Conjectures No More?

Consensus Forming on the Proof of the Poincaré and Geometrization Conjectures

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Have the Poincaré Conjecture and the Thurston Geometrization Conjecture been proved?

This question has been on the minds of mathematicians for more than three years, ever since Grigory Perelman posted his now-famous papers on the Web. In midsummer 2006, as the International Congress of Mathematicians in Madrid approaches and speculation about the Fields Medals is buzzing, some experts who had been making cautious statements for the past three years sound increasingly confident that the conjectures are finally yielding. In particular, many believe the Poincaré Conjecture is now a bona fide theorem. The picture is slightly less clear for the Geometrization Conjecture, but there is much optimism that this result will soon be established as well.

Of the Dollars and the Glory

For mathematicians, the million dollars that the Clay Mathematics Institute (CMI) has offered for the solution of the Poincaré Conjecture is mere icing on the cake. The real prize is the glory of settling a question that has tantalized mathematicians for more than a century. The statement dates back to 1904, when Henri Poincaré conjectured that it is the property of being simply connected that topologically distinguishes the three-sphere from other compact three-manifolds. Since that time there have been many incorrect attempts to prove the Poincaré Conjecture, some of them by such well-known mathematicians as Edwin Moise, Christos Papakyriakopolous, Valentin Poenaru, and Colin Rourke. A recent incorrect proof, by Martin Dunwoody of Southampton University, came in 2002, about six months before Perelman posted his first paper on the subject. Almost as soon as news stories started to appear about Dunwoody's proof (an April 2002 article in the *New York Times* carried the headline "UK Math Wiz May Have Solved Problem"), the proof fell apart.

In fact, there have been so many wrong proofs of the Poincaré Conjecture that John Stallings of the University of California, Berkeley, has posted on his webpage a paper he wrote in 1966 called "How not to prove the Poincaré Conjecture", which describes his own failed attack, as a warning to others who might hit upon the same idea. One characteristic that most of the failed attempts share is a reliance on topological arguments. But, noted John Morgan of Columbia University, "It seems like this problem does not succumb to that type of argument." Rather, he said, one needs tools from outside topology, from geometry and analysis, to tackle this topological question.

In contrast to the multiple failed attempts on Poincaré, it appears that, before Perelman's work appeared, no one had seriously claimed to be able to prove the full Thurston Geometrization Conjecture. In fact, this is a much deeper and more far-reaching statement than the Poincaré Conjecture and includes Poincaré as a special case. First proposed in the 1970s by William Thurston, who is now at Cornell University, the Geometrization Conjecture provides a way to classify all three-manifolds. Thurston's great insight was to see how geometry could be used to understand the topology of

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three-manifolds. The Geometrization Conjecture states that any three-manifold can be split into pieces in an essentially unique way and that each of these pieces carries a geometric structure given by one of eight model geometries. The conjecture was not wide open before the work of Perelman; it had been established in many cases. Thurston himself proved the conjecture for manifolds that are sufficiently large. Several mathematicians contributed to establishing the full conjecture for six of the eight geometries. The two remaining geometries are the spherical and hyperbolic ones, where the metrics have constant positive and constant negative curvature, respectively. The Poincaré Conjecture comes under the case of metrics of constant positive curvature. (An excellent historical account is [Milnor].)

Against this background, mathematicians were naturally skeptical when Perelman posted his articles on the arXiv, the first in November 2002, the second in March 2003, and the third in July 2003 [Perelman1–3]. Nevertheless, his efforts were from the outset taken quite seriously. One reason is that Perelman is a well-regarded mathematician who had already made distinguished contributions to geometric analysis. He was an invited speaker at the 1994 ICM in Zurich, where he gave a lecture in the geometry section about spaces with curvature bounded below. In 1996 he was awarded one of the ten prizes given to outstanding young mathematicians every four years by the European Mathematical Society (Perelman refused to accept that prize).

Another reason Perelman's work was taken seriously is that it fits into a well-known program to use the Ricci flow to prove the Geometrization Conjecture. The originator of this program is Richard Hamilton, now at Columbia University, who will be a plenary speaker at the 2006 ICM in Madrid. The abstract for Hamilton's talk says that the Ricci flow program was developed by him and Shing-Tung Yau of Harvard University. The idea, first described in a 1982 paper by Hamilton [Hamilton], is to use the Ricci flow, a partial differential equation that is a nonlinear version of the heat equation, to homogenize the geometry of three-manifolds to show that they fit into Thurston's classification. It was generally believed that, philosophically, Hamilton's approach ought to work. This belief strengthened as Hamilton and others worked out much of the analysis that was needed. The toughest obstacle was handling the singularities that could develop in the Ricci flow. It was this obstacle that Perelman, by introducing deep new ideas in geometric analysis, was able to overcome to such spectacular effect. (An excellent expository account about the Ricci flow is [Anderson].)

Poring over Perelman

In the spring of 2003, after his first two papers had appeared on the Web, Perelman gave lectures at several universities in the U.S., including Columbia University, the Massachusetts Institute of Technology, and Princeton University, as well as a series of lectures at Stony Brook University. Soon thereafter he returned to his home base in St. Petersburg, and he has given only a very few lectures on the subject since then. He answered mathematical questions by email, but some mathematicians report that after a while he stopped even that form of communication. It is not clear what Perelman has made of the acclaim that has surrounded his achievements. Many articles about his work have come out in the popular press, though it appears that he never consented to be interviewed by reporters.

As mathematicians began to read the papers carefully, they found them tough going. "Perelman's articles are remarkably carefully written if one takes into account how much new ground he breaks in a relatively few number of pages," explained John Lott of the University of Michigan. "However, they are not written in such a way that one can just sit down and quickly decide whether his arguments are complete." Morgan remarked that Perelman omits certain technicalities that turned out to be standard, but rather tricky, to work out in detail. And, Morgan said, sometimes arguments are justified by a statement that they are analogous to arguments presented earlier, but it is not always clear exactly how the earlier arguments can be adapted. On top of these difficulties, there are some outright mistakes in the paper, though none has proven serious. It appears that Perelman never submitted his articles to any journal. Had he done so, they probably would not have been accepted without substantial revisions.

Soon after Perelman's papers appeared on the Web, mathematicians undertook efforts to understand and verify them. In June 2003 Lott, together with Bruce Kleiner, who is now at Yale University, started a webpage in which they presented notes about Perelman's work as they went carefully through his papers. In late 2003 the American Institute of Mathematics in Palo Alto and the Mathematical Sciences Research Institute in Berkeley jointly sponsored a workshop on Perelman's first article; another workshop, about Perelman's second article, was held in the summer of 2004 at Princeton University. The Clay Institute, which has an obvious interest in knowing whether Perelman's work is correct, provided funding for the Princeton workshop and also sponsored a month-long summer school held at MSRI in the summer of 2005. In addition Clay provided some support to Kleiner and Lott, who continued to add to and post their notes on the Web, as well as Morgan and

Gang Tian of Princeton University, who are collaborating on a book about Perelman's work on the Poincaré Conjecture.

In June 2005 Gérard Besson of the University of Grenoble presented a Bourbaki lecture on the work of Perelman; the lecture will appear in the *Astérisque* series in September 2006. In the fall of 2005 Xi-Ping Zhu of Zhongshan University gave a six-month series of lectures at Harvard University, describing the content of a paper that he has written with Huai-Dong Cao of Lehigh University and that appeared in the June 2006 issue of the *Asian Journal of Mathematics*. There have been other workshops and summer schools on the subject, not to mention the many lectures given in mathematics departments and at conferences. Study groups were formed to go through Perelman's papers in several countries, including China, France, Germany, and the United States.

While it seems that Perelman's papers were never refereed in the traditional sense, they have been subjected to extraordinary scrutiny in the three and a half years since their posting on the Web. The simple passage of time without anyone finding a serious problem in his work has, at least for many nonexperts, led to a conviction that it must be correct. For example, Koji Fujiwara of Tohoku University is not an expert in this area, but he believes Perelman's work must be right, for two reasons. "If there were something philosophically wrong, so that the approach could not work, after three years someone would have found the philosophical problem," he reasoned. And second, Fujiwara said, Perelman is a well-known expert on Ricci curvature, and his previous papers have been reliable and have not been found to contain mistakes. Of course, this kind of confidence is the privilege of the nonexpert. Experts have to work much harder.

Filling in the Details

"They should give [Perelman] a Fields Medal for the Poincaré Conjecture," declared John Morgan in an interview in May 2006. "I believe the argument is correct, as do, I think, all who have looked at it seriously.…This is clearly the most exciting thing that has happened in mathematics in the last four years," since the previous batch of Fields Medals were awarded. Morgan said that the book he is writing with Tian, which is to appear in early 2007, will provide a full exposition of the proof of the Poincaré Conjecture à la Perelman.[1] Morgan said that he has no doubts that Perelman can also prove the Geometrization Conjecture, but Morgan has not personally gone through that proof in detail, as he has done with Poincaré.

Indeed, many mathematicians express more confidence in the proof of Poincaré than in the proof of Geometrization. Perelman himself provided a shortcut to proving Poincaré, and there is a more extensive body of material that is needed for the proof of the full Geometrization Conjecture. Some believe that the best way to ensure that Poincaré has really been proved is to verify the proof of Geometrization. So what is the status of the proof of the Geometrization Conjecture?

In May 2006 Kleiner and Lott posted on the arXiv an article titled "Notes on Perelman's papers". They say that their article, along with a 2005 paper by T. Shioya and T. Yamaguchi, provides details for Perelman's arguments for the Geometrization Conjecture. Lott cautioned that Perelman's work has to be further examined by the mathematical community before there can be any universally accepted verdict. The Kleiner-Lott paper is based on the set of notes they began posting on the Web in the summer of 2003. In the three years over which they developed the notes and made them public, Kleiner and Lott received corrections and comments from many mathematicians. They plan to submit their paper to a journal.

In late April 2006 the *Asian Journal of Mathematics* announced on its website the upcoming publication of the paper by Cao and Zhu, "A complete proof of the Poincaré and Geometrization Conjectures—Application of the Hamilton-Perelman theory of the Ricci flow". The announcement included the paper's abstract, which states in full: "In this paper, we give a complete proof of the Poincaré and the geometrization conjectures. This work depends on the accumulative works of many geometric analysts in the past thirty years. This proof should be considered as the crowning achievement of the Hamilton-Perelman theory of Ricci flow." The 330-page paper appeared in print in the June 2006 issue of the *Asian Journal*. The issue has not been made available electronically on the journal's website and is available only as a printed paper publication. The Cao-Zhu article did not circulate as a preprint, but the work presented there was described in Zhu's lectures at Harvard during the 2005–2006 academic year.

Some have noted the short amount of time between the submission date for the Cao-Zhu paper, December 12, 2005, and the date when it was accepted for publication, April 16, 2006, and wondered whether such an important paper of over 300 pages could have been refereed in a serious way. In a May 2006 interview, Yau, who is one of the editors-in-chief of the *Asian Journal*, said that the manuscript had been around for a year, but "we have been very careful not to distribute it, to make sure everything is right before it is in print." Asked

[1] *On July 25, 2006, Morgan and Tian posted on the arXiv a 473-page manuscript* Ricci Flow and the Poincaré Conjecture, `http://arXiv.org/abs/math/0607607`.

whether the paper had been refereed in the usual way, Yau said that it had and remarked that the *Asian Journal* has very high standards.

Although not enough time has yet passed for the Cao-Zhu paper to have been subjected to much scrutiny by the mathematical community, the paper became widely known because of coverage about it in the Chinese press during June 2006. "Chinese Mathematicians Solve Global Puzzle" read the headline of an article that appeared on the Xinhua news service on June 3, 2006. The article's first sentence stated: "Two Chinese mathematicians have put the final pieces together in the solution to a puzzle that has perplexed scientists around the globe for more than a century." Cao characterized the barrage of media attention to his work with Zhu as "overwhelming". Some of the news articles were translated into English and posted on the Web. In those articles, the achievements of Cao and Zhu, both of whom are Chinese, are emphasized, while the achievements of Perelman are mentioned in a less prominent way. In one story from the Xinhua news agency, which appeared on June 21, 2006, the name of Perelman does not even appear. The coverage began after Yau held a news conference in Beijing on June 3, 2006, in which he announced the work of Cao and Zhu. Yau said that he was misquoted in some of the media accounts and does not endorse what is said there. On June 20, 2006, he presented a public lecture on the subject at the Morningside Center of Mathematics at the Chinese Academy of Sciences in Beijing, the slides of which are available on the center's website at `http://www.mcm.ac.cn/Active/yau_new.pdf`.

Doling Out the Prizes

With so many players, who will get credit for the proof of these monumental results? This is not a simple question. Often in mathematics credit for a result goes to the person who came up with the decisive ideas that really made the proof work, even if that person never wrote up a complete proof. As a historical example, Robion Kirby of the University of California, Berkeley, pointed to Thurston's orbifold theorem. Thurston described this result in a 1982 article in the *Bulletin of the AMS* [Thurston], using an argument that Kirby characterized as "definitely sketchy". The orbifold theorem covers the Geometrization Conjecture when there is a discrete group acting on the three-manifold with fixed points, and this covers a lot of cases, although not the Poincaré Conjecture. After more than a dozen years had passed without a complete proof, Kirby added the orbifold theorem to his well-known problem list in topology and declared it to be an open question. Two different groups of mathematicians independently produced complete proofs of the theorem (one group was Daryl Cooper, Craig Hodgson, and Steven Kerckhoff, and the other was Michel Boileau, Bernhard Leeb, and Joan Porti). "This was a lot of work, some pieces of Thurston's sketch were improved, and the community honors their work," Kirby said. "But it is acknowledged that this is Thurston's theorem."

The mathematical world is waiting to find out whether Perelman will receive a Fields Medal for his work. The traditional rule followed by the Fields Medal committees is that a recipient must not be over forty in the year in which the medal is given. Perelman turned forty in June 2006. Some believe that, even disregarding the Poincaré and Geometrization Conjectures, Perelman may have done enough to deserve a Fields Medal. "What Perelman's work says about singularity development in Ricci flow is an enormous advance that in itself would make him a serious candidate for a Fields Medal," Morgan said.

The Poincaré Conjecture is one of the CMI's seven Millennium Prize Problems, which were announced in 2000. Until Perelman's work, there were no serious solutions proposed to any of the problems, so no prizes have yet been given. The prize rules state that a proposed solution must be published in "a refereed journal of worldwide repute" and that this published solution must be out for two years before the CMI will consider awarding a prize. The rules are worded in such a way that the person considered for the prize need not be the author of the published solution, noted James Carlson, president of the Clay Mathematics Institute. "The fact that Perelman pursued an unorthodox route and posted [his papers] on the arXiv and did not submit them to a journal is not itself an obstacle" to him receiving the prize, Carlson said. At the appropriate time, he said, the Clay Institute will consider all the available materials and make a judgment about whether the proof of Poincaré is correct. Only after that will it consider giving the prize. One question the Clay Institute faces is whether to give the prize solely to Perelman or to include others as joint recipients—perhaps Hamilton? Carlson said it would be premature for him to speculate on such possibilities.

But no doubt the mathematical world will continue to speculate and to discuss the extraordinary saga of Perelman's work. One thing is clear: Perelman has made an enormous contribution to the field. Many of the things he did—not submitting his work to a journal, not lecturing much, completely shunning the limelight—are not easy to understand. "Perelman is a very talented and unusual individual, and this is the route that he has chosen," Carlson remarked. "I think the most important thing is that he wrote those three papers and he posted them on the arXiv, and that has given mathematicians a great gift and lots of new ideas and things to think about."

References

[Anderson] MICHAEL ANDERSON, Geometrization of 3-manifolds via the Ricci flow, *Notices* **2** (2004), 184–93.

[Besson] GÉRARD BESSON, Preuve de le conjecture de Poincaré en déformant le métrique par la courbure de Ricci, d'après G. Perel'man, *Astérisque* **307**, Société Mathématique de France (to appear September 2006).

[Cao-Zhu] HUAI-DONG CAO and XI-PING ZHU, A complete proof of the Poincaré and Geometrization Conjectures—Application of the Hamilton-Perelman Theory of the Ricci flow, *Asian J. Math.* **10** (2006), 145–492.

[Kleiner-Lott] BRUCE KLEINER and JOHN LOTT, Notes on Perelman's papers, arXiv:math.DG/0605667, 25 May 2006.

[Hamilton] R. S. HAMILTON, Three-manifolds with positive Ricci curvature, *J. Differential Geom.* **17** (1982), 695–729.

[Milnor] JOHN MILNOR, Towards the Poincaré Conjecture and the classification of 3-manifolds, *Notices* **10** (2003), 1226–33.

[Perelman1] G. PERELMAN, The entropy formula for the Ricci flow and its geometric applications, `http://arXiv.org/abs/math.DG/0211159`, 11 November 2002.

[Perelman2] ____ , Ricci flow with surgery on three-manifolds, `http://arxiv.org/abs/math.DG/0303109`, 10 March 2003.

[Perelman3] ____ , Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, `http://arxiv.org/abs/math.DG/0307245`, 17 July 2003.

[Shioya-Yamaguchi] T. SHIOYA and T. YAMAGUCHI, Volume-collapsed three-manifolds with a lower curvature bound, *Math. Ann.* **333** (2005), 131–55.

[Thurston] WILLIAM P. THURSTON, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc. (N. S.)* **6** (1982), 357–81.