only *isolated* operating points. Assume further that a certain *transversality condition*[35] is satisfied.

Conclusion:

1. State equation (3.15a) has only *isolated equilibrium points*$\Leftrightarrow\mathfrak{N}$ contains *no capacitor-only cut sets* and *no inductor-only-loops*.

2. There is an *invariant minimal* dynamic subspace $M^*(z_0) \subset \mathbb{R}^n$ which is uniquely determined by the *initial condition* $z(t_0) = z_0$ such that the trajectory $z(t;\, t_0, z_0)$ remains on $M^*(z_0)$ for all $t \geqslant t_0$, where $n = n_C + n_L$, n_C = number of capacitors and n_L = number of inductors in $\mathfrak{N}$.[36]

3. The invariant minimal dynamic subspace $M^*(z_0)$ is an n^*-dimensional affine submanifold of $\mathbb{R}^n$, where $n^* = n - (\bar{n}_C + \bar{n}_L)$, $\bar{n}_C$ = number of *linearly independent* capacitor-only cut sets and $\bar{n}_L$ = number of *linearly independent* inductor-only loops.

4. The set M of *all* equilibrium points of $\mathfrak{N}$ consist, of the union of $M(Q_1), M(Q_2), \cdots,, \cdots, M(Q_m)$, where $M(q_j)$ denotes a continuum of nonisolated equilibrium points corresponding to the *isolated* operating points Q_j of $\mathfrak{N}_R$.

5. *Each* set $M(Q_j)$ of *nonisolated* equilibrium points of (3.15a) intersects the minimal dynamic space M^* at only *isolated* points. Hence, $\mathfrak{N}$ behaves as if it has only isolated equilibrium points. Moreover, the local stability of each equilibrium point of (3.15a), and hence the *observability* of the corresponding operating point Q_j of the associated Resistive network $\mathfrak{N}_R$, can be determined by applying *Theorem 11* to a *reduced-order* state equation in $\mathbb{R}^{n^*}$ when the initial conditions are restricted to lie on $M^*(z_0)$l

[34] More precisely, Fig. 13(c) represents the projection of trajectories $(v_C(t), i_{L_1}(t), i_{L_2}(t), i_{L_3}(t))$ in $\mathbb{R}^4$ *onto* the three-dimensional $i_{L_1} - i_{L_2} - i_{L_3}$ subspace. In the general case, such a projection would normally occupy a three-dimensional region in $\mathbb{R}^3$. In Fig. 13(c), however, it is only two-dimensional.

[35] This corresponds to condition 3 of *Theorem 2* in [46]. This condition is automatically satisfied if all capacitors and inductors are *strongly locally passive*.

[36] $M^*(z_0)$ can be interpreted as obtained by translating (and possibly rotating) the n^*-dimensional Euclidean space $\mathbb{R}^{n^*}$ to the point z_0 in $\mathbb{R}^n$.

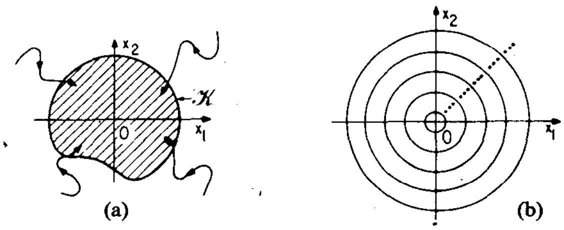


Fig. 30. (a) All trajectories must eventually enter and remain inside the closed and bounded set $\mathcal{X}$. (b) A continuum of periodic orbits, one for each initial condition.

VI. Qualitative Property 3: Eventual Uniform Boundedness of Solutions

Examples 14–17 from Section 2.4 show that a network having *no finite-forward-escape-time solutions* may still have solutions which tend to ∞ as $t \to \infty$. Moreover, *Examples 14 and 15* show that the boundedness of solutions with respect to the state variables $z=(q_C, \phi_L)$ does not guarantee the boundedness of solutions with respect to $x=(v_C, i_L)$, and vice versa. Furthermore, *Example 16* shows that even if both $z(t)$ and $x(t)$ are bounded, the associated current $i_a(t)$ and voltage $v_b(t)$ of the Resistive n-port N in Fig. 20 need not be bounded. Finally, *Examples 15 and 17* show that a network containing only *strictly locally passive* elements can have unbounded solutions. Our objective in this section is to present simple circuit-theoretic criteria for avoiding unbounded solutions.

Definition 9. Eventually Uniformly Bounded Network

The *RLC* network $\mathfrak{N}$ in Fig. 20 is said to be *eventually uniformly bounded* with respect to state variables $x=(v_C, i_L)$ {resp.; $z=(q_C, \phi_L)$} *iff* given any *bounded* source vector $u_s(t)$, there exists a *closed and bounded set* $\mathcal{X} \subset \mathbb{R}^n$ such that any solution $x(t)$ of (3.14) {resp.; $z(t)$ of (3.15)} enters and remains inside $\mathcal{X}$ for all time $t \geq T$, where $T < \infty$ may depend on the initial condition and on the initial time.

Remarks.

1. The source vector $u_s(t)$ must be bounded for *Definition 9* to make sense. Otherwise, most networks can be made unbounded by driving it with an unbounded source waveform, say $v_s(t)=e^t$.

2. The adjective "eventually" is used to suggest that all trajectories must eventually be "sucked" into the set $\mathcal{X}$ and remain inside it thereafter, as depicted in Fig. 30(a).

3. The adjective "uniform" is used to emphasize that the closed and bounded set $\mathcal{X}$ does *not* depend on the initial condition, or on the initial time. It is fixed once the source vector $u_s(t)$ is specified. Observe that this property is stronger than mere "boundedness" of solutions. For example, *each* solution of an *LC* harmonic oscillator is bounded as shown by the infinite continuum of concentric periodic orbits in Fig. 30(b). But since it is impossible to *prescribe* a closed and bounded set $\mathcal{X}$ which contains all the orbits, the solutions are not eventually uniformly bounded. Note that it is possible to find $\mathcal{X}$ if we allow it to depend on the initial *condition*.[37]

[37]Criteria for guaranteeing this weaker form of boundedness are given in [25], [31]. Due to space limitation, only results on eventual uniform boundedness will be presented in this section.

Theorem 15. Eventual-Uniform-Boundedness Criteria

The *RLC* network in Fig. 20 is *eventually uniformly bounded* with respect to either $z=(q_C, \phi_L)$ or $x=(v_C, i_L)$ if the following conditions are satisfied:

1. The *Resistive* n-port N is *eventually strictly passive* (*Definition 6*).
2. All *capacitors* and *inductors* are *Reciprocal* and *eventually strongly locally passive* (*Definition 7*).
3. The Resistor function $h(\cdot)$ and the capacitor–inductor function $g(\cdot)$ are C^1-functions.

Proof: The proof with respect to the state variable $z=(q_C, \phi_L)$ is given in [25]. The same proof holds with respect to $x=(v_C, i_L)$ because $g(\cdot)$ is "eventually" a C^1-bijective function inview of conditions 2 and 3. □

Remarks.

1. Since the Resistor function $h(\cdot)$ is C^1, conditions 1, 2, and 3 actually guarantee that *all solution waveforms*, including $i_a(t)$ and $v_b(t)$ are eventually uniformly bounded.

2. Condition 2 can be replaced by a weaker "growth condition" [25].

Theorem 16. Eventual-Uniform-Boundedness Criteria at the Element Level

The *RLC* network in Fig. 20 is *eventually uniformly bounded* with respect to either $z=(q_C, \phi_L)$ or $x=(v_C, i_L)$ if the following conditions are satisfied:

1. There is no *loop* {resp.; no *cut set*} formed exclusively by capacitors, inductors, and/or voltage sources {resp.; current sources}.
2. All internal *Resistors* are eventually strongly locally passive (*Definition 7*)
3. All *Capacitors* and *Inductors* are *Reciprocal* and *eventually strongly locally passive* (*Definition 7*).
4. The Resistor function $h(\cdot)$ and the Capacitor–Inductor function $g(\cdot)$ are C^1-functions.

Proof: See the proof of *Theorem 2* in [25].

Remarks.

1. Condition 2 can be replaced by a slightly weaker condition. See *Theorem 2* of [25].

2. Condition 4 is *unnecessarily* strong. Except in contrived and highly pathological situations, Condition 4 can be replaced by the requirement that all element constitutive relations are *continuous* functions.

To show *continuity* is an important condition,[38] note the capacitor voltage in *Example 14* is unbounded because the capacitor function is discontinuous at $q_C=0$. Likewise, the resistor current in *Example 16* is unbounded because the resistor function is discontinuous at $v_R=0$.

3. Conditions 1, 2, and 3 are reasonably sharp in the sense that there exists networks which are *not* eventually uniformly bounded and which violates *only one* of these 3 conditions: (a) The network in Fig. 28(a) is *not* eventually-uniformly bounded (only Condition 1 is violated). (b) The inductor current in *Example 17* is unbounded (only Con-

[38]We can further weaken this condition to allow "finite" discontinuities. However, the statement and proof become awkward.

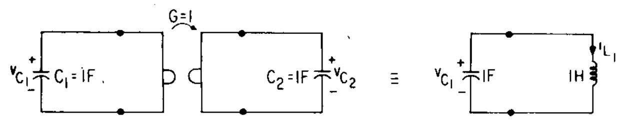


Fig. 31. An *RC* network which oscillates.

dition 2 is violated). (c) The capacitor charge in *Example 15* is unbounded (only Condition 3 is violated).

4. Most realistically modeled networks are eventually uniformly bounded.

VII. Qualitative Property 4: Complete Stability and Global Asymptotic Stability of Autonomous Networks

An *eventually uniformly bounded* network may oscillate or display bizarre and chaotic behavior [15]. Except for oscillators, all properly designed networks must have at least one of the following qualitative properties:

Definition 10. Completely Stable Network

An autonomous *RLC* network is said to be *completely stable* [47], [48], [31], [35] *iff each* solution of state equation (3.14a) or (3.15a) through *any* initial state at t_0 exists for all time $t \geqslant t_0$ and tends to an *equilibrium point* as $t \to \infty$.

Definition 11. Global-Asymptotically Stable Network

A *completely stable* network is said to be *globally asymptotically stable* [49], [50], [31] if *all* solutions tend to a *unique* equilibrium point.

Theorem 17. Complete Stability Criteria for RC Network

Any *RC* network $\mathfrak{N}_{RC}$ (containing only capacitors, resistors, and dc sources) described by either (3.14a) or (3.15a) is *completely stable* if the following conditions are satisfied:

1. All *Resistors* are *Reciprocal* and *eventually strongly locally passive* (*Definition 7*).
2. All *Capacitors* are *strongly locally passive* (*Definition 1*).
3. The Resistor function $\boldsymbol{h}(\cdot)$ and *capacitor function* $\boldsymbol{g}(\cdot)$ are C^1 functions.

Proof: $\mathfrak{N}_{RC}$ is described by (3.14a) or (3.15a) implies there are no loops made of capacitors and voltage sources. It follows from *Theorem 16* that $\mathfrak{N}_{RC}$ is eventually uniformly bounded. The remaining proof is standard. See [31],[48], [49]. □

Remarks.

1. Condition 1 is simply condition 2 of *Theorem 16* plus *Reciprocity*. Condition 2 is simply condition 3 of *Theorem 16* with the word "eventually" deleted.

2. To show that *condition 1* of *Theorem 17* is not superflous, consider the linear passive *RC* network shown in Fig. 31(a). Since this network is equivalent to the *LC* parallel circuit in Fig. 31(b), it is *not* completely stable. Note that the gyrator is neither reciprocal nor eventually strongly locally passive. To show that reciprocity is *essential*, let us connect a 2-terminal resistor having the i_{R_1}–v_{R_1} curve shown in Fig. 28(c) in series with the capacitor C_1. The "composite" 2-port resistor consisting of the gyrator and the inserted resistor is eventually strongly locally passive but still *nonreciprocal*. Since $v_{R_1}=0$ for $|i_{L_1}|<2$, the equivalent *LC* circuit in Fig. 31 still holds, and can support an oscillation with $|i_{L_1}|<2$. Hence the network is *not* completely stable.

To show that *condition 3* of *Theorem 17* is not superflous, we only need to look at the network in *Example 15* (Fig. 14). Note that the capacitor is *not* strongly locally passive.

3. A more general version of this theorem is given in [20], [31].

4. A *dual* theorem holds for *RL* networks.

5. The hypotheses of all theorems stated so far are "qualitative" in nature (e.g., graph- and circuit-theoretic conditions). Unfortunately, it is not possible to derive any general complete stability theorem for networks containing *both* capacitors and inductors, and/or *nonreciprocal* resistors (e.g., transistors) without introducing conditions of a "quantitative" nature involving element parameters. This is because an *RLC* network may be completely stable for certain element values but become oscillatory when some element value is changed.

6. The first *complete stability theorem* applicable to a subclass of *Reciprocal RLC* network is given in a classic paper by Brayton and Moser [48]. The main theorem requires a certain topological matrix to have *full rank*, and a certain matrix involving element parameters to be *nonsingular*. These conditions are not always satisfied (the network in Fig. 13 is a case in point) and can be replaced by weaker conditions. The most recent generalization of this theorem is given in [36].

7. There is as yet no genuine *completely stability theorem* applicable to a *nonreciprocal RLC* network (say transistor networks) having *multiple* equilibrium points.

8. For networks having a *unique* equilibrium point, a complete stability theorem becomes a *global-asymptotic-stability* theorem. Here, the following general results can be stated.

Theorem 18. Global-Asymptotic-Stability Criterion 1

The *RLC* network in Fig. 20 is *Globally asymptotically stable* if the following conditions are satisfied:

1. There is *no loop* (resp.; *no cut set*} formed exclusively by capacitors, inductors, and/or voltage sources {resp.; current sources}.
2. All Resistors are *strongly locally passive* (*Def. 1*).
3. All Capacitors and Inductors are *Reciprocal* and *strongly locally passive*.

Proof: See the proof of *Theorem 9* in [31].

Theorem 19. Global-Asymptotic-Stability Criterion 2

The *RLC* network in Fig. 20 is *Globally asymptotically stable* if the following conditions are satisfied:

1. There is *no loop* {resp.; *no cut set*} formed exclusively by capacitors, inductors, and/or voltage sources {resp.; current sources}.

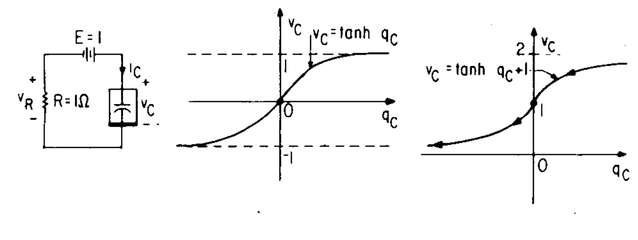


Fig. 32. An RC network whose capacitor charge $q_C \to -\infty$ as $t \to \infty$.

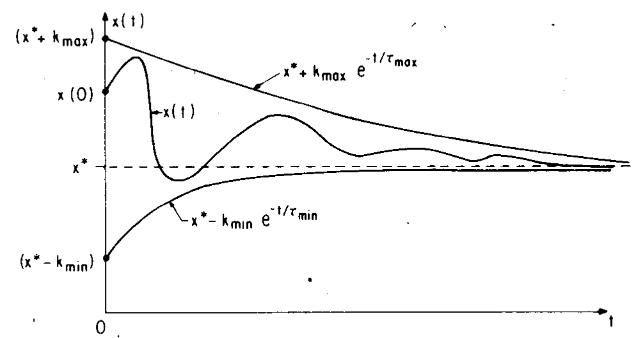


Fig. 33. A geometrical interpretation of *Theorem 21* for a first order network: The solution $x(t)$ is bounded by 2 exponentials having time constants $\tau_{\min}$ and $\tau_{\max}$, respectively.

2. *Every loop* {resp.; cut set} containing a voltage source {resp.; current source} also contains a capacitor {resp.; inductor}.

3. All Resistors are *strictly passive*.

4. All capacitors and inductors are *Reciprocal* and *eventually strongly locally passive*.

Proof: The proof is based on a repeated use of the *colored branch theorem*, the *v-shift theorem*, and the *i-shift theorem* [43]. □

Remarks.

1. In *Theorem 19*, the equilibrium point corresponds to $v_R = i_R = 0$ for all *resistor* voltages and currents.

2. Condition 3 of *Theorem 19* is much weaker than condition 2 of *Theorem 18* because *locally active* resistors such as tunnel diodes, transistors, etc., are allowed in *Theorem 19*.

3. Condition 4 of *Theorem 19* is much weaker than condition 3 of *Theorem 18* because *locally active* capacitors and inductors are allowed. To show that condition 4 is reasonably sharp, consider the network shown in Fig. 32(a). Note that all conditions of *Theorem 19*, except the *eventual-strong local-passivity* requirement, are satisfied. To show that this network is *not* globally asymptotically stable, let us combine the $E=1$ V battery in series with the capacitor q_C–v_C curve in Fig. 32(b) to obtain the composite q_C–v_C curve shown in Fig. 32(c). Since $v_C > 0$, we have $\dot{q}_C < 0$ for all t and hence $q_C(t) \to -\infty$. In fact, note that this network does *not* have an equilibrium point for $E \geq 1$.

Theorem 20. Exponential Transient Decay Property

If $\mathfrak{N}$ satisfies the hypotheses of *Theorem 18*, then all solutions $\boldsymbol{x}(t) = (\boldsymbol{v}_C(t), \boldsymbol{i}_L(t))$ and $\boldsymbol{z}(t) = (\boldsymbol{q}_C(t), \boldsymbol{\phi}_L(t))$ *decay* at an *exponential rate* to the unique globally asymptotically stable equilibrium point $\boldsymbol{x}^*$ and $\boldsymbol{z}^*$, respectively.

In particular, there exist 2 *time constants* $\tau_{\min}$ and $\tau_{\max}$ $(0 < \tau_{\min} < \tau_{\max})$, such that (see Fig. 33)

$$k_{\min} e^{-t/\tau_{\min}} \leq \|\boldsymbol{x}(t) - \boldsymbol{x}^*\| \leq k_{\max} e^{-t/\tau_{\max}} \tag{7.1}$$

$$\hat{k}_{\min} e^{-t/\tau_{\min}} \leq \|\boldsymbol{z}(t) - \boldsymbol{z}^*\| \leq \hat{k}_{\max} e^{-t/\tau_{\max}} \tag{7.2}$$

where $k_{\min}$, $k_{\max}$, $\hat{k}_{\min}$, and $\hat{k}_{\max}$ are constants which, along with $\tau_{\min}$ and $\tau_{\max}$, can be estimated from the constants γ and $\bar{\gamma}$ associated with $\boldsymbol{h}(\cdot)$ and $\boldsymbol{g}(\cdot)$ as defined in (3.17).[39]

Proof: See the Proof of *Theorem 11* in [31].

VIII. Qualitative Property 5: Existence of a DC or Periodic Steady-State Solution

A nonautonomous network driven by T-periodic sources (i.e., all sources have the same fundamental frequency $\omega = 2\pi/T$) may *not* have a *periodic* steady-state response [13]–[18]. Even if there is a periodic response, it need not have the same fundamental frequency as the input. Indeed, *Example 18* shows a network having an *infinitely* many *nonisolated dc* periodic (fundamental frequency = 0) steady-state solutions.

Since most realistically modeled networks satisfy the hypotheses of *Theorems 15 and 16*, and hence are eventually uniformly bounded, the weakest *qualitative* property to be sought next is whether such a network has *at least* one *periodic* steady-state solution when driven by T-periodic sources, and if so, whether it has the same fundamental frequency. The following theorems are the latest circuit-theoretic results on this subject.[40]

Theorem 21. Existence of a T-Periodic Steady-State Solution

Under the same hypotheses as *Theorems 15 or 16*, the following additional properties are true:

(a) If the *RLC* network in Fig. 20 contains only *dc* sources, then the *autonomous* state equations (3.14a) and (3.15a) have *at least one* equilibrium point (i.e., a dc steady-state solution).

(b) If the *RLC* network in Fig. 20 contains only *T-periodic* sources, then the *T*-periodic *nonautonomous* state equations (3.14b) and (3.15b) have *at least one* periodic solution of the *same period T.*

Proof: See the Proofs of *Theorem 4* of [31] and *Theorem 1* of [25]. □

To show that the hypotheses of *Theorem 21* are reasonably sharp, note that the diode network in Fig. 17 (*Example 18*) violates condition 1 of *Theorem 15* and condition 2 of *Theorem 16*. Indeed the steady-state solutions do *not* have the same fundamental frequency as the input. This

[39] $k_{\min} \triangleq (\gamma_g/\bar{\gamma}_g)^{3/2}\|\boldsymbol{x}(0) - \boldsymbol{x}^*\|$, $k_{\max} \triangleq (\bar{\gamma}_g/\gamma_g)^{3/2}\|\boldsymbol{x}(0) - \boldsymbol{x}^*\|$, $\hat{k}_{\min} \triangleq (\gamma_g/\bar{\gamma}_g)^{1/2}\|z(0) - z^*\|$, $\hat{k}_{\max} \triangleq (\bar{\gamma}_g/\gamma_g)^{1/2}\|z(0) - z^*\|$, $\tau_{\min} \triangleq \gamma_g/\bar{\gamma}_g^2\bar{\gamma}_h$, and $\tau_{\max} \triangleq \bar{\gamma}_g/\gamma_g^2\gamma_h$.

[40] For some earlier related results, see [54].

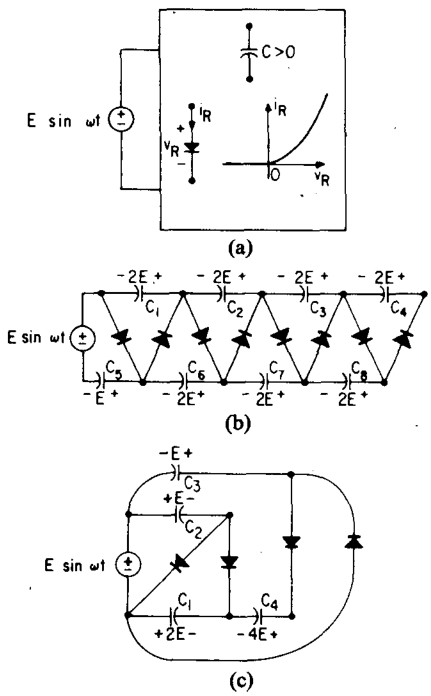


Fig. 34. (a) A diode-capacitor network driven by a sinusoidal voltage source. (b) A ladder-type dc voltage multiplier. (c) A voltage-quadrupler having the smallest "output" resistance.

network turns out to be the prototype of a small but important subclass of practical networks specifically designed to have a *dc* steady-state solution when driven by a single sinusoidal voltage source; namely, *dc voltage multipliers* [51]. Clearly, any such network must violate at least one of the hypotheses of *Theorem 15 or 16*. Our next theorem gives a general graph-theoretic characterization of this somewhat specialized class of nonautonomous network.

Theorem 22. Existence of Unique DC Steady-State Solution

Let $\mathcal{N}$ be an *RC* network containing one sinusoidal voltage source of *amplitude E* and *frequency* ω, linear passive capacitors, and diodes modeled by the *continuous* i_R–v_R curve shown in Fig. 34(a), where $i_R=0$ for all $v_R \leq 0$, and $i_R \to \infty$ as $v_R \to \infty$.[41] Assume the following topological conditions are satisfied:

1. The voltage source and the capacitors form a *tree*.
2. The voltage source and the diodes form a *tree*.
3. The voltage source and the diodes form a *cut set*.
4. The diodes, possibly with the voltage source, form a *similarly directed path*.

Conclusion.

If the capacitors are initially uncharged, then $\mathcal{N}$ has a *unique dc* steady-state solution. In particular, each dc capacitor voltage in steady state is given explicitly by

$$v_{C_j} = k_j E \tag{8.1}$$

where k_j is the number of diodes contained in the *fundamental* loop defined by capacitor C_j with respect to the voltage-source-diode tree (Condition 2), provided the reference polarity of v_{C_j} is aligned with the diode's forward direction.

Proof: See [51]. □

To illustrate the application of *Theorem 22*, consider the networks shown in Fig. 34(b) and (c). Note that conditions 1–4 are satisfied in each case and hence the dc steady-state voltage across each capacitor can be trivially determined by *inspection*, as indicated in Fig. 34(b) and (c).

Note that the diodes in *Theorem 21* violate condition 1 of *Theorem 15* and condition 2 of *Theorem 16.*

IX. Qualitative Property 6: Unique Steady-State Response and Spectrum Conservation

Examples 18–20 of Section 2.5 show that even if a *T*-periodic network has a periodic solution, it need not have the same fundamental frequency as the input source. Moreover, if a network $\mathcal{N}$ is driven by several sources whose frequency spectrum contains several *base* frequencies $\{\omega_1, \omega_2, \cdots, \omega_k\}$, the frequency spectrum of the solution need *not* be an *integer combination* of these base frequencies—as would be the case if $\mathcal{N}$ is purely *resistive* —if $\mathcal{N}$ has subharmonic or other more exotic modes of steady-state solutions. The following qualitative properties are important to have in many practical networks:

Definition 12. Unique Steady-State Solution

A network $\mathcal{N}$ is said to have a *unique steady-state solution* with respect to state variable $z=(q_C, \phi_L)$ *iff any* 2 solutions $z'(\cdot)$ and $z''(\cdot)$ of $\mathcal{N}$ satisfy the property

$$\lim_{t\to\infty} \|z'(t)-z''(t)\|=0 \tag{9.1}$$

regardless of the initial conditions.

Remark. For the sake of generality, *Definition 12* does *not* require $z'(\cdot)$ and $z''(\cdot)$ in (9.1) to be periodic functions. It only demands *all* solution waveforms to tend toward a *unique* limiting waveform.

To avoid dealing with "chaotic" phenomena [15]–[18], we shall assume that all input and output waveforms are of the following type:

Definition 13. Asymptotically Almost-Periodic Function

A continuous vector time function $x(\cdot)$ is said to be asymptotically almost-*periodic* iff

$$x(t)=x_0(t)+x_{\mathrm{ap}}(t) \tag{9.2}$$

where

$$x_0(t) \to \mathbf{0} \text{ as } t\to\infty \tag{9.3}$$

is called the *transient component*, and

$$x_{\mathrm{ap}}(t)=\sum_{k=1}^{m} A_k e^{j\omega_k t} \tag{9.4}$$

where $\omega_1, \omega_2, \cdots, \omega_k \cdots \omega_m$ are real frequencies (m may be ∞), is called the *almost-periodic component*.[42]

Definition 14. Spectrum Combination[43]

[41] The exact shape of the $i_R - v_R$ curve is irrelevant.

[42] Almost periodic function is often defined in terms of an ϵ-translation number. Every almost periodic function has a "generalized" Fourier series in the form of (9.4) [42].

[43] Also called a module in the mathematical literature [42].

We define the *spectrum combination* $\mathbb{S}_{x(\cdot)}$ associated with the almost-periodic function (9.4) to be the set of *all possible* integer combinations of the *base frequencies* $\{\omega_1, \omega_2, \cdots, \omega_k \cdots\}$; i.e.,

$$\omega \in \mathbb{S}_{x(\cdot)} \Leftrightarrow \omega = \sum_k n_k \omega_k \tag{9.5}$$

where n_k is any positive or negative integer.

Roughly speaking, the *spectrum combination* $\mathbb{S}_{x(\cdot)}$ is simply the collection of all *harmonic* and *intermodulation* frequency components generated by $x(t)$ using a *Resistive* nonlinearity whose Taylor series has infinitely many terms.

Definition 15. Spectrum Conservation Property

Let $z(\cdot)$ be the response of the network $\mathfrak{N}$ in Fig. 20 when driven by $u_s(\cdot)$. Assume that $u_s(\cdot)$ and $z(\cdot)$ are asymptotically almost periodic. Let $\mathbb{S}_{z(\cdot)}$ and $\mathbb{S}_{u_s(\cdot)}$ denote the *spectrum combination* of the *steady-state* waveform of $z(\cdot)$ and $u_s(\cdot)$, respectively. We say $\mathfrak{N}$ has the *spectrum-conservation property* iff

$$\mathbb{S}_{z(\cdot)} \subset \mathbb{S}_{u_s(\cdot)}. \tag{9.6}$$

Roughly speaking, a dynamic network $\mathfrak{N}$ has the spectrum-conservation property if the frequency spectrum of the output solution $z(\cdot)$ does *not* contain any component which is not a harmonic or intermodulation component of the input base frequencies. This rules out, therefore, any subharmonic or other more bizarre modes of solution.

Definition 16. Exponential-Transient-Decay Property

The *RLC* network $\mathfrak{N}$ in Fig. 20 is said to have the *exponential-transient-decay property* with respect to the state variable $z=(q_C, \phi_L)$ *iff* there exist real numbers $\tau_{\min}$, $\tau_{\max}$, $k_{\min}$, and $k_{\max}$ such that any pair of solutions $z'(\cdot)$ and $z''(\cdot)$ of $\mathfrak{N}$ satisfies the inequality.

$$k_{\min}\|z'(0)-z''(0)\|e^{-t/\tau_{\min}} \leqslant \|z'(t)-z''(t)\| \leqslant k_{\max}\|z'(0)-z''(0)\|e^{-t/\tau_{\max}} \tag{9.7}$$

for all $t \geqslant 0$. □

Any network satisfying the hypotheses of *Theorem 20* clearly has the exponential-transient-decay property.

Since the network in Fig. 19 (*Example 20*) satisfies *all conditions* so far stated in the preceding theorems, and since this network does *not* have a *unique* steady-state solution and does not have the spectrum conservation and the exponential-transient-decay property,[44] it is clear that even stronger hypotheses would be needed to guarantee these 3 properties. The following theorems are the latest circuit-theoretic results available on this subject:

Theorem 23. Networks with Linear Capacitors and Inductors

The *RLC* network $\mathfrak{N}$ in Fig. 20 has a *unique almost periodic steady-state solution*, the *spectrum-conservation property* and the *exponential-transient-decay property* with respect to the state variable $z=(q_C, \phi_L)$ if the following conditions are satisfied:

1. There is *no loop* {resp.; *no cut set*} formed exclusively by capacitors, inductors, and/or voltage sources {resp.; current sources.}
2. All Capacitors and Inductors are *Linear Passive*, and *Reciprocal*.
3. All internal Resistors are *strongly locally passive*.
4. The source vector $u_s(\cdot)$ is C^1 and asymptotically almost-periodic (*Definition 13*).

Proof: See the proof of *Theorem 6* of [25]. □

Theorem 24. RC and RL Networks with Linear Resistors

The conclusions of *Theorem 23* holds if the following conditions are satisfied:

1. There is *no loop* formed exclusively by capacitors and/or voltage sources {resp.; there is no *cut set* formed exclusively by inductors and/or current sources.}
2. $\mathfrak{N}$ contains no inductors (*RC* networks) {resp.; no capacitors (*RL* network)}.
3. All internal Resistors are *Linear*, *Passive*, and *Reciprocal*.
4. The source vector $u_s(\cdot)$ is C^1 and *asymptotically almost-periodic*.

Proof: See the proof of Theorem 8 of [25]. □

Theorem 25. RLC Networks with "Small-Signal Inputs

The *RLC* network $\mathfrak{N}$ in Fig. 20 has a *unique almost-periodic steady-state solution* and the *spectrum-conservation property* with respect to the state variable $z=(q_C, \phi_L)$ if the following conditions are satisfied:

1. There is *no loop* {resp.; *no cut set*} formed exclusively by capacitors, inductors, and/or voltage sources {resp.; current sources}.
2. All capacitors and Inductors are *Reciprocal* and *Strongly locally passive*.
3. All Resistors are *strongly locally passive*.
4. The *time-varying* component
$$\tilde{u}_s(t) \triangleq u_s(t) - u_{dc} \tag{9.8}$$
of the source vector $u_s(t)$ about the "dc bias" u_{dc} is continuous, asymptotically almost-periodic, and $\|\tilde{u}_s(t)\| < \delta$ for all $t \geqslant T$ where δ is a *sufficiently* small number, and T is any real number.
5. The Capacitor–inductor function $g(\cdot)$ is a C^2 function.

Proof: See the proof of *Theorem 10* of [25]. □

Remarks.

1. Conditions 4 and 5 of *Theorem 25* can be replaced by weaker conditions [25].

2. Since every *periodic* function is asymptotically almost -periodic, we can simplify the statement of *Theorems 23–25* by replacing the property "asymptotically almost periodic" by "periodic" to obtain a compact but more restrictive theorem.

[44] Any 2 distinct periodic solutions $z'(\cdot)$ and $z''(\cdot)$ of $\mathfrak{N}$ would violate (9.7).

3. To show that the conditions in *Theorems 23–25* are reasonably sharp, consider the network $\mathfrak{N}$ in Fig. 19 (*Example 20*) again, where it has at least 2 distinct periodic solutions. Note that $\mathfrak{N}$ violates only *Condition 2* of *Theorem 23* (inductor is *nonlinear* here), *Condition 2 of Theorem 24* ($\mathfrak{N}$ has both a capacitor and an inductor), and *Condition 4* of *Theorem 25* (the amplitude E which gave the computer-solution is *not* sufficiently small). Indeed, by decreasing the amplitude E to a sufficiently small value, the computer-simulation gives only a unique steady-state solution, as predicted by *Theorem 25*.

4. Criteria which guarantee unique steady-state response and the exponential-transient-decay property for diode-transistor networks are given in [52]. Expressed in terms of our formulation in Fig. 20, it can be shown that the criteria in [52] essentially requires the associated Resistive n-port N to be "almost" strictly locally passive. In other words, even though transistors are *locally active* resistors, they are "swamped" by the dissipation of the linear passive resistors so much so that the resulting "composite" n-port is "almost" *locally passive*.

X. Concluding Remarks

Any network which exhibits either an *impasse point* or a *finite-forward-escape-time solution* is *nonphysical* and must be remodeled. *Local solvability* is the weakest condition for excluding impasse points. Finite-forward-escape-time solutions can be excluded by imposing "eventual-strong-local passivity" on the capacitors and inductors, and "eventual passivity" on the resistors. By further imposing an "eventual-strong-local passivity" condition on the resistors, we can be assured that all solutions will be *eventually uniformly bounded*. Since all devices have a finite maximum power rating, there is little loss of generality to require that *nonlinear resistors* used in device models be eventually strongly locally passive. After all, *realistically modeled* networks are usually eventually uniformly-bounded.

Whether an *operating point* of a resistive network is observable or not in practice depends on whether its associated *equilibrium point* is "locally" *asymptotically stable*. This is the case if all eigenvalue of the associated Jacobian matrix have negative real parts. If at least one eigenvalue has a positive real part, the operating point is unobservable. We have as yet mentioned nothing about the *critical* case when one or more eigenvalues have a *zero real part*. The analysis of this case requires examination of higher order partial derivatives. The most important case of practical interest occurs when there is a pair of *purely imaginary* eigenvalues. In this case, a *frequency-domain* version of the *Hopf Bifurcation theorem* has been derived recently which completely characterizes the mechanism of "almost sinusoidal" oscillation [53]. This recent result can be considered as a *generalized Nyquist Criterion* for oscillation for multi-loop nonlinear feedback systems. It provides a completely rigorous foundation for the design of electronic sinusoidal oscillators.

Any one of the remaining qualitative properties (complete stability, global asymptotic stability, unique steady-state response, etc.) may *not* apply even for a realistically modeled network simply because many *physical* networks do not have these properties. It is expected therefore that *much stronger* hypotheses will be necessary in order to impose such additional properties on a network. The hypotheses of *Theorem 11–25* involve one or more of the following increasingly stronger conditions:

1. $\mathfrak{N}$ contains only *reciprocal* Resistors, Inductors, and Capacitors.
2. $\mathfrak{N}$ contains only *reciprocal* resistors and capacitors (*RC* network) or *reciprocal* resistors and inductors (*RL* network).
3. $\mathfrak{N}$ contains only *strongly locally passive* Resistors, Inductors, and Capacitors,[45]

Since most useful circuit elements (e.g., transistors, op amps., etc.) are *locally active*, and *nonreciprocal*, *Theorem 11–25* are presently applicable only to a rather restricted class of nonlinear *RLC* networks. Consequently, one of the most significant and challenging future research problems in *nonlinear dynamic networks* is to replace "Reciprocity" by a condition *weaker* than "strong local passivity"[46] so that *complete stability theorems* may be derived for transistor networks containing *multiple* equilibrium points. To dramatize the importance of solving this basic unsolved problem, we shall henceforth refer to it as the "curse of nonreciprocity."

Appendix A.1.

Proof of Theorem 2.

Condition 1 of Theorem 2 allows us to choose capacitor voltages $\boldsymbol{v}_C$ and inductor currents $\boldsymbol{i}_L$ as *independent* variables (without violating KCL or KVL).

Conditions 2–5 guarantee that the *current* through each *voltage-controlled* 2-terminal resistor, terminal pair, or port in parallel with a capacitor is *uniquely* determined by $\boldsymbol{v}_C$. Likewise, the *voltage* across each *current-controlled* 2-terminal resistor, terminal pair, or port in series with an inductor is uniquely determined by $\boldsymbol{i}_L$.

Now let us extract all these voltage-controlled resistors and consider them with their parallel capacitors as *external* elements. Similarly, extract all these current-controlled resistors and consider them with their series inductors as external elements. The remaining (n_C+n_L)-port $\hat{N}$ now contains only *strongly locally passive resistors* and independent sources.

Since the voltage sources {resp.; current sources} do not form loops {resp.; cut sets} with capacitors, inductors,

[45] In most of the theorems involving condition 3, *the "strong-local-passivity"* condition can be replaced by the slightly weaker "strict-local-passivity" condition plus a "C-diffeomorphic" requirement. However, in any closed and bounded region, these two properties are equivalent, except in some contrived cases.

[46] We reiterate that "strong local passivity" is an extremely strong condition. For 2-terminal elements, this implies the characteristic curve must be *monotone increasing* with *finite positive* slope at *all points*, and hence must tend to infinity in both directions. This should not be confused with the "*eventual*"-strong-local-passivity condition" which requires *monotonicity* only in far-out "don't care" regions.

and/or other voltage sources {resp.; current source} it follows from the *colored branch theorem* [30], [34] that each voltage source {resp.; current source} must form a cut set {resp.; loop} exclusively with the ports of one or more strongly locally passive resistors inside $\hat{N}$. Hence, each voltage source {resp.; current source} is either in series {resp.; parallel} or can be shifted in series via the *v-shift theorem* {resp.; in parallel via the *i-shift theorem*} with the ports of one or more strongly locally passive resistors inside $\hat{N}$. Hence, we can eliminate all voltage and current sources by combining them with the internal resistors. Each "composite" resistor R_α (consisting of R_α whose ports are in series with one or more voltage sources and/or in parallel with one or more current sources) will in general be described by a *time-varying* (if at least one source is time-dependent) constitutive relation [30, fig. 6]:

$$\hat{y}_{R_\alpha}=h_{R_\alpha}(\hat{x}_{R_\alpha}+b_\alpha(t))+c_\alpha(t) \triangleq \hat{h}_{R_\alpha}(x_{R_\alpha},t) \quad (A.1)$$

where $y_{R_\alpha} \triangleq h_R(x_R)$ denotes the constitutive relation of the original element R_α. Since R_α is strongly locally passive, and since the source vector $u_s(t)$ is bounded, each *composite resistor R* is also *strongly locally passive at any time t*.

It follows from *Theorem A-3(ii)* and *Theorem 4* of [30] that at any time t, $\hat{N}$ is described by a strictly increasing *onto* function. Hence, the port currents and voltages of $\hat{N}$ are also uniquely determined by v_C and i_L.

Applying KCL to the capacitor ports and KVL to the inductor ports, it follows that at any time t, i_C and v_L are uniquely determined by any $v_C \in \mathbb{R}^{n_C}$ and any $i_L \in \mathbb{R}^{n_L}$. Hence, $y=h(x;\, u_s(t))$ is a single-valued function for all t and $x \in \mathbb{R}^n$. □

A2. ERRATA FOR REFERENCES [31], [25]

Errata for Reference [31]:

1. p. 359, equation (26) should read:

$$\frac{\partial V(x)}{\partial x} f(x) > 0, \qquad \text{for all } \|x\| \geq k_0.$$

2. p. 362, in (46), t is a superscript for e.

3. p. 366, the statement following (80) should read: In this case, all solutions are bounded.

4. p. 368, delete the "inverse" on the left of (94a).

Errata for Reference [25]:

1. p. 536, line 3 below (35) should read: That is, $\omega \in \mathbb{S}$ if and only if $\omega = \Sigma_k n_k \omega_k \cdots$.

2. p. 537, last sentence of *Theorem B.2* should read: If (13) has a unique steady-state solution, then every solution $x(\cdot)$ of (13) is asymptotically almost periodic, and in the steady state $\mathbb{S}_x \subset \mathbb{S}_\xi$, *provided that*: $\xi(\cdot)$ is *asymptotically periodic*, or $\xi(\cdot)$ satisfies the hypotheses of *Theorem A.2*, Corollary A.1, or Corollary A.2, and further there exists open bounded sets $\tilde{D}_\xi \subset \mathbb{R}^m$, $\tilde{D}_x \subset \mathbb{R}^n$ and constant $\gamma > 0$ such that for any solution $x(\cdot)$ of (1) corresponding to input $\xi(\cdot)$, there exists $t_1 \in \mathbb{R}^n$ such that

$$x(t) \in \tilde{D}_x, \qquad \xi(t) \in \tilde{D}_\xi \quad (36b)$$

for all $t \geq t_1$, and

$$\frac{\partial V_\Delta(x',x'')}{\partial x'} f(x',\xi) + \frac{\partial V_\Delta(x',x'')}{\partial x''} f(x'',\xi) \geq \gamma \|x'-x''\|^2 \quad (36c)$$

for all $x', x'' \in D_x$, $\xi \in \tilde{D}_\xi$.

3. p. 537, insert the following after Remark 2). 3) Since (36b) is always true whenever (13) has a unique steady-state solution, this correction requires one additional condition for *Theorem B.2* to be valid in general: Either $\xi(\cdot)$ is *asymptotically periodic* (not asymptotically almost periodic) or (36c) must hold.

It can be proved that the following *Theorems 5–10.* and *Corollary 3* remain valid *as stated* because in each case the additional conditional (36c) holds.

The only statement affected by this correction is *Theorem 11* (p. 546). Here, either $\xi(\cdot)$ is asymptotically periodic "or" the unique steady state is known *a priori*," is needed to guarantee that each $x(\cdot)$ is asymptotically almost periodic with $\mathbb{S}_x \subset \mathbb{S}_\xi$.

4. p. 539, line 3 above (46), the last symbol should read $\mathbb{R}^{n_p}$.

5. p. 549, replace, t_0 in the first 3 lines (2nd column) by t_0 (3 locations).

6. p. 549, line 5 above (A.9), replace condition ii) by condition iii).

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