Euler's Proof That $\mathbf{1+2+3+\cdots = -\frac{1}{12}}$

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*Just as a k-**coloring** of a set S is a function $f\colon S \to k$, where k stands for the k-element set, a k-**pointing** of S is a function $f\colon k \to S$. In other words, it is a way of labelling k not necessarily distinct points of S by the numbers $1, \ldots, k$. Let us call a set equipped with a k-pointing a k-**pointed set**. Warning: a k-pointed set does not need to have k points! The most familiar case is $k = 1$, where people speak of a **pointed set**. This is just a set with a chosen point, often called the **basepoint**. I have also seen people speak of a **bipointed set** in the case $k = 2$.*

Let $P(k)$ be the structure type of 'being a k-pointed totally ordered finite set', so that $P(k)_n$ is the set of ways of totally ordering and k-pointing the n-element set.

1. Compute the cardinality $|P(k)_n|$.

2. Write a formula for the generating function $|P(k)|(z)$ as a formal power series.

3. Show that as formal power series,

$$|P(0)|(z) = \frac{1}{1-z}.$$

*Let A, A^* be the annihilation and creation operators on structure types, respectively. Thus, to put an $A\Psi$-structure on the finite set S is to put a Ψ-structure on $S + 1$, and to put an $A^*\Psi$-structure on S is to choose a point $x \in S$ and put a Ψ-structure on $S - \{x\}$. Copying the usual formula from the quantum harmonic oscillator, define the **number operator** N by*

$$N\Psi = A^*A\Psi.$$

4. What does it mean to put an $N\Psi$-structure on the set S? Give the simplest description you can.

5. Show that

$$NP(k) \cong P(k+1).$$

6. By decategorifying part 5, prove the following equation between formal power series:

$$z\frac{d}{dz}|P(k)|(z) = |P(k+1)|(z)$$

7. Using part 3 and part 6, show that as formal power series,

$$|P(1)|(z) = \frac{z}{(1-z)^2}.$$

Now let's do some dirty tricks invented by Euler!

8. By using the equation in part 7 to evaluate the formal power series $|P(1)|(z)$ at $z = -1$, even though the series diverges, 'show' that

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}.$$

No, I haven't gone insane! This equation can be understood using the concept of 'Abel summation'. The **Abel sum** *of a series is defined by*

$$\mathrm{A}\sum_n a_n := \lim_{t\uparrow 1} \sum_n t^n a_n$$

whenever $\sum_n t^n a_n$ *converges for* $t \in (0,1)$ *and the the limit as* $t \uparrow 1$ *exists. If the series* $\sum_n a_n$ *converges absolutely, its Abel sum exists and equals its usual sum. However, the Abel sum may exist when the usual sum diverges! Abel summation can be very useful, even though Abel himself wrote:*

"The divergent series are the invention of the devil,
and it is a shame to base on them any demonstration whatsoever."

9. Using the equation in part 7, prove rigorously that

$$\mathrm{A}\sum_{n=1}^{\infty}(-1)^{n+1}\, n = \frac{1}{4}.$$

Euler then went on to compute the sum of all natural numbers, as follows. First, he considered what is now called the **Riemann zeta function***:*

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \cdots$$

where the series converges for $\mathrm{Re}(s) > 1$*. Multiplying by* 2^{-s}*, he obtained*

$$2^{-s}\zeta(s) = 2^{-s} + 4^{-s} + 6^{-s} + 8^{-s} + \cdots.$$

Subtracting twice the second equation from the first one, he got

$$(1 - 2\cdot 2^{-s})\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots$$

or in other words

$$(1 - 2^{1-s})(1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \cdots) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots$$

So far so good... but then he evaluated both sides at $s = -1$*, where both series diverge! He got:*

$$-3(1 + 2 + 3 + 4 + \cdots) = 1 - 2 + 3 - 4 + \cdots$$

but since he already knew the right-hand side equals $1/4$*, he concluded that*

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

Again, this may seem like mindless noodling with divergent series, but there is a way to extract a perfectly rigorous argument from these ideas. Using this, one can prove that the Riemann zeta admits a unique analytic continuation to $\mathbb{C} - \{1\}$*, and that this analytic continuation has*

$$\zeta(-1) = -\frac{1}{12}.$$

Unfortunately, doing all this here would make your homework too long! So, instead, just do this:

10. Using parts 6 and 7, work out the generating function of the structure type $P(2)$. Use this to rigorously determine the Abel sum of

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots$$

and then use Euler's nonrigorous argument to compute

$$\zeta(-2) = 1^2 + 2^2 + 3^2 + 4^2 + \cdots$$