# Categorical and Kripke Semantics for Constructive S4 Modal Logic

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**Abstract.** We consider two systems of constructive modal logic which are computationally motivated. Their modalities admit several computational interpretations and are used to capture intensional features such as notions of computation, constraints, concurrency, etc. Both systems have so far been studied mainly from type-theoretic and category-theoretic perspectives, but Kripke models for similar systems were studied independently. Here we bring these threads together and prove duality results which show how to relate Kripke models to algebraic models and these in turn to the appropriate categorical models for these logics.

## 1 Introduction

This paper is about relating traditional Kripke-style semantics for constructive modal logics to their corresponding categorical semantics. Both forms of semantics have important applications within computer science. Our aim is to persuade traditional modal logicians that categorical semantics is easy, fun and useful; just like Kripke semantics. Additionally we show that categorical semantics generates interesting new constructive modal logics, which differ somewhat from the traditional diet of intuitionistic modal logics[WZ95].

The salient feature of the constructive modal logics considered in this paper is the omission of the axioms $\Diamond(A \vee B) \rightarrow \Diamond A \vee \Diamond B$ and $\neg\Diamond\bot$, which are typically assumed for possibility $\Diamond$ not only in classical but also in intuitionistic settings. While in classical (normal) modal logics these principles follow from the properties of necessity $\Box$ there is no a priori reason to adopt them in an intuitionistic setting where the classical duality between $\Box$ and $\Diamond$ breaks down and $\Diamond$ is no longer derivable from $\Box$. In fact, a growing body of work motivated by computer science applications [Wij90,FM97,PD01] rejects these principles from a constructive point of view. In this paper we will study the semantics of two such constructive modal logics, CS4 and PLL, introduced below.

We explore three standard types of semantics, Kripke, categorical, and algebraic semantics for CS4 and PLL. The algebraic semantics (CS4-modal algebra, PLL-modal algebra) is concerned only with equivalence of and the relative strength of formulas in terms of abstract semantic values (eg. truth values, proofs, constraints, etc...). It does not explain why a formula is true or why one formula is stronger than another. If one is interested in a more informative presentation and a concrete analysis of semantics, then a Kripke or categorical semantics may be more useful. The former explains 'meaning' in terms of worlds (in models) and validity of assertions at worlds (in models) in a classical Tarski-style interpretation. The 'semantic value' is given by the set of worlds at which a formula is valid. This form of semantics has been very successful for intuitionistic and modal logics alike. More recent and less traditional is the categorical approach. Here, we model not only the 'semantic value' of a formula, but also the 'semantic value' of its derivations/proofs, usually in a given natural deduction calculus. Thus, derivations in the logic are studied as entities in their own right, and have their own semantic objects in the models. Many applications of modal logic to computer science rely on having a term calculus for natural deduction proofs in the logic. Such a term calculus is a suitable variant of the $\lambda$-calculus, which is the prototypical functional programming language. From this point of view the semantic value of a formula is given by the collection of normal form programs that witness its assertion. Having a calculus of terms corresponding to derivations in the logic one obtains a direct correspondence between properties of proofs and properties of programs in the functional programming language based on these terms. For a discussion of the necessity modal operator $\Box$ and its interpretation as the 'eval/quote' operator in Lisp the reader is refered to [GL96].

In this sense both Kripke semantics and categorical semantics, presented here for CS4 and PLL, should be seen as two complementary elaborations of the algebraic semantics. They are both intensional refinements of their corresponding modal algebras, and have important applications within computer science. The natural correspondence between the Kripke models and modal algebras will be stated and proved as a *Stone Duality Theorem*. This turns out to require a different approach compared to other more standard intuitionistic modal logics, in particular as regards the $\Diamond$ modality. The other correspondence, between modal algebras and corresponding categorical structures, is essentially that between natural deduction proofs and the appropriate $\lambda$-calculus. This is known as the *Extended Curry-Howard Isomorphism*. Whereas the extended Curry-Howard isomorphism between intuitionistic propositional logic and the simply-typed $\lambda$-calculus has been known since the late 60s, establishing such isomorphisms for modal logics is a more recent development. In this paper we develop a suitable categorical semantics and associated $\lambda$-calculus for CS4 and PLL. It should be mentioned that the results for PLL are not new (see [FM97] for the Kripke and [BBdP98] for categorical semantics for PLL). Our contribution here is to show how PLL is related to CS4 and how these known results for PLL can be derived from those from CS4, or, to put it the other way round, how the known constructions for PLL may be generalised to CS4.

## 2 The Constructive Modal Systems CS4 and PLL

In this paper we take a fresh look at two prominent constructive modal extensions to intuitionistic propositional logic (IPL), which are particularly interesting because of their various applications in computer science.

To give the reader a taste for these applications, we list a few. Davies and Pfenning [DP96] use the $\Box$-modality to give a $\lambda$-calculus for computation in stages. The idea is that a term $\Box t$ represents a delayed computation. Ghani et al. [GdPR98] investigate refinements of this calculus which are suitable for the design of abstract machines. Similar ideas relating $\Box$ with staged evaluation and the distinction between run-time and compile-time semantics have been developed by Moggi et.al. [BMTS99]. Despeyroux and Pfenning [DPS97] use a box modality to encode higher-order abstract syntax in theorem-provers like Elf and Isabelle. Still another use of the $\Box$ modality, to model the `quote` mechanism of Lisp, is proposed by Goubault-Larrecq [GL96]. A $\Diamond$-style modality has been extensively used to distinguish a computation from its result in the $\lambda$-calculus: Moggi's [Mog91] influential work on computational monads describes the computational $\lambda$-calculus, which corresponds to an intuitionistic modal type theory with a $\Diamond$-like modality (see [BBdP98]). Fairtlough and Mendler [Men93,FMW97,Men00] use the same modality, which they call $\bigcirc$, in their work on lax logic for constraints and hardware verification. The calculus has also been used for denotational semantics of exception handling mechanisms, continuations, etc. On the syntactic side, it has been used, in the monadic-style of functional programming to add a notion of 'encapsulated state' to functional languages.

Despite their relevance for computer science these modal extensions of IPL seem to be less well investigated as modal logics in their own right, perhaps because of the "unusual properties" of their associated modal operators.

### 2.1 Constructive S4

The first modal system, which we call Constructive S4 (CS4), is a version of the intuitionistic S4 first introduced by Prawitz in his 1965 monograph [Pra65]. The Hilbert-style formulation of CS4 is obtained by extending IPL by a pair $\Box, \Diamond$ of S4-like intuitionistic modalities satisfying the axioms and the necessitation rule listed in Figure 1. The normal basis of CS4, *i.e.*, consisting only of axioms $\Box K$ and $\Diamond K$ plus the axiom $\neg\Diamond\bot$ (which we reject, see below) has been introduced[1] and motivated by Wijesekera [Wij90] as a predecessor to constructive concurrent dynamic logic. The practical importance of CS4 as a type system for functional programming is evident from the literature, e.g. as cited in the beginning of this section, though most applications so far focus on the $\Box$ modality. The formal role of $\Diamond$ and its interaction with $\Box$ has recently been studied systematically by Pfenning and Davies [PD01].

| | |
|---|---|
| $\Box K : \Box(A \to B) \to (\Box A \to \Box B)$ | $\Diamond K : \Box(A \to B) \to (\Diamond A \to \Diamond B)$ |
| $\Box T : \Box A \to A$ | $\Diamond T : A \to \Diamond A$ |
| $\Box 4 : \Box A \to \Box\Box A$ | $\Diamond 4 : \Diamond\Diamond A \to \Diamond A$ |
| $Nec$ : If $A$ is a theorem then $\Box A$ is a theorem. | |

**Fig. 1.** Hilbert-style system for Constructive S4

The natural deduction formulation of CS4 is subject to some controversy. We recall it in the style of Bierman and de Paiva [BdP96]. The naive introduction rule for $\Box$ (corresponding to the necessitation rule $Nec$) insists that all of the undischarged assumptions at the time of application are modal, i.e. they are all of the form $\Box A_i$. However, the

[1] Wijesekera considers a first order system, to be precise.

fundamental feature of natural deduction is that it is *closed under substitution* and this naive rule will not be closed under substitution, i.e. substituting a correct derivation in another correct derivation will yield an incorrect one (if this substitution introduces non-modal assumptions). We conclude that $\Box_{\mathcal{I}}$ must be formulated as in Figure 2, where the substitutions are given explicitly. The same sort of problem arises in the rules for $\Diamond_{\mathcal{E}}$ and the same solution (of explicit substitutions) can be used, see the rule $\Diamond_{\mathcal{E}}$ in Figure 2.

Both problems were first observed by Prawitz, who proposed a syntactically more complicated way of solving it [Pra65]. An interesting alternative approach has recently been presented by Pfenning and Davies [PD01], which (essentially) involves two kinds of variables, and two kinds of substitution. Note that in our solution the discharging brackets are used in a slightly different way from traditional natural deduction. In the introduction rule for $\Box$ they mean, discharge *all* assumptions (which must be all boxed in this rule).

$$\frac{\begin{array}{c}\vdots\\ \Box B\end{array}}{B}\ (\Box_{\mathcal{E}}) \qquad \frac{\begin{array}{c}\vdots\\ \Box A_1\end{array}\ \ldots\ \begin{array}{c}\vdots\\ \Box A_n\end{array}\qquad \begin{array}{c}[\Box A_1 \cdots \Box A_n]\\ \vdots\\ B\end{array}}{\Box B}\ (\Box_{\mathcal{I}})$$

$$\frac{\begin{array}{c}\vdots\\ B\end{array}}{\Diamond B}\ (\Diamond_{\mathcal{I}}) \qquad \frac{\begin{array}{c}\vdots\\ \Box A_1\end{array}\ \ldots\ \begin{array}{c}\vdots\\ \Box A_n\end{array}\ \begin{array}{c}\vdots\\ \Diamond B\end{array}\qquad \begin{array}{c}[\Box A_1 \cdots \Box A_n\ B]\\ \vdots\\ \Diamond C\end{array}}{\Diamond C}\ (\Diamond_{\mathcal{E}})$$

**Fig. 2.** Natural Deduction rules for Constructive S4

The system CS4 is the weakest among the variants of intuitionistic S4 discussed in the literature. In particular, it does not prove the distribution of the possibility operator over disjunction $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$, nor does it assume $\neg\Diamond\bot$, *i.e.*, that possibly falsum ($\Diamond\bot$) and falsum ($\bot$) are equiprovable (which is the nullary form of the distribution). This version of non-classical S4 without distributivity of $\Diamond$ over $\vee$ is extremely well-behaved. As we will see there is a complete version of the Curry-Howard Isomorphism for it.

### 2.2 Propositional Lax Logic

The second constructive modal logic we consider is an extension of IPL that features a single modality $\Diamond$ satisfying the axioms

$$\begin{array}{l} \Diamond T : A \to \Diamond A \\ \Diamond 4\ : \Diamond\Diamond A \to \Diamond A \\ \Diamond F : (A \to B) \to \Diamond A \to \Diamond B. \end{array}$$

The third axiom is known (categorically) as 'functorial strength'. This system is discussed under different names and in slightly differing but equivalent axiomatic presentations, such as *Computational Logic* [BBdP98] or *Propositional Lax Logic* (PLL) [FM97]. Henceforth we shall call it PLL. The natural deduction system contains the following rules for $\diamond$ ([Men93]):

$$\dfrac{\begin{array}{c}\vdots\\ B\end{array}}{\diamond B}\,(\diamond\mathcal{I}) \qquad \dfrac{\begin{array}{c}\vdots\\ \diamond A\end{array} \qquad \begin{array}{c}[A]\\ \vdots\\ \diamond B\end{array}}{\diamond B}\,(\diamond\mathcal{E})$$

PLL also has a colourful history. As a modal logic it was invented in the forties by Curry [Cur57] (who seems to have dropped it again because of its wild properties) and independently rediscovered in the nineties by Benton et al. and Fairtlough and Mendler, who used the symbol $\bigcirc$ for the modality, as the Curry-Howard isomorphic version of Moggi's computational lambda-calculus. As an algebra the system PLL is well known in abstract topology. The operator $\bigcirc$ arises naturally as a (strong, or multiplicative) closure operator on the lattice of open sets, or more generally as a so-called nucleus in the theory of topoi and sheafification [Joh82]. From this topological perspective, Goldblatt studied a system identical to PLL accommodating Lawvere's suggestion that the $\bigcirc$ modality means "it is locally the case that" by interpreting this in various ways to mean "at all nearby points" [Gol81,Gol93]. The algebraic properties of such operators (on complete Heyting algebras) have been explored by Macnab [Mac81], who calls them "modal operators".

In this paper we show how PLL can be naturally seen as a special CS4 theory or CS4 algebra in the sense that it can be obtained from CS4 by adding the axiom $A \rightarrow \Box A$. These results identify $\bigcirc$ as a constructive modality of possibility and provide a satisfactory explanation for why in PLL a modality $\Box$ is missing: it is implicitly built into the semantics already.

## 3 Kripke models

Our first step is to develop a suitable Kripke model theory for CS4. While it is easy to agree that a Kripke model of constructive modal logic should consist of a set of worlds $W$ and two accessibility relations, one intuitionistic $\leq$ and the other modal $R$, it is not so clear how these relations should interact (frame conditions) and just how they should be used to interpret specifically the $\diamond$ modality. The mainstream approach as exemplified by Ewald [Ewa86], Fischer-Servi [FS80], Plotkin and Stirling [PS86], Simpson [Sim94] is based on the analogy of $\Box$ with $\forall$ and of $\diamond$ with $\exists$-quantification over the modal accessibility $R$. Reading these quantifiers intuitionistically, relative to $\leq$, one arrives at the semantic interpretation $w \models \Box A$ iff $\forall v.\ w \leq v \Rightarrow \forall u.\ v\ R\ u \Rightarrow u \models A$ for necessity, and

$$w \models \diamond A \text{ \textit{iff} } \exists u.\, w\, R\, u \ \&\ u \models A \tag{1}$$

for possibility. Indeed, as the shown in the literature, this gives a fruitful basis for intuitionistic modal logics. Unfortunately, it is not suitable for CS4, since it forces the axiom $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$ to hold, which we want to avoid. It also requires an extra frame condition to ensure hereditariness of truth, *viz.*, that $w \models \Diamond A$ and $w \leq v$ implies $v \models \Diamond A$. Hereditariness, however, can also be achieved simply by $\forall$-quantifying over all $\leq$-successors in the interpretation of $\Diamond$:

$$w \models \Diamond A \textit{ iff } \forall u.\ w \leq u \Rightarrow \exists v.\ u\ R\ v\ \ \&\ \ v \models A. \tag{2}$$

Not only does this away with the extra frame condition to force $\Diamond$ hereditary along $\leq$, it also eliminates the unwanted axiom $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$. In fact, as it turns out this works for CS4. This interpretation (2) of $\Diamond$, as far as we are aware, has been introduced by Wijesekera [Wij90] to capture non-deterministic computations and independently in [FM97] as an adequate Kripke interpretation of truth "up to constraints". In both cases the absence of the axioms $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$ is a natural consequence of the semantics.

Wijesekera only considered the normal base $\Box K, \Diamond K$ of CS4, yet included the axiom $\neg\Diamond\bot$. To eliminate the axiom $\neg\Diamond\bot$ we follow [FM97] in permitting explicit fallible worlds in our models. What remains, then, is to find suitable frame conditions on $\leq$ and $R$ that are characterised by the CS4 axioms $\Box T, \Box 4, \Diamond T, \Diamond 4$. These are incorporated into the following notion of CS4 model:

**Definition 1.** *A* Kripke model of CS4 *is a structure $M = (W, \leq, R, \models)$, where $W$ is a non-empty set, $\leq$ and $R$ are reflexive and transitive binary relations on $W$, and $\models$ a relation between elements $w \in W$ and propositions $A$, written $w \models A$ ("$A$ satisfied at $w$ in $M$") such that:*

- *$\leq$ is hereditary with respect to propositional variables, that is, for every variable $p$ and worlds $w, w'$, if $w \leq w'$ and $w \models p$, then $w' \models p$.*
- *$R$ and $\leq$ are related as follows: if $wRw'$ and $w' \leq v$ then there exists $v'$ such that $w \leq v'$ and $v'Rv$. In other words: $(R\ ;\leq)\ \subseteq\ (\leq\ ; R)$.*
- *The relation $\models$ has the following properties:*
  $w \models \top$;
  $w \models A \wedge B$ *iff* $w \models A$ *and* $w \models B$;
  $w \models A \vee B$ *iff* $w \models A$ *or* $w \models B$;
  $w \models A \to B$ *iff* $\forall w'.\ w \leq w' \Rightarrow (w' \models A \Rightarrow w' \models B)$
  $w \models \Box A$ *iff* $\forall w'.\ w \leq w' \Rightarrow \forall u.\ w'Ru \Rightarrow u \models A$
  $w \models \Diamond A$ *iff* $\forall w'.\ w \leq w' \Rightarrow \exists u.\ w'Ru \wedge u \models A$
  *Notice that we do not have the clause $w \not\models \bot$, i.e., we allow inconsistent worlds. Instead, we have*
- *if $w \models \bot$ and $w \leq w'$, then $w' \models \bot$, and*
- *if $w \models \bot$, then for every propositional variable $p$, $w \models p$ (to make sure that $\bot \to A$ is still valid).*

As usual, a formula $A$ is *true* in a model $M = (W, \leq, R, \models)$ if for every $w \in W$, $w \models A$. We sometimes write $M, w \models A$ when we want to make the model explicit. A formula $A$ is *valid* ($\models A$) if it is true in all models; a formula is satisfiable if there is a model and a *consistent* world where it is satisfied. A formula $A$ is a *logical consequence* of a set of formulae $\Gamma$ if for every $M, w$ if $M, w \models \Gamma$, then $M, w \models A$.

Observe that under the translation of intuitionistic logic into classical S4 which introduces a modality $\Box_I$ corresponding to the intuitionistic accessibility relation $\leq$, our modalities $\Box$ and $\Diamond$ are translated as $\Box_I \Box_M$ and $\Box_I \Diamond_M$, respectively (where $\Box_M$ and $\Diamond_M$ are modalities corresponding to $R$). This means that our variant of S4 does not fall directly in the scope of Wolter and Zakharyaschev's analysis of intuitionistic modal logics as classical bimodal logics in [WZ97] since they assume $\Diamond$ to be a normal modality. However, analogous techniques could probably be used to give a new proof of decidability and finite modal property of CS4 and PLL.

**Theorem 1.** CS4 *is sound and strongly complete with respect to the class of models defined above, that is, for every set of formulae $\Gamma$ and formula $A$, we have $\Gamma \vdash_{\mathsf{CS4}} A \Leftrightarrow \Gamma \models A$.*

We can use Theorem 1 to give a new soundness and completeness theorem for PLL. This is based on the observation that PLL models are a sub-class of CS4 models:

**Definition 2.** *A* Kripke model for PLL *is a Kripke model for* CS4 *where $R$ is hereditary, that is, for every formula $A$, if $w \models A$ and $wRv$, then $v \models A$.*

The latter requirement corresponds to the strength axiom. It is in fact equivalent to the axiom $A \to \Box A$, so that $\Box$ becomes redundant in Kripke models for PLL. An alternative (slightly stronger) definition to the same effect given by Fairtlough and Mendler requires that $R$ is a subset of $\leq$.

**Theorem 2.** PLL *is sound and strongly complete with respect to the class of models defined above.*

*Proof.* Soundness of PLL follows from soundness of CS4 and the fact that PLL-models satisfy the axiom scheme $A \to \Box A$, which renders the strength $\Diamond F$ axiom derivable from $\Diamond K$ of CS4.

For completeness consider an arbitrary set $\Gamma$ of PLL-formulas, and a PLL-formula $B$ such that $\Gamma \not\vdash_{\mathsf{PLL}} B$. Then, it is not difficult to see that $\Gamma^* \not\vdash_{\mathsf{CS4}} B$ where $\Gamma^*$ is the theory $\Gamma$ extended by all instances of the scheme $A \to \Box A$. For otherwise, if $\Gamma^* \vdash_{\mathsf{CS4}} B$, we could transform this derivation into a derivation $\Gamma \vdash_{\mathsf{PLL}} B$ simply by dropping all occurrences of $\Box$ in any formula, which means that every use of a CS4-axiom becomes an application of a PLL-axiom, and any use of an axiom $A \to \Box A$ or rule $Nec$ becomes trivial. Note, this holds since if we drop all $\Box$ in a CS4 axiom, we get a PLL-axiom. By strong completeness of CS4 we conclude there exists a CS4-model $M$ such that $M \models \Gamma^*$ but $M \not\models B$. But then not only $M \models \Gamma$ but also $M$ validates all instances of $A \to \Box A$, which means that $M$ is a PLL-model.

## 4 Modal Algebras and Duality

There is no unique 'right' Kripke semantics for a given system of modal logic. In general, the fit between modal (intuitionistic or classical) logics and Kripke structures is not perfect: apart from several versions of Kripke semantics for the same logic, which already seems suspect to category theorists, there are logics which are not complete for any Kripke semantics ([Fin74,Tho74]). *Modal algebras* have the definite advantage of fitting the logics much better.

One can think of an algebra as a collection of syntactic objects, e.g. formulae of a logic. Representation theorems for algebras show how given an algebra one can build a 'representation' for it - a structure which is a 'concrete' set-theoretic object, e.g. a Kripke model[2].

We define modal algebras corresponding to PLL and CS4 below and show how to construct representations for them. Since the modal algebras can be directly obtained from the respective categorical models, and modal algebras can be shown (see below) to be Stone-dually related to our Kripke models, we obtain an algebraic link (albeit a weak one) between Kripke models and categorical models for the two constructive modal systems considered.

Recall that a *Heyting algebra* $H$ is a structure of the form $\langle A, \leq, \times, +, \Rightarrow, 0\rangle$ where $A$ is a set of objects (one example would be formulae), $\leq$ is a partial order (for formulae, $a \leq b$ means 'a implies b'), $\times$ is a product (which corresponds to $\wedge$ in intuitionistic logic), $+$ a sum (corresponds to $\vee$), $\Rightarrow$ pseudocomplement (corresponds to $\rightarrow$) and $0$ the least element ($\perp$).

We introduce two additional operators, corresponding to the modalities. Note that $\Box$ distributes over $\times$, but $\Diamond$ does not distribute over $+$.

**Definition 3.** *A* CS4-modal algebra $\mathcal{A} = \langle A, \leq, \times, +, \Rightarrow, 0, \Box, \Diamond\rangle$ *consists of a Heyting algebra* $\langle A, \leq, \times, +, \Rightarrow, 0\rangle$ *with two unary operators* $\Box$ *and* $\Diamond$ *on* $A$*, such that for every* $a, b \in A$*,*

$$\begin{array}{lll} \Box(a \times b) = \Box a \times \Box b & \Box a \leq a & a \leq \Diamond a \\ \Diamond a \leq \Diamond(a + b) & \Box a \leq \Box\Box a & \Diamond\Diamond a \leq \Diamond a \\ 1 \leq \Box 1 & \multicolumn{2}{c}{\Box a \times \Diamond b \leq \Diamond(\Box a \times b).} \end{array}$$

Next, we identify the corresponding algebraic structure for PLL, which are also known, in a somewhat different axiomatisation, as "local algebras" [Gol76]:

**Definition 4.** *A* PLL-modal algebra $\mathcal{A} = \langle A, \leq, \times, +, \Rightarrow, 0, \Diamond\rangle$ *consists of a Heyting algebra* $\langle A, \leq, \times, +, \Rightarrow, 0\rangle$ *with a unary operator* $\Diamond$ *on* $A$*, such that for every* $a, b \in A$*,*

$$\Diamond a \leq \Diamond(a + b) \quad a \leq \Diamond a \quad \Diamond\Diamond a \leq \Diamond a \quad a \times \Diamond b \leq \Diamond(a \times b).$$

Obviously, every Kripke model $M$ for CS4 or PLL gives rise to a corresponding modal algebra $M^+$ (take the set of all definable sets of possible worlds).

Conversely, every modal algebra gives rise to a so-called *general frame*. A general frame is a structure which consists of a set of possible worlds $W$, two accessibility relations and a collection $\mathcal{W}$ of subsets of $W$ *which can serve as denotations of formulae*. Intuitively, $\mathcal{W}$ should contain $\{w{:}w \models p\}$ for every propositional variable $p$ and be closed under intersection, union and operations which give the set of worlds satisfying $\Box\varphi$ ($\Diamond\varphi$) from the set of worlds satisfying $\varphi$. (For more background, see for example [Ben83].)

[2] More precisely, a general frame; see the discussion below.

Here, we will be somewhat sloppy and identify elements of the algebra with logical formulae straightaway. We assume that some subset $P$ of $A$ is arbitrarily designated as a set of propositional variables; $\times$, $+$, $\Rightarrow$ and $0$ are interpreted as $\wedge$, $\vee$, $\rightarrow$ and $\perp$. Then we can formulate the representation theorem for models instead of general frames:

**Theorem 3 (Representation for CS4).** *Let $\mathcal{A}$ be a CS4-modal algebra. Then the Stone representation of $\mathcal{A}$, $SR(\mathcal{A}) = (W^*, R^*, \leq^*, \models^*)$ is a Kripke model for CS4, where*

1. *$W^*$ is the set of all pairs $(\Gamma, \Theta)$ where $\Gamma \subseteq \mathcal{A}$ is a prime filter, and $\Theta \subseteq \mathcal{A}$ an arbitrary set of elements such that for all finite, nonempty, choices of elements $c_1, \ldots, c_n \in \Theta$, $\Diamond(c_1 + \cdots + c_n) \notin \Gamma$.*
2. *$(\Gamma, \Theta) \leq^* (\Gamma', \Theta')$ iff $\Gamma \subseteq \Gamma'$.*
3. *$(\Gamma, \Theta) R^* (\Gamma', \Theta')$ iff $\forall a.\ \Box a \in \Gamma \Rightarrow a \in \Gamma'$ and $\Theta \subseteq \Theta'$.*
4. *For all $a \in \mathcal{A}$, $(\Gamma, \Theta) \models^* a$ iff $a \in \Gamma$.*

Let us call pairs $(\Gamma, \Theta)$ with $\Gamma, \Theta \subseteq \mathcal{A}$ *consistent theories* if for any, possibly empty, choice of elements $b_1, \ldots, b_m$ in $\Gamma$ and any non-empty choice of elements $c_1, \ldots, c_n \in \Theta$, $b_1 \times \ldots \times b_m \not\leq \Diamond(c_1 + \cdots + c_n)$. Then, the worlds of $SR(\mathcal{A})$ are simply the consistent theories $(\Gamma, \Theta)$ where $\Gamma$ is a prime filter. In the completeness proof we also need a slightly stronger notion of consistency as follows: For $a \in \mathcal{A}$, a theory $(\Gamma, \Theta)$ is *$a$-consistent* if for any choice of elements $b_1, \ldots, b_m$ in $\Gamma$ and $c_1, \ldots, c_n \in \Theta$, $b_1 \times \ldots \times b_m \not\leq (a + \Diamond(c_1 + \cdots + c_n))$. This includes the degenerate case $n = 0$ where we simply require $b_1 \times \ldots \times b_m \not\leq a$.

The proof of our Stone Representation Theorem 3 relies on the following lemma.

**Lemma 1 (Saturation Lemma).** *Let $a \in \mathcal{A}$ and $(\Gamma, \Theta)$ an $a$-consistent theory in the CS4-algebra $\mathcal{A}$. Then $(\Gamma, \Theta)$ has a saturated $a$-consistent extension $(\Gamma^*, \Theta)$, such that $\Gamma^*$ is a prime filter and $\Gamma \subseteq \Gamma^*$.*

We can now extract without extra effort a Stone Representation for PLL algebras from that for CS4 algebras, identical to the one implicit in the completeness proof given in Fairtlough and Mendler [FM97].

**Theorem 4 (Representation for PLL).** *Let $\mathcal{A}$ be a PLL-modal algebra. Then the Stone representation of $\mathcal{A}$, $SR(\mathcal{A}) = (W^*, R^*, \leq^*, \models^*)$ is a Kripke model for PLL, where $W^*$, $\leq^*$, $\models^*$ are as above and $(\Gamma, \Theta) R^* (\Gamma', \Theta')$ iff $\Gamma \subseteq \Gamma'$ and $\Theta \subseteq \Theta'$.*

*Proof.* Observe that every PLL algebra $\mathcal{A}$ is at the same time a CS4 algebra $\mathcal{A}'$ where the operator $\Box$ is taken to be the identity function. Hence, we can construct its CS4 Stone representation $SR(\mathcal{A}')$ as in Theorem 3, which is a CS4 algebra. Now, what properties does the relation $R^*$ have in $SR(\mathcal{A}')$? Well, $(\Gamma_1, \Theta_1)\ R^*\ (\Gamma_2, \Theta_2)$ iff $\forall a.\ \Box a \in \Gamma_1 \Rightarrow a \in \Gamma_2$ and $\Theta_1 \subseteq \Theta_2$. But since $\Box$ is the identity operator, this is the same as $\Gamma_1 \subseteq \Gamma_2$ and $\Theta_1 \subseteq \Theta_2$ as defined in Theorem 4. Observe further that $R^*$ is a subrelation of $\leq^*$, which means that $R^*$ is hereditary. Thus, $SR(\mathcal{A}')$ is a PLL model.

Section 6 introduces categorical models for CS4 and PLL. Observe that one can view categorical models as modal algebras where the partial order relation $\leq$ is replaced by a collection of morphisms. Intuitively, (again thinking of objects as formulae) while $a \leq b$ in an algebra means that $b$ is implied by $a$, the category has possibly several morphisms from $a$ to $b$ labelled by encodings of corresponding derivations of $b$ from $a$.

## 5 Discussion on Kripke Semantics

Since our Kripke semantics for CS4 is new it deserves some further justification and discussion, which we give in this section.

First, how do our models relate to Wijesekera's? Let us call the class of structures $M = (W, \leq, R, \models)$ with $\leq$ reflexive and transitive but arbitrary $R$ CK-*models* (*i.e.*, drop the requirement that $R$ is reflexive and transitive as well as the frame condition $R;\leq \subseteq \leq;R$), and further those in which for all worlds $w \not\models \bot$ *infallible* CK models. Then, Wijesekera [Wij90] showed[3] that the theory $\mathsf{IPL} + \Box K + \Diamond K + \neg\Diamond\bot$ with the rules of Modus Ponens and $Nec$ is sound and complete for the class of infallible CK models. The proof of Wijesekera can be modified to show that $\mathsf{CK} = \mathsf{IPL} + \Box K + \Diamond K$ is sound and complete for all CK models. Our CS4-models may then be seen as the special class of CK models characterised by the additional axioms $\Diamond T, \Box T, \Diamond 4, \Box 4$.

Following [FM97] we permitted *fallible worlds* to render the formula $\neg\Diamond\bot$ invalid. This makes CS4 different from traditional intuitionistic modal logics which invariably accept this axiom. Fallible worlds were used originally to provide an intuitionistic meta-theory for intuitionistic logic, *e.g.*,[TvD88,Dum77]. For intuitionistic propositional logics, with a classical meta-theory, fallible worlds are redundant. However, this is no longer true for modal logics. There, the presence or absence of fallible worlds is reflected in the absence or presence of the theorem $\neg\Diamond\bot$. In particular note that in the standard classical setting, i.e., without fallible worlds and $w \models \Diamond A$ meaning $\exists v.\ w\ R\ v\ \ \&\ \ v \models A$, the axiom $\neg\Diamond\bot$ (as well as $\Diamond(A \vee B) \rightarrow \Diamond A \vee \Diamond B$) is automatically validated.

It is not only the fallible worlds but also the extension by sets $\Theta$, capturing hereditary refutation information, that distinguishes the representation of constructive modal logic, such as CS4, from that for standard intuitionistic modal logics, such as those of [PS86,FS80,Ewa86]. Indeed, if the axioms $\neg\Diamond\bot$ and $\Diamond(\phi \vee \psi) \rightarrow \Diamond\phi \vee \Diamond\psi$ are adopted the sets $\Theta$ and fallible worlds become redundant. Without these axioms, however, we also need the "negative" information in $\Theta$ to characterise truth at a world fully. It is also worthwhile to note that the model representation of Thm. 3 for CS4 is simpler than the one given by Wijesekera [Wij90] in the completeness proof for $\mathsf{CK} + \neg\Diamond\bot$. There, the $\Theta$ are (essentially) *sets of sets* of propositions, in which every element in $\Theta$ is a *set* of all possible future worlds for $(\Gamma, \Theta)$ that are accessible through $R^*$. This too, expresses negative information, though of a second-order nature. A quite different, but still second-order representation of CK models has been proposed by Hilken [Hil96]. As we have shown, however, the representation for CS4 can be done in a first-order fashion.

Our constructive S4 models satisfy the inclusion $R;\leq \subseteq \leq;R$, a frame condition that is typically assumed in standard intuitionistic modal logic already for system IK. One may wonder about the converse $\leq;R \subseteq R;\leq$ of this inclusion. One can show that in our models it generates the independent axiom scheme $((\Box A \rightarrow \Diamond B) \wedge \Box(A \vee \Diamond B)) \rightarrow \Diamond B$, thus inducing a proper extension of CS4.

[3] Actually, Wijesekera also lists the axiom $\Box A \wedge \Diamond(A \rightarrow B) \rightarrow \Diamond B$, but this is derivable already.

As pointed out before, traditional intuitionistic modal logics such as those considered by Fischer-Servi [FS80] or Plotkin and Stirling [PS86] adopt a fundamentally different interpretation of $\Diamond$, defining $w \models \Diamond A$ *iff* $\exists v.\ w\ R\ v\ \ \&\ \ v \models A$. This enforces validity of $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$ but requires a frame condition $\leq^{-1};R \subseteq R;\leq^{-1}$ (confluence of $\leq$ and $R$) to make $\Diamond$ hereditary along $\leq$. It is not surprising, then, that for our constructive modal models, where hereditariness is built in by the semantic interpretation, this frame condition obtains the axiom scheme $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$, again inducing a proper extension.

We leave it as an open question if the above-mentioned axioms $((\Box A \to \Diamond B) \wedge \Box(A \vee \Diamond B)) \to \Diamond B$ or $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$ are complete for the frame conditions $\leq;R \subseteq R;\leq$ or $\leq^{-1};R \subseteq R;\leq^{-1}$, respectively. At least for PLL [FM97] it is known that $\leq^{-1};R \subseteq R;\leq^{-1}$ is completely captured by the axiom $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$, and in [Wij90] this axiom is linked with sequentiality of $R$.

## 6 Categorical models

Categorical models distinguish between different proofs of the same formula. A category consists of objects, which model the propositional variables, and for every two objects $A$ and $B$ each morphism in the category from $A$ to $B$, corresponds to a proof of $B$ using $A$ as hypothesis.

*Cartesian closed categories* (with coproducts) are the categorical models for intuitionistic propositional logic. For a proper explanation the reader should consult Lambek and Scott [LS85]; Here we just outline the intuitions. Conjunction is modelled by cartesian products, a suitable generalisation of the products in Heyting algebras. The usual logical relationship between conjunction and implication

$$A \wedge B \longrightarrow C \text{ if and only if } A \longrightarrow (B \to C)$$

is modelled by an adjunction and this defines categorically the implication connective. Thus we require that for any two objects $B$ and $C$ there is an object $B \to C$ such that there is a bijection between morphisms from $A \wedge B$ to $C$ and morphisms from $A$ to $B \to C$. Disjunctions are modelled by coproducts, again a suitable generalisation of the sums of Heyting algebras. True and false are modelled by the empty product (called a terminal object) and co-product (the initial object), respectively. Finally negation, as traditional in constructive logic, is modelled as implication into falsum. A cartesian closed category (with coproducts) is sometimes shortened to a ccc (respectively a bi-ccc). $\mathbf{Set}$, the category where the objects are sets and morphisms between sets are functions, is the standard example of a bi-cartesian closed category.

To present a categorical model of constructive S4 we must add to a bi-ccc the structure needed to model the modalities. In previous work [BdP96] it was shown that to model the S4 necessity $\Box$ operator one needs a *monoidal comonad*. Such a monoidal comonad consists of an endofunctor $\Box\colon \mathcal{C} \longrightarrow \mathcal{C}$ together with natural transformations $\delta_A\colon \Box A \longrightarrow \Box\Box A$ and $\epsilon_A\colon \Box A \longrightarrow A$ and $m_{A,B}\colon \Box A \times \Box B \longrightarrow \Box(A \times B)$ and a map $m_1\colon 1 \to \Box 1$, satisfying some commuting conditions. These natural transformations model the axioms $\mathbf{4}$ and $\mathbf{T}$ together with the necessitation rule and the $\mathbf{K}$ axiom.

Here we assume that the modal operator $\Diamond$ is dually modelled by a *monad* with certain special characteristics: namely we want our monad to be *strong* with respect to the $\Box$ operator, i.e. we assume a natural transformation $st_{A,B}\colon \Box A \times \Diamond B \longrightarrow \Diamond(\Box A \times B)$ satisfying the conditions detailed in [Kob97]. The strength is needed to model the explicit substitution in the $\Diamond_{\mathcal{E}}$-rule.

**Definition 5.** *A* CS4*-category consists of a cartesian closed category $\mathcal{C}$ with coproducts, a monoidal comonad $(\Box, \delta, \epsilon, m_{-,-}, m_1)$ where $\Box\colon \mathcal{C} \longrightarrow \mathcal{C}$ and a $\Box$-strong monad $(\Diamond, \mu, \eta)$ where $\Diamond\colon \mathcal{C} \longrightarrow \mathcal{C}$.*

The soundness theorem shows in detail how the categorical semantics models the modal logic.

**Theorem 5 (Soundness).** *Let $\mathcal{C}$ be any* CS4*-category. Then there is a canonical interpretation $[\![\_]\!]$ of* CS4 *in $\mathcal{C}$ such that*

- *a formula $A$ is mapped to an object $[\![A]\!]$ of $\mathcal{C}$;*
- *a natural deduction proof $\psi$ of $B$ using formulae $A_1, \ldots, A_n$ as hypotheses is mapped to a morphism $[\![\psi]\!]$ from $[\![A_1]\!] \times \cdots \times [\![A_n]\!]$ to $[\![B]\!]$;*
- *each two natural deduction proofs $\phi$ and $\psi$ of $B$ using formulae $A_1, \ldots, A_n$ as hypotheses which are equal (modulo normalisation of proofs) are mapped to the same morphism, in other words $[\![\phi]\!] = [\![\psi]\!]$.*

A trivial degenerate example of an CS4-category consists of taking any bi-ccc, say $\mathbf{Set}$ for example and considering the identity functor (both as a monoidal comonad and as monad) on it. Less trivial, but still degenerate models are Heyting algebras (the poset version of a bi-ccc) together with a closure and a co-closure operator. Non-degenerate models (but quite complicated ones) can be found in [GL96]. To prove categorical completeness we use a term model construction.

**Theorem 6 (Completeness).**

*(i) There exists a* CS4*-category such that all morphisms are interpretations of natural deduction proofs.*
*(ii) If the interpretation of two natural deduction proofs is equal in all* CS4*-categories, then the two proofs are equal modulo proof-normalisation in natural deduction.*

A categorical model of PLL consists of a cartesian closed category with a strong monad. These models were in fact the original semantics for Moggi's computational lambda-calculus and PLL can be seen as reverse engineering from that [BBdP98]. Hence we refrain from stating categorical soundness and completeness for this system, but of course they hold as expected [Kob97].

In the logic, PLL arises as a special case of CS4 when we assume the derivability of $A \to \Box A$. A similar statement holds in category theory. We have an inclusion functor from the category of PLL-categories into the category of CS4-categories: each PLL-category is a CS4-category where the co-monad is the identity functor. Conversely, each CS4-category such that $\Box A$ is isomorphic to $A$ is a CS4-category.

## 7 Conclusions

This paper shows how traditional Kripke semantics for two systems of intuitionistic modal logic, CS4 and PLL, can be related via duality theory to the categorical semantics of (natural deduction) proofs for these logics. The associated notions of modal algebras serve as an intermediate reference point. From this point of view the results of this paper may be seen as presenting two kinds of representations for these modal algebras.

The first representation explains the semantics of an element in the algebra in terms of sets of worlds and truth within Kripke models. To this end we have developed an appropriate class of Kripke models for CS4 and proved a Stone representation theorem for it. As far as we are aware the model representation for CS4 is new. Its essential first-order character contrasts with the second order representations for the weaker system CK given by Wijesekera and Hilken. We have also shown how the canonical model construction of [FM97] for PLL follows from that for CS4 as a special case. Goldblatt [Gol76] proved a standard representation theorem for PLL algebras in terms of $\mathcal{J}$-frames, that only requires prime filters rather than pairs $(\Gamma, \Theta)$. However, Goldblatt's work explains $\bigcirc$ as a constructive modality of *necessity*, which is an altogether different way to look at $\bigcirc$.

The contribution of this paper regarding PLL lies in showing that the modality $\bigcirc$ of PLL is a constructive modality of *possibility*, in the sense that it can be obtained by adding to CS4 the axiom $A \to \Box A$. This is not the only way to derive PLL from CS4, but probably the most simple one so far proposed. Pfenning and Davies [PD01] give a full and faithful syntactic embedding PLL $\hookrightarrow$ CS4 that reads $\bigcirc A$ as $\Diamond\Box A$ and $A \to B$ as $\Box A \to B$. Both possibilities can be used to generate different semantics for PLL from that of CS4. The embedding discussed in this paper most closely reflects the notion of constraint models for PLL introduced in [FM97].

The second representation given in this paper explains the semantics of an element in the algebra in terms of provability in a natural deduction calculus. The representation theorem establishes a $\lambda$-calculus and Curry-Howard correspondence for CS4. In general, modal algebras can be extended to categorical models by adding information about proofs (replacing $\leq$ of the algebra by the collection of morphisms of the category), but this process is not trivial.

This extra information about proofs is crucial in applications of logic to model computational phenomena. While $\lambda$ terms (encodings of proofs in intuitionistic propositional logic) can be seen as semantic counterparts of functional programs, addition of modalities to intuitionistic propositional logic makes it possible to obtain more sophisticated semantics of programs reflecting such computational phenomena as, for example, non-termination, non-determinism, side effects, etc. [Mog91]. Information about proofs can also be necessary in other applications of logic to computer science, where not just the truth (or falsity) of a formula is important, but also the justification (proof) of the claimed truth (see e.g. [Men93,FMW97,Men00]). One example we are considering is the verification of protocols.

The results in this paper partially depend on having a natural deduction presentation of the logic following the standard Prawitz/Dummett pattern of logical connectives described by introduction and elimination rules. This is true for CS4 and for PLL, but not for weaker logics, for example for a modal logic where $\Box$ satisfies only the $K$-axiom. Thus, our main challenge is to extend this work on categorical semantics to other modal logics.

Next we would like to apply our techniques to constructive temporal logics. Another direction we would like to pursue is providing concrete mathematical models for CS4. Some such applications might be generated as generalisation of our previous work on constraint verification in PLL. Meanwhile we shall continue our work on applications of constructive modal logics to programming.

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## References

[BBdP98] N. Benton, G. Bierman, and V. de Paiva. Computational types from a logical perspective. *Journal of Functional Programming*, 8(2), 1998.

[BdP96] G. M. Bierman and V. de Paiva. Intuitionistic necessity revisited. Technical Report CSR-96-10, University of Birmingham, School of Computer Science, June 1996.

[Ben83] J. Benthem, van. *Modal logic and classical logic*. Bibliopolis, Naples, 1983.

[BMTS99] Z. Benaissa, E. Moggi, W. Taha, and T. Sheard. Logical modalities and multi-stage programming. In *Workshop on Intuitionistic Modal Logics and Application (IMLA'99)*, Satellite to FLoC'99, Trento, Italy, 6th July 1999. Proceedings available from `http://www.dcs.shef.ac.uk/~floc99im`.

[Cur57] H. B. Curry. *A Theory of Formal Deducibility*, volume 6 of *Notre Dame Mathematical Lectures*. Notre Dame, Indiana, second edition, 1957.

[DP96] R. Davies and F. Pfenning. A modal analysis of staged computation. In Guy Steele, Jr., editor, *Proc. of 23rd POPL*, pages 258–270. ACM Press, 1996.

[DPS97] J. Despeyroux, F. Pfenning, and C. Schürmann. Primitive recursion for higher-order abstract syntax. In P. de Groote and J. Roger Hindley, editors, *Proc. of TLCA'97*, pages 147–163. LNCS 242, Springer Verlag, 1997.

[Dum77] M. Dummett. *Elements of Intuitionism*. Clarendon Press, Oxford, 1977.

[Ewa86] W. B. Ewald. Intuitionistic tense and modal logic. *Journal of Symbolic Logic*, 51, 1986.

[Fin74] K. Fine. An incomplete logic containing S4. *Theoria*, 39:31 – 42, 1974.

[FM97] M. Fairtlough and M. Mendler. Propositional lax logic. *Information and Computation*, 137, 1997.

[FMW97] M. Fairtlough, M. Mendler, and M. Walton. First-order lax logic as a framework for constraint logic programming. Technical Report MIP-9714, University of Passau, July 1997. Postscript available through `http://www.dcs.shef.ac.uk/~michael`.

[FS80] G. Fischer-Servi. Semantics for a class of intuitionistic modal calculi. In M. L. Dalla Chiara, editor, *Italian Studies in the Philosophy of Science*, pages 59–72. Reidel, 1980.

[GdPR98] Neil Ghani, Valeria de Paiva, and Eike Ritter. Explicit Substitutions for Constructive Necessity. In *Proceedings ICALP'98*, 1998.

[GL96] J. Goubault-Larrecq. Logical foundations of eval/quote mechanisms, and the modal logic S4. Manuscript, 1996.

[Gol76] R. Goldblatt. Metamathematics of modal logic. *Reports on Mathematical Logic*, 6,7:31 – 42, 21 – 52, 1976.

[Gol81] R. Goldblatt. Grothendieck Topology as Geometric Modality. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 27:495–529, 1981.

[Gol93] R. Goldblatt. *Mathematics of Modality*. CSLI Lecture Notes No. 43. Center for the Study of Language and Information, Stanford University, 1993.

[Hil96] B. P. Hilken. Duality for intuitionistic modal algebras. *Journal of Pure and Applied Algebra*, 148:171 – 189, 2000.

[Joh82] P. T. Johnstone. *Stone Spaces*. Cambridge University Press, 1982.

[Kob97] S. Kobayashi. Monad as modality. *Theoretical Computer Science*, 175:29 – 74, 1997.

[LS85] J. Lambek and Ph. J. Scott. *Introduction to Higher-Order Categorical Logic*. Cambridge University Press, 1985.

[Mac81] D. S. Macnab. Modal operators on Heyting algebras. *Algebra Universalis*, 12:5–29, 1981.

[Men93] M. Mendler. *A Modal Logic for Handling Behavioural Constraints in Formal Hardware Verification*. PhD thesis, Department of Computer Science, University of Edinburgh, ECS-LFCS-93-255, March 1993.

[Men00] M. Mendler. Characterising combinational timing analyses in intuitionistic modal logic. *The Logic Journal of the IGPL*, 8(6):821–852, November 2000.

[Mog91] E. Moggi. Notions of computation and monads. *Information and Computation*, 93(1):55–92, July 1991.

[PD01] F. Pfenning and R. Davies. A judgemental reconstruction of modal logic. *Mathematical Structures in Computer Science*, 2001.

[Pra65] D. Prawitz. *Natural Deduction: A Proof-Theoretic Study*. Almqvist and Wiksell, 1965.

[PS86] G. Plotkin and C. Stirling. A framework for intuitionistic modal logics. In *Theoretical aspects of reasoning about knowledge*, Monterey, 1986.

[Sim94] A.K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, 1994.

[Tho74] S.K. Thomason. An incompleteness theorem in modal logic. *Theoria*, 40:30 – 34, 1974.

[TvD88] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics*, volume II. North-Holland, 1988.

[Wij90] D. Wijesekera. Constructive modal logic I. *Annals of Pure and Applied Logic*, 50:271–301, 1990.

[WZ95] F. Wolter and M. Zakharyaschev. Intuitionistic Modal Logics. In *Logic in Florence*, 1995.

[WZ97] F. Wolter and M. Zakharyaschev. Intuitionistic modal logics as fragments of classical bimodal logics. In E. Orlowska, editor, *Logic at Work*. Kluwer, 1997.


## Appendix

The full proofs of the main theorems are collected in this appendix.

**Theorem 1** CS4 *is sound and complete with respect to the class of models defined above, that is, for every set of formulae* $\Gamma$ *and formula* $A$, $\Gamma \vdash_{\mathsf{CS4}} A \Leftrightarrow \Gamma \models A$.

*Proof.* The soundness proof goes by induction on the length of a derivation of $A$ from $\Gamma$. We show that all axioms are valid and inference rules preserve validity. The intuitionistic part is not problematic. As for the modal axioms, $\Box K$ and $\Diamond K$ are valid just due to truth definitions and transitivity of $\leq$. $\Diamond T$ and $\Box T$ are valid because $R$ is reflexive.

$\Diamond 4$ and $\Box 4$ are valid because of transitivity of $R$. The latter also depends on transitivity of $\leq;R$, which follows from the frame condition $R;\leq\ \subseteq\ \leq;R$ and the fact that both $R$ and $\leq$ are transitive. In these proofs we also need that $\leq$ is hereditary, reflexive and transitive. The necessitation rule $Nec$ follows from the fact that if a formula is true in all models then it must be satisfied at all worlds in all models since every world induces a model. Completeness follows from the Stone Representation Theorem 3.

**Lemma 1 [Saturation Lemma]** *Let $a$ be element of the algebra, and $(\Gamma, \Theta)$ an $a$-consistent theory. Then $(\Gamma, \Theta)$ has a saturated $a$-consistent extension $(\Gamma^*, \Theta)$, such that $\Gamma^*$ is a prime filter and $\Gamma \subseteq \Gamma^*$.*

In the proof of the Saturation Lemma and the following proof of the Stone Representation Theorem we abbreviate consistency of a theory $(\Gamma, \Theta)$ as $\Gamma \not\leq \Diamond\Theta$, and *$a$-consistency* by $\Gamma \not\leq a + \Diamond\Theta$, remembering that only in the second case we permit the choice from $\Theta$ to be empty, in which case the disjunct $\Diamond\Theta$ disappears rather than being taken as $\Diamond\bot$.

*Proof.* We obtain $(\Gamma^*, \Theta)$ in the usual way by enumerating all elements of the algebra (therefore, we assume that this is possible)

$$c_0, c_1, \ldots, c_n, c_{n+1}, \ldots$$

with *infinite repetition* of every element, and by building up a hierarchy of $a$-consistent theories

$$(\Gamma_0, \Theta)\ \subseteq\ (\Gamma_1, \Theta)\ \subseteq\ \cdots\ \subseteq\ (\Gamma_n, \Theta)\ \subseteq\ (\Gamma_{n+1}, \Theta)\ \subseteq\ \cdots$$

starting with $\Gamma_0 =_{df} \Gamma$ and such that $\Gamma_{n+1} =_{df} (\Gamma_n \cup \{c_n\}$ if the theory $(\Gamma_n \cup \{c_n\}, \Theta)$ is $a$-consistent, otherwise $\Gamma_{n+1} =_{df} \Gamma_n$. Then, put $\Gamma^* =_{df} \bigcup_{n\in\omega} \Gamma_n$.

• First observe that $a$-consistency of $(\Gamma^*, \Theta)$ follows from $a$-consistency of each pair $(\Gamma_n, \Theta)$.

• We show that $\Gamma^*$ is upward closed. To this end suppose $b \in \Gamma$ and $b \leq c$. For some $n$, $b \in \Gamma_n$. Since our enumeration is with infinite repetition $c = c_m$ for some $m \geq n$. Then, we claim that $c_m \in \Gamma_{m+1}$. For otherwise, $(\Gamma_m \cup \{c_m\}, \Theta)$ would have to be $a$-inconsistent, or $(\Gamma_m \cup \{c_m\}) \leq a + \Diamond\Theta$. But since $\Gamma_n \subseteq \Gamma_m$, we also have $b \in \Gamma_m$ and $b \leq c$, which would imply $\Gamma_m \leq a + \Diamond\Theta$, contradicting $a$-consistency of $(\Gamma_m, \Theta)$. Hence, $c = c_m \in \Gamma_{m+1} \subseteq \Gamma^*$ as desired.

• It remains to be seen that $\Gamma^*$ is prime, i.e. if $c + c' \in \Gamma^*$ then $c \in \Gamma^*$ or $c' \in \Gamma^*$. Suppose $c + c' \in \Gamma^*$, i.e. $c + c' \in \Gamma_n$ for some $n$. Again, we can find indices $m \geq n$ and $m' \geq n$ such that $c = c_m$ and $c' = c'_{m'}$. Let $k$ be the maximum of both. We claim that $c_m \in \Gamma_{k+1}$ or $c_{m'} \in \Gamma_{k+1}$. Suppose otherwise, i.e. both $(\Gamma_k \cup \{c_m\}, \Theta)$ and $(\Gamma_k \cup \{c_{m'}, \Theta)$ are $a$-inconsistent. Thus, $(\Gamma_k^1 \cup \{c_m\}) \leq a + \Diamond\Theta^1$ and $(\Gamma_k^2 \cup \{c_{m'}\}) \leq a + \Diamond\Theta^2$, where $\Gamma_k^i$ and $\Theta^i$ are some subsets of propositions from $\Gamma_k$ and $\Theta$, respectively. Let $\Gamma_k^3 = \Gamma_k^1 \cup \Gamma_k^2$ and $\Theta^3 = \Theta^1 \cup \Theta^2$. Then, we can derive $(\Gamma_k^3 \cup \{c_m\}) \leq a + \Diamond\Theta^3$ and $(\Gamma_k^3 \cup \{c_{m'}\} \leq a + \Diamond\Theta^3$ From this, we get $(\Gamma_k^3 \cup \{c_m + c_{m'}\}) \leq a + \Diamond\Theta^3$ But since $c_m + c_{m'} = c + c' \in \Gamma_n \subseteq \Gamma_k$ by assumption, finally $\Gamma_k \leq a + \Diamond\Theta$ in contradiction to $a$-consistency of $(\Gamma_k, \Theta)$. This proves our claim that $c_m \in \Gamma_{k+1}$ or $c_{m'} \in \Gamma_{k+1}$, hence $c \in \Gamma^*$ or $c' \in \Gamma^*$.

**Theorem 3 [Representation for CS4]** *Let $\mathcal{A}$ be a* CS4*-modal algebra. Then the Stone representation of $\mathcal{A}$, $SR(\mathcal{A}) = (W^*, R^*, \leq^*, \models^*)$ is a Kripke model for* CS4*, where*

1. *$W^*$ is the set of all pairs $(\Gamma, \Theta)$ where $\Gamma \subseteq \mathcal{A}$ is a prime filter, and $\Theta \subseteq \mathcal{A}$ an arbitrary set of elements such that for all finite, nonempty, choices of elements $c_1, \ldots, c_n \in \Theta$, $\Diamond(c_1 + \cdots + c_n) \notin \Gamma$.*
2. *$(\Gamma, \Theta) \leq^* (\Gamma', \Theta')$ iff $\Gamma \subseteq \Gamma'$*
3. *$(\Gamma, \Theta) R^* (\Gamma', \Theta')$ iff $\forall a.\ \Box a \in \Gamma \Rightarrow a \in \Gamma'$ and $\Theta \subseteq \Theta'$.*
4. *For all $a \in \mathcal{A}$, $(\Gamma, \Theta) \models^* a$ iff $a \in \Gamma$.*

*Proof.* Consider $SR(\mathcal{A})$ as defined in the theorem. We must show that it satisfies the definition of a Kripke model for constructive S4.

It is easy to see that $R$ is reflexive and transitive (inequalities corresponding to the axioms T and 4 take care of that). Obviously, $\leq$ is reflexive, transitive and hereditary.

Finally, to verify the inclusion of $R^*; \leq^*$ in $\leq^*; R^*$ let the accessibilities

$$(\Gamma_1, \Theta_1) R^* (\Gamma_2, \Theta_2) \leq^* (\Gamma_3, \Theta_3)$$

in $W^*$ be given. Consider the pair $(\Gamma_1, \emptyset) \in W^*$. We are going to show that

$$(\Gamma_1, \Theta_1) \leq^* (\Gamma_1, \emptyset) R^* (\Gamma_3, \Theta_3).$$

Trivially, $(\Gamma_1, \Theta_1) \leq^* (\Gamma_1, \emptyset)$. Moreover, by definition of $R^*$ and $\leq^*$, $\Gamma_1^{\Box} \subseteq \Gamma_2 \subseteq \Gamma_3$, where $\Gamma^{\Box}$ is $\{a{:}\Box a \in \Gamma\}$. This proves $(\Gamma_1, \emptyset) R^* (\Gamma_3, \Theta_3)$, whence $R^*; \leq^* \subseteq \leq^*; R^*$ overall.

Now we need to show that $(\Gamma, \Theta) \models^* a$ satisfies the properties of a constructive modal validity relation.

If $a$ is of the form $b \times c$ or $b + c$, the proof is easy (for disjunction, we use the fact that $\Gamma$ is a prime filter). If $a$ is of the form $b \Rightarrow c$, the proof uses the fact that $SR(\mathcal{A})$ contains pairs $(\Gamma, \Theta)$ for *all* prime filters $\Gamma$.

Suppose $\Box a \in \Gamma$, $(\Gamma, \Theta) \leq^* (\Gamma_1, \Theta_1)$ and $(\Gamma_1, \Theta_1) R^* (\Gamma_2, \Theta_2)$. We want to show that $a \in \Gamma_2$. Since $(\Gamma, \Theta) \leq^* (\Gamma_1, \Theta_1)$, $\Box a \in \Gamma_1$. Since $(\Gamma_1, \Theta_1) R^* (\Gamma_2, \Theta_2)$, $a \in \Gamma_2$ as desired.

Suppose $\forall (\Gamma_1, \Theta_1)((\Gamma, \Theta) \leq^* (\Gamma_1, \Theta_1) \Rightarrow \forall (\Gamma_2, \Theta_2)((\Gamma_1, \Theta_1) R^* (\Gamma_2, \Theta_2) \Rightarrow a \in \Gamma_2))$. We want to show $\Box a \in \Gamma$. Consider the theory $(\Gamma^{\Box}, \emptyset)$. If it is $a$-consistent, then by the saturation lemma it has a saturated $a$-consistent extension $(\Gamma_2, \emptyset) \in W^*$. It is easy to check that $(\Gamma, \Theta) \leq^* (\Gamma, \emptyset) R^* (\Gamma_2, \emptyset)$ and $a \notin \Gamma_2$. This contradicts our assumption, hence $(\Gamma^{\Box}, \emptyset)$ is not $a$-consistent. For some $b_1, \ldots, b_m \in \Gamma^{\Box}$, $b_1 \times \ldots \times b_m \leq a$; by monotonicity of $\Box$ and the filter property, $\Box a \in \Gamma$.

Suppose $\Diamond a \in \Gamma$ and $(\Gamma, \Theta) \leq^* (\Gamma_1, \Theta_1)$, i.e. $\Gamma \subseteq \Gamma_1$. We want to show that there exists $(\Gamma_2, \Theta_2)$ such that $a \in \Gamma_2$ and $(\Gamma_1, \Theta_1) R^* (\Gamma_2, \Theta_2)$. Consider the pair $(\Gamma_1^{\Box} \cup a, \Theta_1)$, which must be consistent. Otherwise we would have, for some $\Box b_1, \ldots, \Box b_m \in \Gamma_1$, $b_1 \times \ldots \times b_m \times a \leq \Diamond \Theta_1$. Hence by monotonicity $\Diamond(b_1 \times \ldots \times b_m \times a) \leq \Diamond\Diamond\Theta_1$ and $\Diamond(b_1 \times \ldots \times b_m \times a) \leq \Diamond\Theta_1$ (by $\Diamond\Diamond a \leq \Diamond a$). On the other hand, $\Box b_1 \times \ldots \times \Box b_m \times \Diamond a \leq \Diamond(\Box b_1 \times \ldots \times \Box b_m \times a)$ by $\Box c \times \Diamond d \leq \Diamond(\Box c \times d)$ and $\Diamond(\Box b_1 \times \ldots \times \Box b_m \times a) \leq \Diamond(b_1 \times \ldots \times b_m \times a)$ by monotonicity of $\Diamond$, hence our assumption implies that $(\Gamma_1, \Theta_1)$ is inconsistent: $\Box b_1 \times \ldots \times \Box b_m \times \Diamond a \leq \Diamond\Theta_1$.

Since $(\Gamma_1^{\Box} \cup a, \Theta_1)$ is consistent, it has a saturated consistent extension $(\Gamma_2, \Theta_1)$ such that $a \in \Gamma_2$. It is easy to check that $(\Gamma_1, \Theta_1)R^*(\Gamma_2, \Theta_1)$.

Suppose $\Diamond a \notin \Gamma$. Consider the theory $(\Gamma, \{a\}) \in W^*$. It holds that $(\Gamma, \Theta) \leq^* (\Gamma, \{a\})$. Now let $(\Gamma_2, \Theta_2) \in W^*$ be any theory such that $(\Gamma, \{a\})R^*(\Gamma_2, \Theta_2)$. Then, by definition of $R^*$, $a \in \Theta_2$. But this implies $a \notin \Gamma_2$, for otherwise $\Diamond a \in \Gamma_2$ by the filter property and $a \leq \Diamond a$, which would contradict consistency of theory $(\Gamma_2, \Theta_2)$. This proves that for all $(\Gamma_2, \Theta_2)$ with $(\Gamma, \{a\})R^*(\Gamma_2, \Theta_2)$, we have $a \notin \Gamma_2$, as desired.

**Theorem 5** *Let $\mathcal{C}$ be any* CS4*-category. Then there is a canonical interpretation $[\![\_]\!]$ of* CS4 *in $\mathcal{C}$ such that*

- *a formula $A$ is mapped to an object $[\![A]\!]$ of $\mathcal{C}$;*
- *a natural deduction proof $\psi$ of $B$ using formulae $A_1, \ldots, A_n$ as hypotheses is mapped to a morphism $[\![\psi]\!]$ from $[\![A_1]\!] \times \cdots \times [\![A_n]\!]$ to $[\![B]\!]$;*
- *each two natural deduction proofs $\psi$ and $\psi$ of $B$ using formulae $A_1, \ldots, A_n$ as hypotheses which are equal (modulo normalisation of proofs) are mapped to the same morphism, in other words $[\![\phi]\!] = [\![\psi]\!]$.*

*Proof.* We use an induction over the structure of natural deduction proofs.

We describe the modality rules, starting with the $\Box_{\mathcal{I}}$-rule. Consider a proof $\psi$

$$\dfrac{\begin{matrix}\Gamma_1 & & \Gamma_n & [\Box A_1 \cdots \Box A_n] \\ \vdots\ \phi_1 & & \vdots\ \phi_n & \vdots\ \phi \\ \Box A_1 & \cdots & \Box A_n & B\end{matrix}}{\Box B}\ \Box_{\mathcal{I}}$$

By induction hypothesis, let $f_1, \ldots, f_n, f$ be the interpretation of $\phi_1, \ldots, \phi_n, \phi$ respectively. Then the interpretation of $\psi$ is

$$(\Box f) \circ m_{A_1,\ldots,A_n} \circ (\delta_{A_1} \times \cdots \delta_{A_n}) \circ (f_1 \times \cdots \times f_n)$$

where $m_{A_1,\ldots,A_n}$ is inductively defined by

$$m_{A_1,\ldots,A_{m-1},A_m} = m_{A_1 \times \cdots \times A_{m-1},A_m} \circ (m_{A_1,\ldots,A_{m-1}} \times \mathsf{Id}_{A_m})$$

The $\Box_{\mathcal{E}}$-rule is modelled by the morphism $\epsilon$.

Dually, the $\Diamond_{\mathcal{I}}$-rule is modelled by the morphism $\eta_A$. Last, we consider the $\Diamond_{\mathcal{E}}$-rule. Consider a proof $\theta$

$$\dfrac{\begin{matrix}\Gamma_1 & & \Gamma_n & \Gamma & [\Box A_1 \cdots \Box A_n\ B] \\ \vdots\ \phi_1 & & \vdots\ \phi_n & \vdots\ \phi & \vdots\ \psi \\ \Box A_1 & \cdots & \Box A_n & \Diamond B & \Diamond C\end{matrix}}{\Diamond C}\ \Diamond_{\mathcal{E}}$$

By induction hypothesis, let $f_1, \ldots, f_n, f, g$ be the interpretation of $\phi_1, \ldots, \phi_n, \phi, \psi$ respectively. Then the interpretation of $\theta$ is

$$\mu_C \circ \Diamond g \circ st_{A_1,\ldots,A_n,B} \circ (f_1 \times \cdots \times f_n \times f)$$

where the morphism $st_{A_1,\ldots,A_n,B}$ is inductively defined by

$$st_{A_1,A_2,\ldots,A_{n+1},B} = \mathsf{Id}_{A_1} \times st_{A_2,\ldots,A_{n+1},B}$$

We omit the routine verification that the desired equalities hold.

**Theorem 6**

*(i) There exists a* CS4*-category such that all morphisms are interpretations of natural deduction proofs.*
*(ii) If the interpretation of two natural deduction proofs is equal in all* CS4*-categories, then the two proofs are equal modulo proof-normalisation in natural deduction.*

*Proof.* We show both statements by constructing a CS4-category $\mathcal{C}$ out of the natural deduction proofs. We give here only the morphisms, and omit the verification that the required equalities between proofs hold. We write a natural deduction proof

$$\begin{array}{c} A \\ \vdots \\ \hline B \end{array}$$

as $A \vdash B$. The objects of the category are formulae, and a morphism between $A$ and $B$ is a proof of $B$ using $A$ as a hypothesis. The identity morphism is the basic axiom $A \vdash A$, and composition is given by cut. The bi-cartesian closed structure of $\mathcal{C}$ follows in the usual way from the conjunction, disjunction and implication in intuitionistic logic.

The $\Box$-modality gives rise to a monoidal comonad. The natural transformations $\delta_A \colon \Box A \longrightarrow \Box\Box A$ and $\epsilon_A \colon \Box A \longrightarrow A$ are given by the $\Box\mathcal{I}$- and $\Box\mathcal{E}$-rules applied to the identity axioms $\Box A \vdash \Box A$, respectively. The functor $\Box$ sends an object $A$ to $\Box A$ and a morphism $f \colon A \vdash B$ to the morphism $\Box f \colon \Box A \vdash \Box B$. This is obtained by applying the $\Box\mathcal{I}$-rule to the composition of $f$ and $\Box A \vdash A$. Dually, the $\Diamond$-modality gives rise to a monad on $\mathcal{C}$. The strength is given by the proof obtained thus

$$\dfrac{\dfrac{[\Box A] \qquad [B] \qquad \dfrac{\dfrac{[\Box A][B]}{\Box A \wedge B}\wedge I}{\Diamond(\Box A \wedge B)}\Diamond I}{\Diamond(\Box A \wedge B)}\Diamond E}{\Box A \wedge \Diamond B \to \Diamond(\Box A \wedge B)} \to I$$

This category $\mathcal{C}$ shows now the claim: Assume an equation between proofs holds in all CS4-categories. Because $\mathcal{C}$ is a CS4-category, it holds in $\mathcal{C}$. But equality in $\mathcal{C}$ is equality between natural deduction proofs, hence the two proofs are equal.