Chief Reader Report on Student Responses:

2017 AP® Calculus AB and Calculus BC Free-Response Questions

Number of Readers (Calculus AB/Calculus BC): 971

Calculus AB

- Number of Students Scored: 316,099
- Score Distribution

Exam Score	N	%At
5	59,250	18.7
4	56,775	18.0
3	65,851	20.8
2	69,631	22.0
1	64,592	20.4

- Global Mean: 2.93

Calculus BC

- Number of Students Scored: 132,514
- Score Distribution

Exam Score	N	%At
5	56,422	42.6
4	23,987	18.1
3	26,341	19.9
2	18,694	14.1
1	7,070	5.3

- Global Mean: 3.78

Calculus BC Calculus AB Subscore

- Number of Students Scored: 132,505
- Score Distribution

Exam Score	N	%At
5	64,197	48.4
4	29,862	22.5
3	18,684	14.1
2	13,277	10.0
1	6,485	4.9

- Global Mean: 4.00

The following comments on the 2017 free-response questions for AP® Calculus AB and Calculus BC were written by the Chief Reader, Stephen Davis of Davidson College. They give an overview of each free-response question and of how students performed on the question, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student preparation in these areas are also provided. Teachers are encouraged to attend a College Board workshop to learn strategies for improving student performance in specific areas.

Question AB1/BC1 **Topic:** Modeling Volume/Related Rates-Tabular/Analytic
Max. Points: 9 **Mean Score: AB1:** 3.32; **BC1:** 5.00

What were responses expected to demonstrate in their response to this question?

In this problem students were presented with a tank that has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by a continuous and decreasing function A, where $A(h)$ is measured in square feet. Values of $A(h)$ for heights $h = 0,\ 2,\ 5,$ and 10 are supplied in a table. In part (a) students were asked to approximate the volume of the tank using a left Riemann sum and indicate the units of measure. Students needed to respond by incorporating data from the table in a left Riemann sum expression approximating $\int_0^{10} A(h)\,dh$ using the subintervals $[0, 2],\ [2, 5],$ and $[5, 10].$ [LO 3.2B/EK 3.2B2] In part (b) students needed to explain that a left Riemann sum approximation for the definite integral of a continuous, decreasing function overestimates the value of the integral. [LO 3.2B/EK 3.2B2] In part (c) the function f given by $f(h) = \frac{50.3}{e^{0.2h} + h}$ is presented as a model for the area, in square feet, of the horizontal cross section at height h feet. Students were asked to find the volume of the tank using this model, again indicating units of measure. Using the model f for cross-sectional areas of the tank, students needed to express the volume of the tank as $\int_0^{10} f(h)\,dh$ and use the graphing calculator to produce a numeric value for this integral. [LO 3.4D/EK 3.4D2] In part (d) water is pumped into the tank so that the water's height is increasing at the rate of 0.26 foot per minute at the instant when the height of the water is 5 feet. Students were asked to use the model from part (c) to find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet, again indicating units of measure. Students needed to realize that the volume of water in the tank, as a function of its height h, is given by $V(h) = \int_0^h f(x)\,dx$ and then use the Fundamental Theorem of Calculus to find that the rate of change of the volume of water with respect to its height is given by $V'(h) = f(h).$ Then, using the chain rule for derivatives, students needed to relate the rates of change of volume with respect to time and height and the rate of change of the water's height with respect to time. Information in the problem suffices to be able to find these rates when the water's height is 5 feet. [LO 2.3C/EK 2.3C2, LO 3.3A/EK 3.3A2] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

How well did the response address the course content related to this question? How well did the responses integrate the skills required on this question?

Students did reasonably well at supplying units for parts (a) and (c); both are volume measurements. Students who had correct units in parts (a) and (c) sometimes supplied no units or incorrect units for part (d). In part (a) most students were able to construct a left Riemann sum, although some students confused left with right. Most students were able to recognize volume as the integral of cross-sectional area and correctly completed part (c). In part (d) many students were not able to get the correct start on this part, which required recognizing that the volume V of the water in the tank at height h is given by $V(h) = \int_0^h f(x)\,dx$ or by starting with the correct relationship for rates of change, $\frac{dV}{dt} = f(h) \cdot \frac{dh}{dt}.$

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some students confused left with right, while some other students constructed their own subintervals so that $[0, 10]$ was divided into three subintervals of equal lengths.	• $\text{Volume} = \int_0^{10} A(h)\,dh \approx (2-0)\cdot A(0) + (5-2)\cdot A(2) + (10-5)\cdot A(5)$
• In part (b) some students used vague language or insufficient or speculative reasoning to explain why the approximation in part (a) is an overestimate. Students needed to focus upon the given information that the function A is decreasing.	• Because A is decreasing, a left Riemann sum overestimates the value of $\int_0^{10} A(h)\,dh$.
• In part (c) some students used an incorrect integrand (e.g., $f(h)^2$ or $\pi f(h)^2$) in an integral to compute the volume of the tank.	• Using the model that $f(h)$ describes the area, in square feet, of a horizontal cross section at height h feet, the volume of the tank is $\int_0^{10} f(h)\,dh$ cubic feet.
• In part (d) students struggled to find a correct expression for the volume of water when the height of the water is h feet, e.g., stating that the volume is $h \cdot f(h)$, or $f(h)$, or the constant $\int_0^{10} f(h)\,dh$.	• $V(h) = \int_0^h f(x)\,dx$
• In part (d) students made errors in differentiating their water volume expression with respect to time.	• Using the Fundamental Theorem of Calculus and the chain rule, $\frac{dV}{dt} = f(h)\cdot\frac{dh}{dt}$.

Based on your experience at the AP® Reading with student responses, what advice would you offer to teachers to help them improve the student performance on the exam?

Students should experience a variety of problems in which units are required: area, volume, and various rates of change.

Teachers can work with students to be wary of making unwarranted assumptions. For example, in part (b) some students treated "A is decreasing" to be equivalent to "A' is negative." The latter is not supported by the information given in the problem (see the MPAC reasoning with definitions and theorems) and was not an applicable reason that the left Riemann sum is an overestimate for the volume of the tank.

Teachers should work with students to set up their own functions that describe the behavior of some object. This is particularly true when those functions are given by an integral with a variable limit of integration as in part (d). Teachers should then ask students to verify particular things about those functions, which include analyzing first and second derivatives of the functions that the students have set up.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

In general, review of previous Chief Reader reports (known previously as the *Student Performance Q&A*), as well as applicable sections of the *AP Calculus AB and AP Calculus BC Course and Exam Description* (particularly the Mathematical Practices for AP Calculus) will serve teachers well. Also, previously released exam questions and participation on the Online Teacher Community are of high value for practice and for advice.

Question AB2 **Topic:** Modeling Rates

Max. Points: 9 **Mean Score:** 4.01

What were responses expected to demonstrate in their response to this question?

The context for this problem is the removal and restocking of bananas on a display table in a grocery store during a 12-hour period. Initially, there are 50 pounds of bananas on the display table. The rate at which customers remove bananas from the table is modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \quad \text{for } 0 < t \le 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. Three hours after the store opens, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \le 12,$$

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened. In part (a) students were asked how many pounds of bananas are removed from the display table during the first 2 hours the store is open. Students needed to realize that the amount of bananas removed from the table during a time interval is found by integrating the rate at which bananas are removed across the time interval. Thus, students needed to express this amount as $\int_0^2 f(t)\,dt$ and use the graphing calculator to produce a numeric value for this integral. [LO 3.4E/EK 3.4E1] In part (b) students were asked to find $f'(7)$ and, using correct units, explain the meaning of $f'(7)$ in the context of the problem. Students were expected to use the graphing calculator to evaluate the derivative, and explain that the rate at which bananas are being removed from the display table 7 hours after the store has been open is decreasing by 8.120 pounds per hour per hour. [LO 2.3A/EK 2.3A1, LO 2.3D/EK 2.3D1] In part (c) students were asked to determine, with reason, whether the number of pounds of bananas on the display table is increasing or decreasing at time $t = 5$. This can be determined from the sign of the difference between the rate at which bananas are added to the table and the rate at which they are removed from the table. Thus, students needed to evaluate the difference $g(5) - f(5)$ on the graphing calculator and report that the number of pounds of bananas on the display table is decreasing because this value is negative. [LO 2.2A/EK 2.2A1] In part (d) students were asked how many pounds of bananas are on the display table at time $t = 8$. The number of pounds of bananas added to the table by time $t = 8$ is given by $\int_3^8 g(t)\,dt$, and the number of pounds of bananas removed from the table by that time is given by $\int_0^8 f(t)\,dt$. Thus, using that there were initially 50 pounds of bananas on the table, the expression $50 + \int_3^8 g(t)\,dt - \int_0^8 f(t)\,dt$ gives the number of pounds of bananas on the table at time $t = 8$. Students needed to evaluate this expression using the numeric integration capability of the graphing calculator. [LO 3.4E/EK 3.4E1] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

How well did the response address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) some students misinterpreted the function $f(t)$, leading them to respond with an answer of $f(2)$ or some similar variant.

In part (b) some students had difficulty with the specific context of the problem, inappropriately invoking the vocabulary (“velocity,” “acceleration”) of rectilinear motion, or writing about “slope” in attempting to explain the meaning of $f'(7)$. Students also had difficulty determining the correct units for $f'(7)$, often giving the units for $f(t)$, pounds per hour.

In part (c) some students read the question superficially, focusing either on the phrases "number of pounds of bananas" or "increasing or decreasing," and attempted to answer the question by considering definite integrals or derivatives, respectively, instead of, or in addition to, the function values $f(5)$ and $g(5)$.

Generally, students did well in part (d), although those students that were incorrectly interpreting the original functions in part (a) often carried that incorrect interpretation into part (d).

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) students had difficulty dealing with $f(t)$ as a <u>rate</u> at which pounds of bananas were removed so responded with a value for $f(2)$. Others erroneously incorporated the number of pounds of bananas on the display table when the store opened.	• The number of pounds of bananas removed from the table during the first two hours after the store opened is given by $\int_0^2 f(t)\,dt$.
• In part (b) students made errors calculating a symbolic derivative of $f(t)$, whereas they could use the graphing calculator to evaluate $f'(7)$ numerically.	• $f'(7) = -8.120$
• For the explanation in part (b), students again had difficulty with the meaning of $f(t)$, or failed to tie the meaning to the time $t = 7$.	• After the store has been open 7 hours, the rate at which bananas are being removed from the display table is decreasing by 8.120 pounds per hour per hour.
• In part (c) many students assumed that a question about "increasing or decreasing" must be answered using a derivative, either ignoring or unable to deal with $f(t)$ and $g(t)$ given as <u>rates</u>.	• Whether the number of pounds of bananas on the display table is increasing or decreasing at time $t = 5$ can be determined by comparing the rate of removal of the bananas, $f(5)$ pounds per hour, against the rate at which bananas are added to the table, $g(5)$ pounds per hour. Because $f(5) > g(5)$, the number of pounds of bananas on the display table is decreasing at time $t = 5$.
• In part (d) a common error was to overlook that bananas are not added to the table until time $t = 3$, incorrectly computing the net change in the pounds of bananas on the table as $\int_0^8 (g(t) - f(t))\,dt$ (a convergent improper integral). Also, some students failed to account for the 50 pounds of bananas on the display table when the store opened.	• The number of pounds of bananas on the table at time $t = 8$ is given by: The number of pounds of bananas on the table initially, plus the number of pounds added, minus the number of pounds removed: $50 + \int_3^8 g(t)\,dt - \int_0^8 f(t)\,dt$

Based on your experience at the AP® Reading with student responses, what advice would you offer to teachers to help them improve the student performance on the exam?

Many students failed to make effective use of their graphing calculator on this problem, either because they lacked skill in its use, or because the time allotted for calculator use on the exam had elapsed. The former can be addressed by finding opportunities for students to make appropriate use of calculators throughout the course, and not just during exam preparation time. The latter may be helped some by nurturing time management skills for tests and assignments.

Many students had issues with a context that moves beyond geometry (area/volume/slope) or rectilinear motion (velocity/acceleration). Further, students are challenged by a function that describes a rate. More practice is needed with a variety of applied contexts, especially with interpretation within those contexts.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

In general, review of previous Chief Reader reports (known previously as the *Student Performance Q&A*), as well as applicable sections of the *AP Calculus AB and AP Calculus BC Course and Exam Description* (particularly the Mathematical Practices for AP Calculus) will serve teachers well. Also, previously released exam questions and participation on the Online Teacher Community are of high value for practice and for advice. For this problem, the Teaching and Assessing Calculus Modules 8 (Interpreting Context) and 5 (Analyzing Problems in Context) would be the most help.

Question AB3/BC3 **Topic:** Graphical Analysis of *f* prime /FTC
Max. Points: 9 **Mean Score: AB3:** 3.63; **BC3:** 5.42

What were responses expected to demonstrate in their response to this question?

In this problem students were given that a function f is differentiable on the interval $[-6, 5]$ and satisfies $f(-2) = 7.$ For $-6 \le x \le 5$, the derivative of f is specified by a graph consisting of a semicircle and three line segments. In part (a) students were asked to find values of $f(-6)$ and $f(5).$ For each of these values, students needed to recognize that the net change in *f*, starting from the given value $f(-2) = 7,$ can be computed using a definite integral of $f'(x)$ with a lower limit of integration -2 and an upper limit the desired argument of *f*. These integrals can be computed using properties of the definite integral and the geometric connection to areas between the graph of $y = f'(x)$ and the *x*-axis. Thus, students needed to add the initial condition $f(-2) = 7$ to the values of the definite integrals for the desired values. [LO 3.2C/EK 3.2C1] In part (b) students were asked for the intervals on which f is increasing, with justification. Since f' is given on the interval $[-6, 5]$, f is differentiable, and thus also continuous, on that interval. Therefore, f is increasing on closed intervals for which $f'(x) > 0$ on the interior. Students needed to use the given graph of f' to see that $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5),$ so f is increasing on the intervals $[-6, -2]$ and $[2, 5],$ connecting their answers to the sign of $f'.$ [LO 2.2A/EK 2.2A1-2.2A2, LO 2.2B/EK 2.2B1] In part (c) students were asked for the absolute minimum value of f on the closed interval $[-6, 5],$ and to justify their answers. Students needed to use the graph of f' to identify critical points of f on the interior of the interval as $x = -2$ and $x = 2.$ Then they can compute $f(-2)$ and $f(2),$ similarly to the computations in part (a), and compare these to the values of f at the endpoints that were computed in part (a). Students needed to report the smallest of these values, $f(2) = 7 - 2\pi$ as the answer. Alternatively, students could have observed that the minimum value must occur either at a point interior to the interval at which f' transitions from negative to positive, at a left endpoint for which f' is positive immediately to the right, or at a right endpoint for which f' is negative immediately to the left. This reduces the options to $f(-6) = 3$ and $f(2) = 7 - 2\pi.$ [LO 2.2A/EK 2.2A1-2.2A2, LO 2.2B/EK 2.2B1, LO 3.3A/EK 3.3A3] In part (d) students were asked to determine values of $f''(-5)$ and $f''(3),$ or to explain why the requested value does not exist. Students needed to find the value $f''(-5)$ as the slope of the line segment on the graph of f' through the point corresponding to $x = -5.$ The point on the graph of f' corresponding to $x = 3$ is the juncture of a line segment of slope 2 on the left with one of slope -1 on the right. Thus, students needed to report that $f''(3)$ does not exist, and explain why the given graph of f' shows that f' is not differentiable at $x = 3.$ Student explanations could be done by noting that the left-hand and right-hand limits at $x = 3$ of the difference quotient $\frac{f'(x) - f'(3)}{x - 3}$ have differing values (2 and $-1,$ respectively), or by a clear description of the relevant features of the graph of f' near $x = 3.$ [LO 1.1A(b)/EK 1.1A3] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

How well did the response address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) students generally included the initial value $f(-2) = 7$ in their answers and could make the geometric connection between definite integrals and areas between the integrand and the horizontal axis. However, some students had difficulty with the needed properties of the definite integrals, namely that reversing the limits of integration results in an integral value of opposite sign, or that when the integrand dips below the horizontal axis, the area between the curve and the axis contributes negatively to the integral value.

In part (b) most students did a good job of focusing on where the graph of f' is above the x-axis to conclude that $f'(x) > 0$, so f is increasing. Some students incorrectly considered the figure to be the graph of f and gave an incorrect answer on that basis.

In part (c) many students could identify $x = 2$ as a critical point for f, but there were problems in recognizing that endpoints need to be considered and a global argument given to justify an absolute extremum.

In part (d) students were successful finding $f''(-5)$ by computing a slope on the graph of f' but had difficulty explaining why $f''(3)$ does not exist in a mathematically satisfying way.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) students made errors integrating right-to-left.	• $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = f(-2) - \int_{-6}^{-2} f'(x)\,dx$
• In part (b) some students pulled the incorrect information from the graph of f' (e.g., where f' is increasing, rather than where f' is positive).	• The graph of f' is above the x-axis on $[-6, -2)$ and on $(2, 5)$, so f is increasing on $[-6, -2]$ and on $[2, 5]$.
• In part (c) students explained why f has a relative minimum at $x = 2$, rather than justifying an absolute minimum there.	• Candidates for the absolute minimum are endpoints ($x = -6$ and $x = 5$) and critical points ($x = -2$ and $x = 2$). The smallest of the values $f(-6)$, $f(-2)$, $f(2)$, or $f(5)$ is the absolute minimum value for f on $[-6, 5]$.
• In part (d) students used imprecise language (e.g., the graph of f' has a "vertex") in attempting an explanation of why $f''(3)$ does not exist.	• For $2 < x < 3$, $\frac{f'(x) - f'(3)}{x - 3} = 2$, whereas for $3 < x < 5$, $\frac{f'(x) - f'(3)}{x - 3} = -1$. Thus, the limit of $\frac{f'(x) - f'(3)}{x - 3}$ as $x \to 3$ does not exist, so $f''(3)$ does not exist.

Based on your experience at the AP® Reading with student responses, what advice would you offer to teachers to help them improve the student performance on the exam?

In general, teachers can place more emphasis on students' communication skills in converting graphical information to convey mathematically salient reasons and justifications. In particular, students can practice using mathematical notation to explain, from a function's graph, how a function fails to be differentiable at a particular point.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

In general, review of previous Chief Reader reports (known previously as the *Student Performance Q&A*), as well as applicable sections of the *AP Calculus AB and AP Calculus BC Course and Exam Description* (particularly the Mathematical Practices for AP Calculus) will serve teachers well. Also, previously released exam questions and participation on the Online Teacher Community are of high value for practice and for advice. For this problem, the Teaching and Assessing Calculus Module 4 (Justifying Properties and Behaviors of Functions) would be the most help.

Question AB4/BC4 **Topic:** Modeling with Separable Differential Equation
Max. Points: 9 **Mean Score: AB4:** 1.54; **BC4:** 3.13

What were responses expected to demonstrate in their response to this question?

The context for this problem is the internal temperature of a boiled potato that is left to cool in a kitchen. Initially at time $t = 0$, the potato's internal temperature is 91 degrees Celsius, and it is given that the internal temperature of the potato exceeds 27 degrees Celsius for all times $t > 0$. The internal temperature of the potato at time t minutes is modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$. In part (a) students were asked for an equation of the line tangent to the graph of H at $t = 0$, and to use this equation to approximate the internal temperature of the potato at time $t = 3$. Using the initial value and the differential equation, students needed to find the slope of the tangent line to be $H'(0) = -\frac{1}{4}(91 - 27) = -16$ and report the equation of the tangent line to be $y = 91 - 16t$. Students needed to find the approximate temperature of the potato at $t = 3$ to be $91 - 16 \cdot 3 = 43$ degrees Celsius. [LO 2.3B/EK 2.3B2] In part (b) students were asked to use $\frac{d^2H}{dt^2}$ to determine whether the approximation in part (a) is an underestimate or overestimate for the potato's internal temperature at time $t = 3$. Students needed to use the given differential equation to calculate $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \frac{1}{16}(H - 27)$. Then using the given information that the temperature always exceeds 27 degrees Celsius, students needed to conclude that $\frac{d^2H}{dt^2} > 0$ for all times t. Thus, the graph of H is concave up, and the line tangent to the graph of H at $t = 0$ lies below the graph of H (except at the point of tangency), so the approximation found in part (a) is an underestimate. [LO 2.1D/EK 2.1D1, LO 2.2A/EK 2.2A1] In part (c) an alternate model, G, is proposed for the internal temperature of the potato at times $t < 10$. $G(t)$ is measured in degrees Celsius and satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$ with $G(0) = 91$. Students were asked to find an expression for $G(t)$ and to find the internal temperature of the potato at time $t = 3$ based on this model. Students needed to employ the method of separation of variables, using the initial condition $G(0) = 91$ to resolve the constant of integration, and arrive at the particular solution $G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$. Students should then have reported that the model gives an internal temperature of $G(3) = 54$ degrees Celsius for the potato at time $t = 3$. [LO 3.5A/EK 3.5A2] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

How well did the response address the course content related to this question? How well did the responses integrate the skills required on this question?

In general, students—particularly Calculus AB students—continue to struggle with questions that present a differential equation in which the derivative is expressed in terms of only the dependent variable. See 2011 question AB5/BC5 ($\frac{dW}{dt} = \frac{1}{25}(W - 300)$; AB mean score 1.63, BC mean score 3.53) and 2012 AB5/BC5 ($\frac{dB}{dt} = \frac{1}{5}(100 - B)$; AB mean score 2.87, BC mean score 4.75). The lack of appearance of the independent variable, t, seems to stymie many students.

In part (a) most students knew they had to use the differential equation to get the slope of the tangent line but were confused about what value to substitute for H, some opting for $H = 0$ instead of $H = 91$. As a result, many students had a tangent line passing through the correct point, $(0, 91)$, but with an incorrect slope, $\frac{27}{4}$.

In part (b) many students failed to apply the chain rule, resulting in $\frac{d}{dH}\left(\frac{dH}{dt}\right)$, instead of $\frac{d^2H}{dt^2} = \frac{d}{dt}\left(\frac{dH}{dt}\right) = \frac{d}{dH}\left(\frac{dH}{dt}\right) \cdot \frac{dH}{dt}$. Also, many students argued for an overestimate or underestimate from the value of $\frac{d^2H}{dt^2}$ at a single test point, failing to understand the need to consider an interval spanning the times from $t = 0$ to $t = 3$ (and thus missing the significance of the statement in the stem that the potato's internal temperature is greater than $27°C$ for all times $t > 0$).

In part (c) many students stumbled attempting to separate variables, resulting in either no separation or an incorrect separation. Some students omitted a constant of integration at the antiderivative step, thus those students were not eligible to earn remaining points. Finally, some students went straight from the equation $3(G - 27)^{1/3} = 12 - t$ to compute the value of G when $t = 3$ without giving an explicit expression for $G(t)$.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) computing the slope of the tangent line as $\left.\frac{dH}{dt}\right\|_{t=0} = -\frac{1}{4}(0 - 27) = \frac{27}{4}$	• $\left.\frac{dH}{dt}\right\|_{t=0} = -\frac{1}{4}(91 - 27) = -16$
• In part (b) failing to apply the chain rule: $\frac{d}{dt}\left(-\frac{1}{4}(H - 27)\right) = -\frac{1}{4}$	• $\frac{d}{dt}\left(-\frac{1}{4}(H - 27)\right) = -\frac{1}{4} \cdot \frac{dH}{dt} = \frac{1}{16}(H - 27)$
• In part (b) failing to ascertain concavity of the graph of H on an interval containing $t = 0$ and $t = 3$; e.g., basing a response on just $\left.\frac{d^2H}{dt^2}\right\|_{H=43} > 0$.	• $\frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$ because $H > 27$ for $t > 0$. Thus, the graph of H is concave up on an interval containing $t = 0$ and $t = 3$, so using a line tangent to the graph of H at $t = 0$ to approximate $H(3)$ must result in an underestimate.
• In part (c) from $\frac{dG}{dt} = -(G - 27)^{2/3}$, incorrectly moving to $G(t) = \int -(G - 27)^{2/3}\, dt$ $= -\frac{3}{5}(G - 27)^{5/3} + C$	• From $\frac{dG}{dt} = -(G - 27)^{2/3}$, $(G - 27)^{-2/3} \frac{dG}{dt} = -1$, so $\int (G - 27)^{-2/3}\, dG = \int (-1)\, dt$ and $3(G - 27)^{1/3} = -t + C$.

Based on your experience at the AP® Reading with student responses, what advice would you offer to teachers to help them improve the student performance on the exam?

Teachers can add this problem as a practice problem to 2011 AB5/BC5 and 2012 AB5/BC5 and expand this collection with more differential equations problems involving separation of variables in which the derivative is expressed only in terms of the dependent variable. Further, teachers can adapt many of their separation of variable problems involving variables x, y, and/or t to equivalent problems using other letters for the variables. Students should have the flexibility to transfer skills learned in x's and y's to equivalent environments with other variable names.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

In general, review of previous Chief Reader reports (known previously as the *Student Performance Q&A*), as well as applicable sections of the *AP Calculus AB and AP Calculus BC Course and Exam Description* (particularly the Mathematical Practices for AP Calculus) will serve teachers well. Also, previously released exam questions and participation on the Online Teacher Community are of high value for practice and for advice. For this problem, the Teaching and Assessing Calculus Modules 6 (Interpreting Notational Expressions) and 7 (Applying Procedures) would be the most help.

Question AB5

Topic: Particle Motion

Max. Points: 9 **Mean Score:** 3.59

What were responses expected to demonstrate in their response to this question?

In this problem, two particles, P and Q, are moving along the x-axis. For $0 \le t \le 8$, the position of particle P is given by $x_P(t) = \ln\left(t^2 - 2t + 10\right)$, while particle Q has position 5 at $t = 0$ and velocity $v_Q(t) = t^2 - 8t + 15$. In part (a) students were asked for those times t, $0 \le t \le 8$, when particle P's motion is to the left. Using the chain rule, students needed to find an expression for the velocity of particle P at time t by differentiating the position $x_P(t)$. By analyzing the sign of this derivative to determine those times t with $x'_P(t) < 0$, students should have concluded that particle P is moving left for $0 \le t < 1$. [LO 2.1C/EK 2.1C2-2.1C4, LO 2.3C/EK 2.3C1] In part (b) students were asked for all times t, $0 \le t \le 8$, during which both particles travel in the same direction. Using the velocity of particle P, $v_P(t) = x'_P(t)$, found in part (a), and the given velocity $v_Q(t)$ for particle Q, students needed to find those subintervals of $0 \le t \le 8$ on which both $v_P(t)$ and $v_Q(t)$ have the same sign. Students should have responded that for $1 < t < 3$ and for $5 < t \le 8$, noting that both $v_P(t)$ and $v_Q(t)$ are positive on these intervals, both particles travel in the same direction (to the right). There is no time when both velocities are negative. [LO 2.3C/EK 2.3C1] In part (c) students were asked for the acceleration of particle Q at time $t = 2$, and to determine, with explanation, whether particle Q's speed is increasing, decreasing, or neither at time $t = 2$. Students needed to differentiate $v_Q(t)$ to find that the acceleration of particle Q is given by $a_Q(t) = v'_Q(t) = 2t - 8$, and report that particle Q's acceleration at time $t = 2$ is $a_Q(2) = -4$. Students should have explained that particle Q's speed is decreasing at time $t = 2$ because the velocity and acceleration of particle Q have opposite signs at that time. [LO 2.1C/EK 2.1C2, LO 2.3C/EK 2.3C1] In part (d) students were asked to find the position of particle Q the first time it changes direction. Using the analysis of the sign of $v_Q(t)$ done in part (b), students should have concluded that the first change of direction of particle Q's motion occurs at time $t = 3$. The net change in position of particle Q across the time interval $[0, 3]$ is given by $\int_0^3 v_Q(t)\,dt$. Students needed to evaluate this integral using the Fundamental Theorem of Calculus and use the initial position of particle Q to find that particle Q's position at time $t = 3$ is $5 + \int_0^3 v_Q(t)\,dt = 23$. [LO 3.3B(b)/EK 3.3B2, LO 3.4C/EK 3.4C1] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

How well did the response address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) the vast majority of students knew to consider $v_P(t) = x'_P(t)$ to determine the direction of motion of particle P during the given time interval. Many students, however, were unable to correctly apply the chain rule in computing $x'_P(t)$.

In part (b) students seemed to be aware that the two particles travel in the same direction when their velocity values have the same sign but were challenged to do a sign analysis of the velocity functions, or to merge two sign analyses when the subintervals of one had endpoints differing from those of the other.

In part (c) students could correctly compute the acceleration of particle Q as $v'_Q(t) = 2t - 8$ but failed to take into account the sign of $v_Q(2)$, as well as the sign of acceleration at time $t = 2$, to determine whether the speed of particle Q is increasing, decreasing, or neither at time $t = 2$.

In part (d) students could antidifferentiate $v_Q(t)$, but many failed to consider the particle's initial position to find its position at the desired time.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

In general, students were challenged by notation involving subscripts, so that they often dropped subscripts P or Q, or redefined the functions $x_P(t)$ and $v_Q(t)$ with new names to avoid subscripts. This could lead to a great deal of confusion over which particle the student was discussing. Some students used inappropriate interval notation, e.g., $1 \le 3$ instead of $1 \le t \le 3$ or $[1, 3]$.

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many students had parentheses errors, computing $x'_P(t) = \frac{1}{t^2 - 2t + 10} \cdot 2t - 2 = \frac{2t}{t^2 - 2t + 10} - 2.$	• $x'_P(t) = \frac{1}{t^2 - 2t + 10} \cdot (2t - 2) = \frac{2t - 2}{t^2 - 2t + 10}.$
• In part (b) not synthesizing correct sign analysis of $v_P(t)$ and $v_Q(t)$, stopping at $v_P(t) < 0$ for $0 \le t < 1$ and $v_P(t) > 0$ for $1 < t \le 8$, while $v_Q(t) > 0$ for $0 \le t < 3$, $v_Q(t) < 0$ for $3 < t < 5$, and $v_Q(t) > 0$ for $5 < t \le 8$.	• $v_P(t) < 0$ for $0 \le t < 1$ and $v_P(t) > 0$ for $1 < t \le 8$, while $v_Q(t) > 0$ for $0 \le t < 3$, $v_Q(t) < 0$ for $3 < t < 5$, and $v_Q(t) > 0$ for $5 < t \le 8$. Thus, $v_P(t)$ and $v_Q(t)$ are both positive for $1 < t < 3$ and for $5 < t \le 8$. There is no time, $0 \le t \le 8$, when both velocities are negative. (Note that a well organized and labeled sign chart can aid this synthesis.)
• In part (c) arguing (just) from $a_Q(2) < 0$ that particle Q's speed is decreasing.	• Because $a_Q(2) < 0$ and $v_Q(2) > 0$, particle Q's speed is decreasing at time $t = 2$.
• In part (d) not accounting for the initial position: from $v_Q(t) = t^2 - 8t + 15$, $x_Q(t) = \frac{1}{3}t^3 - 4t^2 + 15t$, so $x_Q(3) = 9 - 36 + 45.$	• $x_Q(3) = x_Q(0) + \int_0^3 v_Q(t)\,dt = 5 + \left[\frac{1}{3}t^3 - 4t^2 + 15t\right]_{t=0}^{t=3}$ $= 5 + (9 - 36 + 45)$

Based on your experience at the AP® Reading with student responses, what advice would you offer to teachers to help them improve the student performance on the exam?

Teachers can emphasize correct and clear communication and find opportunities to increase students' comfort with subscripts as a clarifying notation.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

In general, review of previous Chief Reader reports (known previously as the *Student Performance Q&A*), as well as applicable sections of the *AP Calculus AB and AP Calculus BC Course and Exam Description* (particularly the Mathematical Practices for AP Calculus) will serve teachers well. Also, previously released exam questions and participation on the Online Teacher Community are of high value for practice and for advice. For this problem, the Teaching and Assessing Calculus Module 4 (Justifying Properties and Behaviors of Functions) would be the most help.

Question AB6 **Topic:** Analysis of Functions-Tabular/Graphical
Max. Points: 9 **Mean Score:** 3.32

What were responses expected to demonstrate in their response to this question?

This problem deals with multiple functions. Function f is defined by $f(x) = \cos(2x) + e^{\sin x}$. Function g is differentiable and values of $g(x)$ and $g'(x)$ corresponding to integer values of x from $x = -5$ to $x = 0$, inclusive, are given in a table. Function h is defined on $[-5, 5]$ and the graph of h, comprised of five line segments, is given. In part (a) students were asked for the slope of the line tangent to the graph of f at $x = \pi$. Using the sum and chain rules for differentiation and the derivatives of trigonometric and exponential functions to differentiate $f(x)$, students needed to evaluate $f'(\pi)$ to find the slope of the tangent line. [LO 2.1C/EK 2.1C2-2.1C4, LO 2.3B/EK 2.3B1] In part (b) the function k is defined by $k(x) = h(f(x))$, and students were asked to find $k'(\pi)$. Students needed to apply the chain rule and determine the value of $h'(2)$ from the graph of h to arrive at the value for $k'(\pi)$. [LO 2.1C/EK 2.1C4, LO 2.2A/EK 2.2A2] In part (c) the function m is defined by $m(x) = g(-2x) \cdot h(x)$, and students were asked to find $m'(2)$. Students needed to apply the product and chain rules for differentiation, find values for $g(-4)$ and $g'(-4)$ in the table for g, and use the graph of h to determine $h(2)$ and $h'(2)$, to find $m'(2) = -2g'(-4) \cdot h(2) + g(-4) \cdot h'(2) = -3$. [LO 2.1C/EK 2.1C3-2.1C4, LO 2.2A/EK 2.2A2, LO 2.3B/EK 2.3B1] In part (d) students were asked to determine whether there is a number c in the interval $[-5, -3]$ such that $g'(c) = -4$, and to justify their answers. Using the table for g, students should have confirmed that $\frac{g(-3) - g(-5)}{-3 - (-5)} = -4$. Given that g is differentiable, students should have concluded that g is continuous on $[-5, -3]$ and, thus, recognize that the hypotheses for the Mean Value Theorem are satisfied, and answered in the affirmative that a number c exists in the interval $[-5, -3]$ such that $g'(c) = -4$. [LO 1.2B/EK 1.2B1, LO 2.4A/EK 2.4A1] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

How well did the response address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) most students knew that the slope of the line tangent to the graph of f at $x = \pi$ is given by $f'(\pi)$, but some students made errors using the chain rule, or in stating derivatives of sine and cosine when attempting to find $f'(x)$. Some students also misevaluated $\sin 2\pi$, $\sin \pi$, and/or $\cos \pi$.

In part (b) many students were challenged by the abstract nature of the function k. In general, students could evaluate a derivative of h by calculating the slope of a line segment in the given graph of h. However, students often failed to apply the chain rule to differentiate k, and some students made errors in evaluating $f(\pi)$.

In part (c) students again were challenged with the abstract presentation of a function, in this case m. Some students used an incorrect product rule and/or made a chain rule error in finding the derivative of $g(-2x)$. Finally, some students miscomputed the value of $h'(2)$ from the graph of h.

In part (d) many students attempted to justify a negative response, appealing to a (false) converse of either the Intermediate Value Theorem (with g') or the Mean Value Theorem (with g). Students that recognized -4 as the average rate of change of g across the interval $[-5, -3]$ often failed to declare the full hypotheses needed to apply the Mean Value Theorem, usually omitting that g is continuous on $[-5, -3]$.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) students failed to apply the chain rule, computing $f'(x)$ to be $-\sin(2x) + e^{\sin x}$.	• Applying the chain rule, $f'(x) = (-\sin(2x)) \cdot 2 + e^{\sin x} \cdot \cos x$.
• In part (b) students again made chain rule errors, computing $k'(x)$ to be $h'(f(x))$.	• Applying the chain rule, $k'(x) = h'(f(x)) \cdot f'(x)$.
• In part (c) students made product and chain rule errors, computing $m'(x)$ to be $g'(-2x) \cdot h'(x)$.	• Applying the product and chain rules, $m'(x) = -2g'(-2x) \cdot h(x) + g(-2x) \cdot h'(x)$.
• In part (d) students gave incomplete justifications for an application of the Mean Value Theorem, often omitting the continuity hypothesis.	• g is differentiable, so g is continuous on $[-5, -3]$. The average rate of change of g across $[-5, -3]$ is $\frac{2-10}{-3-(-5)} = -4$. Thus, by the Mean Value Theorem, there is at least one value c, $-5 < c < -3$, for which $g'(c) = -4$.

Based on your experience at the AP® Reading with student responses, what advice would you offer to teachers to help them improve the student performance on the exam?

Part of notational fluency is the ability to apply rules such as the chain or product rules for differentiation to circumstances where function names are other than f and g in what the student may consider "standard" positions. Students can increase their dexterity with such functions through practice finding derivatives of abstractly-defined functions in terms of products, quotients, or compositions, where the constituent functions are presented with a variety of names and in a variety of ways: numerically (table), graphically, or analytically.

Also, teachers can reinforce that justifications of theorem applications require verification that the conditions required by the hypotheses of the theorem are met. Students should get in the practice of highlighting these conditions whenever they state the theorem or apply the theorem to a particular instance.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

In general, review of previous Chief Reader reports (known previously as the *Student Performance Q&A*), as well as applicable sections of the *AP Calculus AB and AP Calculus BC Course and Exam Description* (particularly the Mathematical Practices for AP Calculus) will serve teachers well. Also, previously released exam questions and participation on the Online Teacher Community are of high value for practice and for advice. For this problem, the Teaching and Assessing Calculus Modules 2 (Selecting Procedures) and 3 (Establishing Conditions for Definitions and Theorems) would be the most help.

Question BC2 **Topic:** Polar-Area/Analysis of Curves

Max. Points: 9 **Mean Score:** 3.21

What were responses expected to demonstrate in their response to this question?

In this problem a polar graph is supplied for the curves $f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Regions R, bounded by the graph of $r = f(\theta)$ and the x-axis, and region S, bounded by the graphs of $r = f(\theta)$, $r = g(\theta)$, and the x-axis, are identified on the graph. In part (a) students were asked for the area of R. Students needed to recognize that region R is traced by the polar ray segment from $r = 0$ to $r = f(\theta)$ for $0 \le \theta \le \frac{\pi}{2}$ and use the graphing calculator to evaluate the area of R as the numeric value of $\frac{1}{2}\int_0^{\pi/2} (f(\theta))^2 \, d\theta$. [LO 3.4D/EK 3.4D1] In part (b) students were asked to produce an equation involving one or more integrals that can be solved for k, $0 < k < \frac{\pi}{2}$, such that the ray $\theta = k$ divides S into two regions of equal areas. Students needed to recognize that region S is traced by the polar ray segment from $r = f(\theta)$ to $r = g(\theta)$ for $0 \le \theta \le \frac{\pi}{2}$. The ray $\theta = k$ divides S into two subregions with areas $\frac{1}{2}\int_0^{k} \left((g(\theta))^2 - (f(\theta))^2\right) d\theta$ and $\frac{1}{2}\int_k^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right) d\theta$. Students should have reported an equation equivalent to setting these two expressions equal to each other, or setting one of them equal to half of the area of S, which is given by $\frac{1}{2}\int_0^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right) d\theta$. [LO 3.4D/EK 3.4D1] In part (c) $w(\theta)$ is defined as the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Students were asked to write an expression for $w(\theta)$ and to find w_A, the average value of $w(\theta)$ for $0 \le \theta \le \frac{\pi}{2}$. Students needed to recognize that $w(\theta) = g(\theta) - f(\theta)$ and use the graphing calculator to evaluate the average value $w_A = \dfrac{\int_0^{\pi/2} w(\theta)\, d\theta}{\frac{\pi}{2} - 0}$. [LO 3.4B/EK 3.4B1] In part (d) students were asked to find the value of θ for which $w(\theta) = w_A$, and to determine whether $w(\theta)$ is increasing or decreasing at that value of θ. Importing the value of w_A from part (c), students needed to use the graphing calculator to solve $w(\theta) = w_A$ to obtain $\theta = 0.517688$. Students should have reported this value rounded or truncated to three decimal places. Students should then have reported that $w(\theta)$ is decreasing at this value of θ because the calculator reports a negative value for $w'(0.517688)$. [LO 3.4D/EK 3.4D1] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

How well did the response address the course content related to this question? How well did the responses integrate the skills required on this question?

In parts (a) and (b) students knew to use definite integrals to determine area bounded by curves, but were challenged relating to the polar context, sometimes providing integral expressions that mixed rectangular and polar perspectives.

In part (c) students generally were able to provide the integral setup for the average value of $w(\theta)$ for $0 \le \theta \le \frac{\pi}{2}$. Some students had challenges in recognizing that $w(\theta) = g(\theta) - f(\theta)$ on the interval in question.

In part (d) students knew that, in this context, the sign of the derivative of w can determine whether w is increasing or decreasing at a particular value of θ. However, many students failed to demonstrate the calculator skill to solve $w(\theta) = w_A$ for a value of θ at which they could test w'.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some students expressed the area of R as $\frac{1}{2}\int_0^{\pi/2} f(\theta)\, d\theta$ or as $\frac{1}{2}\int_0^{1}(f(\theta))^2\, d\theta.$	• The area of R is given by $\frac{1}{2}\int_0^{\pi/2}(f(\theta))^2\, d\theta.$
• In part (b) some students represented double the area of S as $\int_0^{\pi/2}(g(\theta) - f(\theta))\, d\theta.$	• Double the area of S is given by $\int_0^{\pi/2}\left((g(\theta))^2 - (f(\theta))^2\right) d\theta.$
• In part (b) some students did not connect the ray $\theta = k$ as indicating k as a limit of integration, instead writing an equation such as $k = \frac{1}{2}\int_0^{\pi/2}\left((g(\theta))^2 - (f(\theta))^2\right) d\theta.$	• If the ray $\theta = k$ divides S into two regions of equal areas, then the area bounded by f and g from $\theta = 0$ to $\theta = k$ must match the area bounded by f and g from $\theta = k$ to $\theta = \frac{\pi}{2}$, i.e., $\frac{1}{2}\int_0^{k}\left((g(\theta))^2 - (f(\theta))^2\right) d\theta = \frac{1}{2}\int_k^{\pi/2}\left((g(\theta))^2 - (f(\theta))^2\right)$
• In part (c) some students confused the average value of w across $0 \le \theta \le \frac{\pi}{2}$ with the average of the endpoint values, $\frac{w(0) + w\left(\frac{\pi}{2}\right)}{2}$.	• The average value of w across $0 \le \theta \le \frac{\pi}{2}$ is $\frac{1}{\frac{\pi}{2} - 0}\int_0^{\pi/2} w(\theta)\, d\theta = \frac{2}{\pi}\int_0^{\pi/2}(g(\theta) - f(\theta))\, d\theta.$
• In part (d) some students who could find a solution to $w(\theta) = w_A$ gave insufficient reason for their conclusions, such as "$w(\theta)$ is decreasing because the graphs are getting closer together."	• $w(\theta) = w_A$ at $\theta = 0.518$. $w(\theta)$ is decreasing at $\theta = 0.518$ because $w'(0.518) < 0$.

Based on your experience at the AP® Reading with student responses, what advice would you offer to teachers to help them improve the student performance on the exam?

Part of the richness of free-response questions is that they often bring together topics from various parts of the course. In a typical course, the notion of average value is rarely encountered in problems involving polar functions, nor is the issue of increasing versus decreasing. One should not strive to expose students to every possible mixture of topics, but it is important to have a variety of surprise encounters so that students are not surprised on the AP Exam. There is no magic resource for this (besides possibly previous free-response questions), but the creativity to write problems that combine topics can be as invigorating for the teacher as it is nurturing for their students.

This problem added to the evidence that many students have difficulty with the radial perspective involved in "thinking polar." This is more a precalculus issue than a calculus one, but polar functions in calculus provide many opportunities to recall and reinforce this perspective. While many aspects of calculus with polar functions involve the connection to a rectangular framework (e.g., the rectangular notion of "slope"), none of the parts of this problem required a conversion to rectangular coordinates. Indeed, such a conversion was an impediment to student progress on these problems. Thus, time spent acclimating students to the polar environment, perhaps aided by relating it to situations such as weather radar or a ship's sonar, can make problems such as these less intimidating.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

In general, review of previous Chief Reader reports (known previously as the *Student Performance Q&A*), as well as applicable sections of the *AP Calculus AB and AP Calculus BC Course and Exam Description* (particularly the Mathematical Practices for AP Calculus) will serve teachers well. Also, previously released exam questions and participation on the Online Teacher Community are of high value for practice and advice.

Question BC5 **Topic:** Analysis of Functions/Improper Integral/Series
Max. Points: 9 **Mean Score:** 3.54

What were responses expected to demonstrate in their response to this question?

In this problem the function f is defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$. In part (a) students were asked to find the slope of the line tangent to the graph of f at $x = 3$. Students needed to differentiate f and find the slope of the line tangent to the graph of f at $x = 3$ by evaluating $f'(3)$. [LO 2.3B/EK 2.3B1] In part (b) students were asked to find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$ and to classify each critical point as the location of a relative minimum, a relative maximum, or neither, justifying these classifications. Students should have observed that f is differentiable on $1 < x < 2.5$ and found that $f'(x) = 0$ has just one solution, $x = \frac{7}{4}$, in this interval. Then students needed to determine that f has a relative maximum at $x = \frac{7}{4}$ by noting that f' changes sign from positive to negative at the critical point $x = \frac{7}{4}$. [LO 2.2A/EK 2.2A1] In part (c) students were given the partial fraction decomposition for $f(x)$ and asked to evaluate $\int_5^\infty f(x)\,dx$ or to show that the integral diverges. Students should have expressed the given improper integral as a limit of proper integrals, $\lim_{b\to\infty} \int_5^b f(x)\,dx$, and used the partial fraction decomposition for $f(x)$ to find that $\int_5^b f(x)\,dx = \ln(2b - 5) - \ln(b - 1) - (\ln 5 - \ln 4) = \ln\left(\frac{2b - 5}{b - 1}\right) - \ln\left(\frac{5}{4}\right)$. Applying limit theorems, students needed to take the limit of this expression as $b \to \infty$ to find that the improper integral converges to $\ln\left(\frac{8}{5}\right)$. [LO 1.1C/EK 1.1C1-1.1C2, LO 3.2D/EK 3.2D1-3.2D2] In part (d) students were asked to determine whether the series $\sum_{n=5}^{\infty} f(n)$ converges or diverges, stating the conditions of the test used for this determination. Students needed to combine the results of part (c) with the integral test or use a limit comparison test to the convergent p-series $\sum_{n=5}^{\infty} \frac{1}{n^2}$ to find that the series $\sum_{n=5}^{\infty} f(n)$ converges. For either test, students should have observed that the necessary conditions hold, namely that f is continuous, positive, and decreasing on $[5, \infty)$. [LO 4.1A/EK 4.1A6] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

How well did the response address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) students were generally successful in knowing to differentiate and computing $f'(3)$ to find the slope of the indicated tangent line.

In part (b) most students could find the critical point $x = \frac{7}{4}$ but were less successful in justifying a classification of the critical point as a relative minimum, relative maximum, or neither.

In part (c) many students were successful in finding the antiderivative, but did not employ limit notation appropriately to describe the process of evaluating an improper integral.

In part (d) many students were able to identify an appropriate test (almost always the integral test or limit comparison test) to use in determining whether the given series converges or diverges. However, the application of the test was often informal, lacking in appropriate limit notation, using language that confused a series with the sequence of its terms, or giving an incomplete description of the conditions necessary to apply the convergence test.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some students had "linkage" errors: using "=" to connect unequal expressions, such as $f'(x) = \frac{-3}{(2x^2 - 7x + 5)^2} \cdot (4x - 7)$ $= f'(3) = \frac{-3}{(18 - 21 + 5)^2} \cdot 5.$	• $f'(x) = \frac{-3}{(2x^2 - 7x + 5)^2} \cdot (4x - 7)$ Thus, the slope of the tangent line is $f'(3) = \frac{-3}{(18 - 21 + 5)^2} \cdot 5.$
• In part (b) some students did not constrain their analysis to the interval $1 < x < 2.5$ and failed to recognize $x = 1$ and $x = 2.5$ as asymptotes for the graph of *f*, resulting in a faulty analysis. For example, a student with a correct derivative could find the only critical point at $x = \frac{7}{4}$ and then explicitly consider intervals $\left(-\infty, \frac{7}{4}\right)$ and $\left(\frac{7}{4}, \infty\right)$. Using test points $f'(-1) > 0$ and $f'(2) < 0,$ then conclude that *f* has a relative maximum at $x = \frac{7}{4}.$	• $f'(x)$ is defined on $1 < x < 2.5$ and $f'(x) = 0$ only at $x = \frac{7}{4}.$ Because f' changes from positive to negative at $x = \frac{7}{4},$ *f* has a relative maximum at $x = \frac{7}{4}.$
• In part (c) students missed the antiderivative of $\frac{2}{2x - 5},$ didn't employ appropriate limit notation, and made errors in evaluation, such as $\int_5^\infty \left(\frac{2}{2x - 5} - \frac{1}{x - 1}\right) dx$ $= 2\ln\lvert 2x - 5\rvert - \ln\lvert x - 1\rvert \Big\rvert_5^\infty$ $= \infty - (\text{number}) = \infty.$ Thus, the integral diverges.	• $\int_5^\infty \left(\frac{2}{2x - 5} - \frac{1}{x - 1}\right) dx = \lim_{b \to \infty}\Big[\ln\lvert 2x - 5\rvert - \ln\lvert x - 1\rvert\Big]_5^b$ $= \lim_{b \to \infty}\left(\ln\left\lvert\frac{2b - 5}{b - 1}\right\rvert - \ln\left(\frac{5}{4}\right)\right) = \ln 2 - \ln\left(\frac{5}{4}\right)$ Thus, the integral converges.
• In part (d) students gave arguments that had a superficial resemblence to correctness but were lacking in good communication of the necessary mathematics. For example, $\sum \frac{3}{2n^2 - 7n + 5} \le \sum \frac{3}{n^2}$ and $\frac{1}{n^2}$ converges, so the series converges by the comparison test.	• The student could use part (c) and apply the integral test: *f* is continuous, positive, and decreasing on $[5, \infty)$. Because $\int_5^\infty f(x)\,dx$ converges, $\sum_{n=1}^{\infty} f(n)$ must also converge by the integral test. Or use the limit comparison test: $f(n) > 0$ and $\frac{1}{n^2} > 0$ for $n \ge 5$.

	Note that $\lim_{n\to\infty} \frac{f(n)}{\frac{1}{n^2}} = \frac{3}{2}$ and the series $\sum_{n=5}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2$), so the series $\sum_{n=5}^{\infty} f(n)$ converges by the limit comparison test.

Based on your experience at the AP® Reading with student responses, what advice would you offer to teachers to help them improve the student performance on the exam?

Teachers need to continue emphasizing correct mathematical notation, especially appropriate use of limit notation. Further, students need to move beyond rote mimicking of textbook notation to an understanding of the role and use of notation as clear and concise communication. Such understanding might clarify when it is inappropriate to join two expressions with a gratuitous "=" sign, or when and where " $\lim_{b\to\infty}$ " should appear in a calculation. Further, arrows to indicate limit evaluation may be fine for quick scratch calculations, but they may provide incomplete communication of the mathematical process being assessed.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

In general, review of previous Chief Reader reports (known previously as the *Student Performance Q&A*), as well as applicable sections of the *AP Calculus AB and AP Calculus BC Course and Exam Description* (particularly the Mathematical Practices for AP Calculus) will serve teachers well. Also, previously released exam questions and participation on the Online Teacher Community are of high value for practice and for advice.

Question BC6 **Topic:** Maclaurin Series/Alternating Series Error Bound
Max. Points: 9 **Mean Score:** 3.23

What were responses expected to demonstrate in their response to this question?

In this problem students were presented with a function f that has derivatives of all orders for $-1 < x < 1$ such that $f(0) = 0$, $f'(0) = 1$, and $f^{(n+1)}(0) = -n \cdot f^{(n)}(0)$ for all $n \geq 1$. It is also stated that the Maclaurin series for f converges to $f(x)$ for $|x| < 1$. In part (a) students were asked to verify that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ and to write the general term of this Maclaurin series. The *n*th-degree term of the Taylor polynomial for f about $x = 0$ is $\frac{f^{(n)}(0)}{n!}x^n$. $f(0) = 0$ and $f'(0) = 1$ are given, and the given recurrence relation for $f^{(n+1)}(0)$ can be readily applied to see that $f''(0) = -1$, $f'''(0) = 2$, $f^{(4)}(0) = -6$, and $f^{(n)}(0) = (-1)^{n+1}(n-1)!$. Using these derivative values, students needed to confirm that the first four nonzero terms of the Maclaurin series for f are as given, and that the general term is $\frac{(-1)^{n+1}x^n}{n}$. [LO 4.2A/EK 4.2A1] In part (b) students were asked to determine, with explanation, whether the Maclaurin series for f converges absolutely, converges conditionally, or diverges at $x = 1$. Substituting $x = 1$, students should have obtained that the Maclaurin series for f evaluated at $x = 1$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Students needed to conclude that this series converges conditionally, noting that the series converges by the alternating series test and that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right|$ is the divergent harmonic series. [LO 4.1A/EK 4.1A4-4.1A6] In part (c) students were asked to find the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$. Students needed to find these terms by integrating the Maclaurin series for f term-by-term. [LO 4.2B/EK 4.2B5] In part (d) using the function g defined in part (c), the expression $P_n\left(\frac{1}{2}\right)$ represents the *n*th-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$. Students were directed to use the alternating series error bound to show that $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}$. Students may have observed that the terms of the Taylor polynomial for g about $x = 0$, evaluated at $x = \frac{1}{2}$, alternate in sign and decrease in magnitude to 0. Thus, the alternating series error bound can be applied to see that $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \leq \left| -\frac{(1/2)^5}{20} \right| = \frac{1}{32 \cdot 20}$, showing that the error in the approximation is less than $\frac{1}{500}$. [LO 4.1B/EK 4.1B2] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

How well did the response address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) students were generally able to use the recursive relationship provided to find the appropriate derivative values and verify the first four nonzero terms of the Maclaurin series for *f*. Many students could also discern a pattern to generate a correct general term for this Maclaurin series.

In part (b) those students that demonstrated understanding of conditional convergence recognized the need to analyze both the harmonic and the alternating harmonic series for convergence. Many students, however, demonstrated an incomplete understanding of conditional versus absolute convergence.

In part (c) most students knew to antidifferentiate the terms from the Maclaurin series for f given in part (a) to generate the first four nonzero terms for the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$. Students did this antidifferentiation correctly. However, students had more difficulty finding the general term for the Maclaurin series for $g(x)$.

In part (d) many students recognized that an application of the alternating series error bound focuses on the first unused term after the truncation of the series to a polynomial.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some students had trouble matching the sign of the coefficient to the general term with the given terms, e.g., presenting $\frac{(-1)^n}{n}x^n$. Other students inserted a gratuitous factorial, e.g., $\frac{(-1)^{n+1}}{n!}x^n$.	• The coefficient of the term involving x^n is positive if n is odd and negative if n is even. The general term is $\frac{(-1)^{n+1}}{n}x^n$.
• In part (b) some students based a conclusion on faulty reasoning, e.g., using a ratio test (which is inconclusive if the limit of the ratios is 1): $\lim_{n\to\infty}\left\lvert \frac{\frac{(-1)^{n+2}}{n+1}x^{n+1}}{\frac{(-1)^{n+1}}{n}x^n} \right\rvert = \lvert x \rvert = 1$ if $x = 1$, so the series converges absolutely at $x = 1$.	• With $x = 1$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. This series is the alternating harmonic series, which converges by the alternating series test. However, $\sum_{n=1}^{\infty}\left\lvert\frac{(-1)^{n+1}}{n}\right\rvert = \sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, which diverges. Thus, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is conditionally convergent.
• In part (c) many students did not include a general term.	• The general term for the Maclaurin series for $g(x)$ is $\int_0^x \frac{(-1)^{n+1}}{n}t^n\,dt = \frac{(-1)^{n+1}}{(n+1)n}x^{n+1}$.

• In part (d) some students looked at the fifth term, arising from the term of degree 6 in the Maclaurin series for $g(x)$, rather than the term of degree 5. • Many students had poor communication, merely presenting a number smaller than $\frac{1}{500}$, without making the crucial connection that the number was a bound on the error of the approximation, e.g., only stating that $\left\lvert -\frac{(1/2)^5}{20} \right\rvert = \frac{1}{640} < \frac{1}{500}$.	• By the alternating series error bound, $\left\lvert P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right\rvert \le \left\lvert -\frac{(1/2)^5}{20} \right\rvert = \frac{1}{640} < \frac{1}{500}$.

Based on your experience at the AP® Reading with student responses, what advice would you offer to teachers to help them improve the student performance on the exam?

Like much of the exam, this problem had several parts in which assessing students' knowledge was dependent upon the students' capacity to communicate clearly both their techniques and the appropriateness of those techniques. For example, to apply a convergence test, such as the alternating series test, it is necessary to know that the conditions for that test are met (terms alternate in sign and decrease in magnitude with limit 0). Showing conditional convergence means showing the series converges, but the sum of the absolute values of its terms diverges, with appropriate reasons for the convergence and divergence. And, showing an approximation has error bounded by some target number involves not just producing a number smaller than the proposed bound, but tying that smaller number to a bound on the error of the approximation. Teachers need to continue to reinforce to students that a collection of calculations does not make for a problem solution; the student needs to communicate how those calculations fit together to yield the desired result. The student needs to "connect the dots," not rely upon someone else to do this for him or her. There is no particular collection of examples for this, just a continued encouragement to students for good communication throughout the course.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

In general, review of previous Chief Reader reports (known previously as the *Student Performance Q&A*), as well as applicable sections of the *AP Calculus AB and AP Calculus BC Course and Exam Description* (particularly the Mathematical Practices for AP Calculus) will serve teachers well. Also, previously released exam questions and participation on the Online Teacher Community are of high value for practice and for advice.