Chapter 3

Connections

Contents

3.1 The idea of parallel transport

A connection is essentially a way of identifying the points in nearby fibers of a bundle. One can see the need for such a notion by considering the following question:

> *Given a vector bundle $\pi : E \to M$, a section $s : M \to E$ and a vector $X \in T_xM$, what is meant by the directional derivative $ds(x)X$?*

If we regard a section merely as a map between the manifolds M and E, then one answer to the question is provided by the tangent map $Ts : TM \to TE$. But this ignores most of the structure that makes a vector bundle interesting. We prefer to think of sections as "vector valued" maps on M, which can be added and multiplied by scalars, and we'd like to think of the directional derivative $ds(x)X$ as something which respects this linear structure.

From this perspective the answer is ambiguous, unless E happens to be the trivial bundle $M \times \mathbb{F}^m \to M$. In that case, it makes sense to think of the section s simply as a map $M \to \mathbb{F}^m$, and the directional derivative is then

$$ds(x)X = \frac{d}{dt}s(\gamma(t))\bigg|_{t=0} = \lim_{t\to 0}\frac{s(\gamma(t)) - s(\gamma(0))}{t} \tag{3.1}$$

for any smooth path with $\dot{\gamma}(0) = X$, thus defining a linear map

$$ds(x) : T_xM \to E_x = \mathbb{F}^m.$$

If $E \to M$ is a nontrivial bundle, (3.1) doesn't immediately make sense because $s(\gamma(t))$ and $s(\gamma(0))$ may be in different fibers and cannot be added. Yet in this case, we'd like to think of different fibers as being equivalent, so that $ds(x)$ can still be defined as a linear map $T_xM \to E_x$. The problem is that there is no *natural* isomorphism between E_x and E_y for $x \ne y$; we need an extra piece of structure to "connect" these fibers in some way, at least if x and y are sufficiently close.

This leads to the idea of *parallel transport*, or *parallel translation*. The easiest example to think of is the tangent bundle of a submanifold $M \subset \mathbb{R}^m$; for simplicity, picture M as a surface embedded in $\mathbb{R}^3$ (Figure 3.1). For any vector $X \in T_xM$ at $x \in M$ and a path $\gamma(t) \in M$ with $\gamma(0) = x$, it's not hard to imagine that the vector X is "pushed" along the path γ in a natural way, forming a smooth family of tangent vectors $X(t) \in T_{\gamma(t)}M$. In fact, this gives a smooth family of isomorphisms

$$P^t_\gamma : T_xM \to T_{\gamma(t)}M$$

with $P^0_\gamma = \mathrm{Id}$, where "smooth" in this context means that in any local trivialization of $TM \to M$ near x, P^t_γ is represented by a smooth path of matrices. The reason this is well defined is that M has a natural connection determined by its embedding in $\mathbb{R}^3$; this is known as the Levi-Cività connection, and is uniquely defined for any manifold with a Riemannian metric (see Chapter 4).

Notice that in general if $\gamma(0) = x$ and $\gamma(1) = y$, the isomorphism $P^1_\gamma : T_xM \to T_yM$ may well depend on the path γ, not just its endpoints. One can easily see this for the example of $M = S^2$ by starting X as a vector along the equator and translating it along a path that moves first along the equator, say 90 degrees of longitude, but then makes a sharp turn and moves straight up to the north pole. The resulting vector Y at the north pole is different from the vector obtained by transporting X

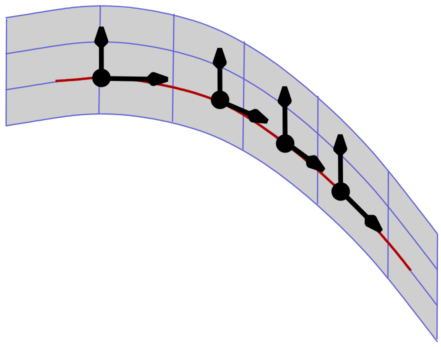

Figure 3.1: Parallel transport of two tangent vectors along a path in a surface.

along the most direct northward path from x to y. Equivalently, one can translate a vector along a *closed* path and find that it returns to a different place in the tangent space than where it began (see Figure 3.2). As we will see in Chapter 5, these are symptoms of the fact that S^2 has nontrivial curvature.

For a general vector bundle $\pi : E \to M$, we now wish to associate with any path $\gamma(t) \in M$ a smooth family of *parallel transport* isomorphisms

$$P_\gamma^t : E_{\gamma(0)} \to E_{\gamma(t)},$$

with $P_\gamma^0 = \mathrm{Id}$. These are far from unique, and as the example of S^2 shows, we must expect that they will depend on more than just the endpoints of the path. But the isomorphisms are also not arbitrary; we now determine what conditions are needed to make this a useful concept.

The primary utility of the family P_γ^t is that, once chosen, it enables us to differentiate sections along paths. Namely, suppose $\gamma(t) \in M$ is a smooth path through $\gamma(0) = x$, and we are given a smooth *section along* γ, i.e. a map $s(t)$ with values in E such that $\pi \circ s(t) = \gamma(t)$ (equivalently, this is a section of the pullback bundle $\gamma^* E$, cf. §2.3). We define the *covariant derivative of* s *along* γ,

$$\left.\frac{D}{dt} s(t)\right|_{t=0} := \nabla_t s(0) := \left.\frac{d}{dt}\left[(P_\gamma^t)^{-1} \circ s(t)\right]\right|_{t=0}. \tag{3.2}$$

This is well defined since each of the vectors $(P_\gamma^t)^{-1} \circ s(t)$ belongs to the same fiber E_x, thus $\nabla_t s(0) \in E_x$. Defining $\nabla_t s(t)$ similarly for all t gives another smooth section of E along γ.

For a smooth section $s : M \to E$, we are also now in a position to define directional derivatives. Reasoning by analogy, we choose a path $\gamma(t) \in M$ with $\gamma(0) = x$ and $\dot\gamma(t) = X \in T_x M$, and define the *covariant derivative*

$$\nabla s(x) X := \nabla_X s := \left.\frac{d}{dt}\left[(P_\gamma^t)^{-1} \circ s(\gamma(t))\right]\right|_{t=0}. \tag{3.3}$$

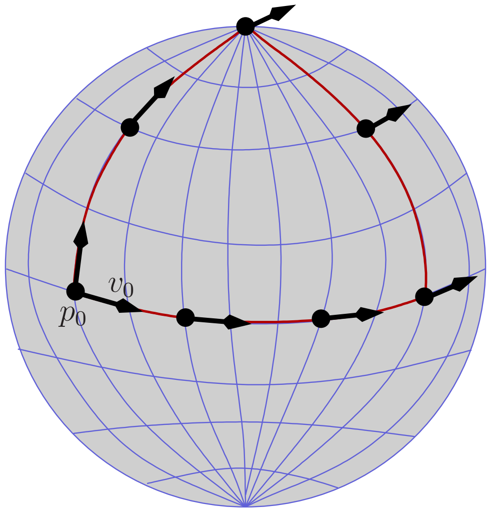

Figure 3.2: Parallel transport of a vector along a closed path in $S^2 \subset \mathbb{R}^3$ leads to a different vector upon return.

Once again we are differentiating a path of vectors in the same fiber E_x, so $\nabla s(x)X \in E_x$. We can now begin to deduce what conditions must be imposed on the isomorphisms P^t_γ: first, we must ensure that the expression (3.3) depends only on s and $X = \dot{\gamma}(0)$, not on the chosen path γ. Let us assume this for the moment. Then for any vector field $X \in \mathrm{Vec}(M)$, the covariant derivative defines another section $\nabla_X s : M \to E$. Any sensible use of the word "derivative" should require that the resulting map

$$\nabla s(x) : T_x M \to E_x$$

be linear for all x. This is not automatic; it imposes another nontrivial condition on our definition of parallel transport. It turns out that from these two requirements, we will be able to deduce the most elegant and useful definition of a connection for vector bundles.

3.2 Connections on fiber bundles

Before doing that, it helps to generalize slightly and consider an arbitrary fiber bundle $\pi : E \to M$, with standard fiber F. Now parallel transport along a path $\gamma(t) \in M$ will be defined by a smooth family of diffeomorphisms $P^t_\gamma : E_{\gamma(0)} \to E_{\gamma(t)}$, and we define covariant derivatives again by formulas (3.2) and (3.3). Now however, we are differentiating paths through the fiber $E_x \cong F$, which is generally not a vector space, so $\nabla s(x)X$ is not in the fiber itself but rather in its tangent space $T_{s(x)}(E_x) \subset T_{s(x)}E$. Remember that the total space E is itself a smooth manifold, and has its own tangent bundle $TE \to E$.

Definition 3.1. Let $\pi : E \to M$ be a fiber bundle. The *vertical bundle* $VE \to E$ is the subbundle of $TE \to E$ defined by

$$VE = \{\xi \in TE \mid \pi_*\xi = 0\}.$$

Its fibers $V_pE := (VE)_p \subset T_pE$ are called *vertical subspaces.*

Then $V_pE = T_p(E_{\pi(p)})$, so the vertical subbundle is the set of all vectors in TE that are tangent to any fiber.

Exercise 3.2. Show that $VE \to E$ is a smooth real vector bundle if $\pi : E \to M$ is a smooth fiber bundle, and the rank of $VE \to E$ is the dimension of the standard fiber F.

By the above definition, the covariant derivative defines for each section $s : M \to E$ and $x \in M$ a map

$$\nabla s(x) : T_xM \to V_{s(x)}E. \tag{3.4}$$

We shall require the definition of parallel transport in fiber bundles to satisfy two (not quite independent) conditions:

(i) The definition of $\nabla_X s$ in (3.3) is independent of γ except for the tangent vector $\dot\gamma(0) = X$.

(ii) The map $\nabla s(x) : T_xM \to V_{s(x)}E$ is linear.

Proposition 3.3. *Suppose $\pi : E \to M$ is a fiber bundle and for every path $\gamma(t) \in M$ there is a smooth family of diffeomorphisms $P_\gamma^t : E_{\gamma(0)} \to E_{\gamma(t)}$ satisfying $P_\gamma^0 = \mathrm{Id}$ and conditions (i) and (ii). Then for each $x \in M$ and $p \in E_x$, there is a unique linear injection*

$$\mathrm{Hor}_p : T_xM \to T_pE$$

such that $\mathrm{Hor}_p(\dot\gamma(0)) = \frac{d}{dt}P_\gamma^t(p)\big|_{t=0}$ for all paths with $\gamma(0) = x$. Moreover, the image of Hor_p is complementary to V_pE in T_pE.

Proof. Fix $x_0 \in M$ and $p_0 \in E_{x_0}$. For any path $\gamma(t) \in M$ with $\gamma(0) = x_0$, there is a unique vector field Y on the total space of the pullback bundle γ^*E such that for any $p \in E_{x_0}$ and any t,

$$Y(P_\gamma^t(p_0)) = \frac{d}{dt}P_\gamma^t(p_0),$$

hence we can write P_γ^t in terms of the flow φ_Y^t of Y as $P_\gamma^t = \varphi_Y^t|_{E_{x_0}} : E_{x_0} \to E_{\gamma(t)}$. The inverse of P_γ^t is then given by reversing the flow of Y, so for a section $s : M \to E$ with $s(x_0) = p_0$,

$$\nabla_{\dot\gamma(0)} s = \frac{d}{dt}\varphi_Y^{-t}(s(\gamma(t)))\bigg|_{t=0}.$$

To compute this, write $F(t_1, t_2) = \varphi_Y^{t_1}(s(\gamma(t_2)))$, so

$$\nabla_{\dot\gamma(0)} s = \frac{d}{dt} F(-t, t)\bigg|_{t=0} = -\frac{\partial F}{\partial t_1}(0,0) + \frac{\partial F}{\partial t_2}(0,0) = -Y(p_0) + Ts(\dot\gamma(0)),$$

and thus

$$\mathrm{Hor}_{p_0}(\dot\gamma(0)) = \frac{d}{dt} P_\gamma^t(p_0)\bigg|_{t=0} = Y(p_0) = Ts(\dot\gamma(0)) - \nabla_{\dot\gamma(0)} s. \tag{3.5}$$

This expression is clearly a linear function of $\dot\gamma(0)$. It is also injective since $\nabla_{\dot\gamma(0)} s \in V_{p_0}E$, and $Ts(\dot\gamma(0)) \in V_{p_0}E$ if and only if $\dot\gamma(0) = 0$, as we can see by applying π_*. The same argument shows $(\operatorname{im} \mathrm{Hor}_{p_0}) \cap V_{p_0}E = \{0\}$, and since any non-vertical vector $\xi \in T_{p_0}E \setminus V_{p_0}E$ can be written as $Ts(\dot\gamma(0))$ for some path γ and section s, clearly

$$\operatorname{im} \mathrm{Hor}_{p_0} \oplus V_{p_0}E = T_{p_0}E.$$

□

The moral is that parallel transport, if defined properly, determines for every $p \in E$ a *horizontal subspace* $H_pE := \operatorname{im} \mathrm{Hor}_p$ complementary to the vertical subspace V_pE. Conversely, it's easy to see that choosing such complimentary subspaces H_pE determines P_γ^t uniquely. This should be sufficient motivation for the following definition.

Definition 3.4. A *connection* on the fiber bundle $\pi : E \to M$ is a smooth distribution HE on the total space such that $HE \oplus VE = TE$. For any $p \in E$, the fiber $H_pE \subset T_pE$ is called the *horizontal subspace* at p.

We can now recast all of the previous concepts in terms of horizontal subspaces. Assume a connection (i.e. a horizontal subbundle) has been chosen. Then for each $x \in M$ and $p \in E_x$, the linear map $\pi_* : T_pE \to T_xM$ restricts to an isomorphism $H_pE \to T_xM$. Its inverse is called the *horizontal lift*

$$\mathrm{Hor}_p : T_xM \to H_pE.$$

A path through the total space E is called *horizontal* if it is everywhere tangent to HE. Then given $x_0 \in M$ and $p_0 \in E_{x_0}$, any path $\gamma(t) \in M$ with $\gamma(0) = x_0$ lifts uniquely to a horizontal path $\tilde\gamma(t) \in E$ with $\tilde\gamma(0) = p_0$. This path is similarly called the *horizontal lift* of γ, and its tangent vectors satisfy

$$\frac{d}{dt}\tilde\gamma(t) = \mathrm{Hor}_{\tilde\gamma(t)}(\dot\gamma(t)).$$

By considering horizontal lifts for all possible $p \in E_{x_0}$, we obtain naturally the parallel transport diffeomorphisms $P_\gamma^t : E_{x_0} \to E_{\gamma(t)}$. Finally, (3.5) yields a convenient formula for the covariant derivative with respect to any vector $X \in T_x M$,

$$\nabla_X s = Ts(X) - \mathrm{Hor}_{s(x)}(X).$$

Note that since $\pi_* Ts(X) = X$, the second term on the right is simply the projection of $Ts(X)$ to the horizontal subspace. We can express this more simply by defining the *vertical projection*

$$K : TE \to VE,$$

which maps each $T_p E$ to the vertical subspace $V_p E$ by projecting along $H_p E$. Then

$$\nabla_X s = K \circ Ts(X), \tag{3.6}$$

so the covariant derivative is literally the "vertical part" of the tangent map. For a section $s(t) \in E$ along a path $\gamma(t) \in M$, we have the analogous formula

$$\nabla_t s(t) = K(\dot{s}(t)). \tag{3.7}$$

As one would expect, it is clear from this formula that $s(t)$ is a horizontal lift of $\gamma(t)$ if and only if $\nabla_t s \equiv 0$.

The projection $K : TE \to VE$ is called a *connection map*, and it gives an equivalent definition for connections on fiber bundles.

Definition 3.5. A *connection* on the fiber bundle $\pi : E \to M$ is a smooth fiberwise linear map $K : TE \to VE$ such that $K(\xi) = \xi$ for all $\xi \in VE$.

The two definitions are related by setting $HE = \ker K$.

Exercise 3.6. Show that every smooth fiber bundle admits a connection. *Hint:* any local trivialization defines a natural connection in its neighborhood. Use a partition of unity to piece together the connection maps. (See the proof of Theorem 3.37 if you need more hints.)

Remark 3.7. The existence of connections for bundles on infinite dimensional manifolds is a far more intricate problem, because such manifolds do not generally admit smooth partitions of unity. However, more direct constructions of connections succeed in many interesting cases, such as for the "manifolds of maps" defined in [Elĭ67].

3.3 Connections on vector bundles

3.3.1 Three definitions

For a vector bundle $\pi : E \to M$, some minor changes in the previous discussion are appropriate in order to exploit the linear structure on the fibers. Most importantly, it is no longer enough for the parallel transport maps $P^t_\gamma : E_{\gamma(0)} \to E_{\gamma(t)}$ to be diffeomorphisms; they should be linear isomorphisms. We thus define a *linear connection* to be any connection on the fiber bundle $E \to M$ for which the induced parallel transport is linear. It will always be assumed that a connection on a vector bundle is a *linear* connection unless otherwise noted. We will prove the existence of such objects later, in the context of principal bundles.

To see more concretely what linearity entails, observe that for any scalar $\lambda \in \mathbb{F}$, there is a fiberwise linear map

$$m_\lambda : E \to E : v \mapsto \lambda v,$$

which is a diffeomorphism if $\lambda \neq 0$. Choose a path $\gamma(t) \in M$, label $x = \gamma(0)$, $X = \dot\gamma(0) \in T_x M$, and choose $v \in E_x$. Then for any linear connection, we have $P^t_\gamma(\lambda v) = m_\lambda(P^t_\gamma(v))$, and differentiating at $t = 0$,

$$\mathrm{Hor}_{\lambda v}(X) = (m_\lambda)_* \mathrm{Hor}_v(X).$$

This implies $H_{\lambda v}E = (m_\lambda)_* H_v E$. Though it may not be obvious just yet, this is enough of a criterion to identify linear connections. We shall prove this below, after giving two new equivalent definitions.

Definition 3.8. A *connection* on the vector bundle $\pi : E \to M$ is a smooth distribution HE on the total space such that $HE \oplus VE = TE$ and for any scalar $\lambda \in \mathbb{F}$ and $v \in E$,

$$H_{\lambda v}E = (m_\lambda)_* H_v E.$$

Observe that the vector space structure on each fiber E_x gives natural isomorphisms

$$\mathrm{Vert}_v : E_x \to V_v E : w \mapsto \frac{d}{dt}(v + tw)\bigg|_{t=0}$$

for each $v \in E_x$. It is thus appropriate to rewrite the projection $K : TE \to VE$ as a map $K : TE \to E$ that takes $T_v E$ to $E_{\pi(v)}$ and satisfies $K(\mathrm{Vert}_v(w)) = w$ for all $w \in E_{\pi(v)}$. This will be called a *connection map* for the vector bundle $\pi : E \to M$. Setting $\ker K = HE$, it is an easy exercise to verify that the following is now equivalent to Definition 3.8.

Definition 3.9. A *connection* on the vector bundle $\pi : E \to M$ is a smooth map $K : TE \to E$ such that

1. For each $v \in E$, K defines a real linear map $T_v E \to E_{\pi(v)}$.
2. $K(\mathrm{Vert}_v(w)) = w$ for all $w \in E_{\pi(v)}$.
3. For all scalars $\lambda \in \mathbb{F}$, $K \circ (m_\lambda)_* = m_\lambda \circ K$.

We now show that these new definitions are equivalent to the notion of a *linear connection* defined above. The following lemma will be of use.

Lemma 3.10. *Let V and W be real, normed vector spaces. Then any map $F : V \to W$ that is differentiable at 0 and satisfies $F(\lambda v) = \lambda F(v)$ for all scalars $\lambda \in \mathbb{R}$ and all $v \in V$ is linear.*

Proof. The key is to show that under this assumption, F is actually equal to its derivative at zero, $dF(0) : V \to W$. Clearly $F(0) = 0$, so we can write

$$F(v) = dF(0)v + |v|\eta(v)$$

for some function $\eta : V \to W$ such that $\lim_{v\to 0} \eta(v) = 0$. Then

$$\begin{aligned} F(v) = \lim_{\lambda\to 0^+} \frac{1}{\lambda} F(\lambda v) = \lim_{\lambda\to 0^+} & \frac{dF(0)\lambda v + \lambda|v|\eta(\lambda v)}{\lambda} \\ &= dF(0)v + \lim_{\lambda\to 0^+} |v|\eta(\lambda v) = dF(0)v. \end{aligned}$$

□

Remark 3.11. The vector spaces V and W need not be finite dimensional—in particular, they could be Banach spaces.

Proposition 3.12. *If $\pi : E \to M$ is a vector bundle and $K : TE \to E$ is a connection as defined above, then the induced parallel transport maps $P^t_\gamma : E_{\gamma(0)} \to E_{\gamma(t)}$ are linear (with respect to $\mathbb{F}$).*

Proof. For any path $\gamma(t) \in M$, denote $\gamma(0) = x$, $\dot\gamma(0) = X \in T_x M$, and choose any $v \in E_x$, $\lambda \in \mathbb{F}$. Denote by $\tilde\gamma(t) \in E$ the horizontal lift of γ with $\tilde\gamma(0) = v$, and similarly let $\tilde\gamma_\lambda(t) \in E$ denote the horizontal lift with $\tilde\gamma_\lambda(0) = \lambda v$. We have,

$$\begin{aligned} \frac{d}{dt} m_\lambda(\tilde\gamma(t)) &= (m_\lambda)_* \frac{d}{dt}\tilde\gamma(t) = (m_\lambda)_* \operatorname{Hor}_{\tilde\gamma(t)}(\dot\gamma(t)) \\ &= \operatorname{Hor}_{\lambda\tilde\gamma(t)}(\dot\gamma(t)) \\ &= \frac{d}{dt}\tilde\gamma_\lambda(t), \end{aligned}$$

hence $\tilde\gamma_\lambda(t) \equiv \lambda\tilde\gamma(t)$. This proves that the diffeomorphisms $P^t_\gamma : E_{\gamma(0)} \to E_{\gamma(t)}$ satisfy

$$P^t_\gamma(\lambda v) = \lambda P^t_\gamma(v) \tag{3.8}$$

for all $\lambda \in \mathbb{F}$, and by Lemma 3.10, P^t_γ is real linear. If $\mathbb{F} = \mathbb{C}$, it is clearly also complex linear since (3.8) holds for $\lambda \in \mathbb{C}$. □

As before, the covariant derivative of a section $s : M \to E$ in the direction $X \in T_x M$ is defined by

$$\nabla_X s = \frac{d}{dt} \left[(P_\gamma^t)^{-1} \circ s(\gamma(t)) \right] \Big|_{t=0}, \tag{3.9}$$

where $\dot{\gamma}(0) = X$, and $\nabla_X s$ can now be regarded as a vector in E_x. In light of the new definition for the connection map $K : TE \to E$, we have also

$$\nabla_X s = K \circ Ts(X). \tag{3.10}$$

Similar remarks apply to sections along paths.

For any smooth section $s : M \to E$ and vector field $X \in \mathrm{Vec}(M)$, $\nabla_X s$ now defines another section of E, while ∇s itself defines a smooth section of the bundle of real linear maps $\mathrm{Hom}_{\mathbb{R}}(TM, E) = T^*M \otimes_{\mathbb{R}} E$. Using (3.9) and the fact that P_γ^t is linear, one sees that the resulting map

$$\nabla : \Gamma(E) \to \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM, E))$$

is also linear (with respect to $\mathbb{F}$).

In this context, we have the following version of the product rule:

Proposition 3.13. *For any section $s : M \to E$ and smooth function $f : M \to \mathbb{F}$,*

$$\nabla(fs) = df(\cdot)s + f\nabla s, \tag{3.11}$$

where both sides are regarded as sections of $\mathrm{Hom}_{\mathbb{R}}(TM, E)$.

This also follows easily from (3.9), using the linearity of parallel transport. Formula (3.11) is called a *Leibnitz rule* for the operator ∇; such relations appear naturally in any context that involves derivatives of bilinear products. We'll see more examples in Chapter 4 when we define connections on the tensor bundles associated with E.

Prop. 3.13 has a converse of sorts, which leads to a third equivalent definition for linear connections. Suppose we have a vector bundle $\pi : E \to M$ and an $\mathbb{F}$-linear operator

$$D : \Gamma(E) \to \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM, E))$$

satisfying the Leibnitz rule $D(fs) = df \cdot s + fDs$. We denote $D_X s := Ds(x)X$ for $X \in T_x M$.

Proposition 3.14. *Given the map D above, there is a unique connection on $\pi : E \to M$ such that $D = \nabla$.*

The proof is based on the observation that any two operators satisfying the same Leibnitz rule must differ by an operator which is *tensorial*:

Lemma 3.15. *Suppose $D, D' : \Gamma(E) \to \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM, E))$ are two $\mathbb{F}$-linear operators that satisfy the Leibnitz rule (3.11) for all smooth functions $f : M \to \mathbb{F}$ and sections $s \in \Gamma(E)$. Then the operator $L : \mathrm{Vec}(M) \times \Gamma(E) \to \Gamma(E)$ defined by*

$$L(X, s) = D_X s - D'_X s$$

determines a bilinear bundle map $TM \oplus E \to E$. The map is real linear in TM and $\mathbb{F}$-linear in E.

Proof. We must verify that $L : \mathrm{Vec}(M) \times \Gamma(E) \to \Gamma(E)$ is C^∞-linear in s, i.e. that $L(X, fs) = fL(X, s)$ for all $f \in C^\infty(M, \mathbb{F})$. Indeed,

$$\begin{aligned} L(X, fs) = D_X(fs) - D'_X(fs) &= df(X)s + fD_X s - df(X)s - fD'_X s \\ &= f(D_X s - D'_X s) = fL(X, s). \end{aligned}$$

Clearly also $L(fX, s) = fL(X, s)$ for all smooth real-valued functions f. This shows that for each $x_0 \in M$, the value of $L(X, s)(x_0)$ depends only on $X(x_0)$ and $s(x_0)$. □

Proof of Prop. 3.14. Uniqueness is easy: if there is such a connection, then the resulting horizontal lift maps $\mathrm{Hor}_v : T_x M \to T_v E$ for $v \in E_x$ must satisfy

$$\mathrm{Hor}_v(X) = Ts(X) - \mathrm{Vert}_v(D_X s)$$

for all $X \in T_x M$ and $s \in \Gamma(E)$ with $s(x) = v$. To prove existence, we must verify that the right hand side of this expression gives a well defined linear map $T_x M \to T_v E$, regardless of the choice of section with $s(x) = v$.

We show this by choosing another connection $\widetilde{K}$, which induces a covariant derivative operator $\widetilde{\nabla} : \Gamma(E) \to \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM, E))$, satisfying the Leibnitz rule (3.11). By Lemma 3.15, there is a bundle map $L : TM \oplus E \to E$ such that $L(X, s) \equiv \widetilde{\nabla}_X s - D_X s$, and for $X \in T_x M$, we have

$$\begin{aligned} Ts(X) - \mathrm{Vert}_v(D_X s) &= Ts(X) - \mathrm{Vert}_v(\widetilde{\nabla}_X s) + \mathrm{Vert}_v(L(X, v)) \\ &= \widetilde{\mathrm{Hor}}_v(X) + \mathrm{Vert}_v(L(X, v)), \end{aligned}$$

where $\widetilde{\mathrm{Hor}}_v : T_x M \to T_v E$ denotes the horizontal lift map defined by $\widetilde{K}$. The right hand side now depends only on X and v, and gives a well defined linear injection $T_x M \to T_v E$. There is a unique connection K such that this map is Hor_v for each $v \in E_x$, and by construction, $\nabla = D$. □

We are now ready for the "quick and dirty" definition of linear connections that is most commonly found in modern introductions to differential or Riemannian geometry. Prop. 3.14 shows that it is equivalent to our previous two definitions.

Definition 3.16. A connection on the vector bundle $\pi : E \to M$ is an $\mathbb{F}$-linear operator

$$\nabla : \Gamma(E) \to \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM, E))$$

satisfying the Leibniz rule

$$\nabla(fs) = df(\cdot)s + f\nabla s$$

for all functions $f \in C^\infty(M, \mathbb{F})$ and sections $s \in \Gamma(E)$.

3.3.2 Christoffel symbols

Let $\pi : E \to M$ denote a vector bundle, and choose a local trivialization

$$\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{F}^m$$

over some open subset $\mathcal{U} \subset M$. Using the trivialization we can identify sections $s : \mathcal{U} \to E|_{\mathcal{U}}$ with maps $\mathcal{U} \to \mathbb{F}^m$. Then Φ determines a natural connection on $E|_{\mathcal{U}}$, for which the covariant derivative acts on sections $s : \mathcal{U} \to \mathbb{F}^m$ by $s \mapsto ds$.

Given another connection $\nabla : \Gamma(E) \to \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM, E))$, Lemma 3.15 implies that ∇ differs from this natural connection on $E|_{\mathcal{U}}$ by a bilinear bundle map

$$\Gamma_\Phi : (TM \oplus E)|_{\mathcal{U}} \to E_{\mathcal{U}};$$

that is, for any section $s \in \Gamma(E|_{\mathcal{U}})$ expressed as a map $\mathcal{U} \to \mathbb{F}^m$, we have

$$\nabla s(x)X = ds(x)X + \Gamma_\Phi(X, s(x)) \qquad \text{for } x \in \mathcal{U}. \tag{3.12}$$

Note that Γ_Φ is real linear in the first factor and $\mathbb{F}$-linear in the second. It must be emphasized that Γ_Φ is not globally defined, and it depends on the choice of trivialization. Of course we're being somewhat sloppy with notation; one can think of Γ_Φ either as a bundle map on $(TM \oplus E)|_{\mathcal{U}}$, or—since we're really working in a trivialization—as a bilinear map $TM|_{\mathcal{U}} \times \mathbb{F}^m \to \mathbb{F}^m$.

One more often sees Γ_Φ expressed in local coordinates as a set of locally defined functions with three indices. Assume $\mathcal{U}$ admits a coordinate system $(x^1, \ldots, x^n)$; this then determines a framing $(\partial_1, \ldots, \partial_n)$ of the tangent bundle $TM|_{\mathcal{U}}$, i.e. a set of linearly independent vector fields that span the tangent space at each point. There is similarly a canonical framing $(e_{(1)}, \ldots, e_{(m)})$ of $E|_{\mathcal{U}}$ determined by Φ. Then the functions $\Gamma^a_{ib} : \mathcal{U} \to \mathbb{F}$ are defined by

$$\Gamma_\Phi(\partial_i, e_{(b)}) = \Gamma^a_{ib} e_{(a)},$$

so that for any $X = X^i\partial_i \in T_xM$ and $v = v^b e_{(b)} \in E_x$, we have

$$\Gamma_\Phi(X, v) = \Gamma_\Phi(X^i\partial_i, v^b e_{(b)}) = X^i v^b \Gamma_\Phi(\partial_i, e_{(b)}) = \Gamma^a_{ib} X^i v^b e_{(a)},$$

i.e. $(\Gamma_\Phi(X,v))^a = \Gamma^a_{ib} X^i v^b$. A section $s : \mathcal{U} \to E|_{\mathcal{U}}$ can now be expressed as $s = s^a e_{(a)}$ for a set of m functions $s^a : \mathcal{U} \to \mathbb{F}$, and writing $\nabla_i := \nabla_{\partial_i}$, equation (3.12) becomes

$$(\nabla_i s)^a = \partial_i s^a + \Gamma^a_{ib} s^b. \tag{3.13}$$

In this context the functions Γ^a_{ib} are called *Christoffel symbols*. Standard treatments of general relativity usually define connections purely in terms of the Christoffel symbols, while the covariant derivative is defined by (3.13).

In order for this definition of a connection to make sense, one must have the right notion of how the symbols Γ^a_{ib} "transform" with respect to changes in coordinates and local trivializations.

Exercise 3.17. Given a bundle $\pi : E \to M$ and a sufficiently small open set $\mathcal{U} \subset M$, use a coordinate system $(x^1, \ldots, x^n)$ and a framing $(e_{(1)}, \ldots, e_{(m)})$ to identify $E|_{\mathcal{U}}$ with the trivial bundle $\mathcal{V} \times \mathbb{F}^m$, where $\mathcal{V}$ is an open subset of $\mathbb{R}^n$. Suppose Γ^a_{ib} are the corresponding Christoffel symbols for some connection ∇ on E. Then another choice of coordinates and framing can be expressed by smooth functions

$$\begin{aligned}
&\mathcal{V} \to \mathbb{R}^n : (x^1, \ldots, x^n) \mapsto (\tilde{x}^1, \ldots, \tilde{x}^n) \\
&\mathcal{V} \to \mathbb{R}^m : (x^1, \ldots, x^n) \mapsto \tilde{e}_{(1)} = (\tilde{e}^1_{(1)}, \ldots, \tilde{e}^m_{(1)}) \\
&\qquad\qquad\qquad\vdots \\
&\mathcal{V} \to \mathbb{R}^m : (x^1, \ldots, x^n) \mapsto \tilde{e}_{(m)} = (\tilde{e}^1_{(m)}, \ldots, \tilde{e}^m_{(m)})
\end{aligned}$$

Let $\widetilde{\Gamma}^a_{ib}$ denote the Christoffel symbols of ∇ with respect to $(\tilde{x}^1, \ldots, \tilde{x}^n)$ and $(\tilde{e}_1, \ldots, \tilde{e}_m)$. Derive the transformation formula

$$\widetilde{\Gamma}^a_{ib} = \frac{\partial x^j}{\partial \tilde{x}^i} \tilde{e}^c_{(b)} \Gamma^a_{jc} + \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial}{\partial x^j} \tilde{e}^a_{(b)}.$$

As a special case when $E = TM$, show that this becomes

$$\widetilde{\Gamma}^i_{jk} = \frac{\partial x^p}{\partial \tilde{x}^j} \frac{\partial x^q}{\partial \tilde{x}^k} \Gamma^i_{pq} + \frac{\partial x^p}{\partial \tilde{x}^j} \frac{\partial}{\partial x^p} \left(\frac{\partial x^i}{\partial \tilde{x}^k} \right).$$

Remark 3.18. The definitions we've stated are not quite strict enough to define connections on an infinite dimensional Banach space bundle. Suppose for instance that the base $\boldsymbol{M}$ is a Banach manifold that looks locally like the Banach space $\mathbf{X}$, and $\boldsymbol{E} \to \boldsymbol{M}$ is a bundle with fibers isomorphic to a Banach space $\mathbf{Y}$. One must now explicitly require that for any choice of smooth chart $\boldsymbol{\varphi} : \boldsymbol{\mathcal{U}} \to \mathbf{X}$ and local trivialization $\boldsymbol{\Phi} : \boldsymbol{E}|_{\boldsymbol{\mathcal{U}}} \to \boldsymbol{\mathcal{U}} \times \mathbf{Y}$, there is a smooth *Christoffel map*

$$\boldsymbol{\Gamma}_{\boldsymbol{\Phi}} : \boldsymbol{\mathcal{U}} \to \mathcal{L}(\mathbf{X} \otimes_{\mathbb{R}} \mathbf{Y}, \mathbf{Y}).$$

In the infinite dimensional case, this is stricter than simply asking for the map $(x, X, v) \mapsto \mathbf{\Gamma}_{\mathbf{\Phi}}(x, X, v)$ to be smooth; we saw an analogous situation in the definition of a Banach space bundle (see Definition 2.60). This technical requirement is needed in order that the covariant derivative should define a continuous linear map

$$\nabla : \Gamma(\boldsymbol{E}) \to \Gamma(\mathrm{Hom}_{\mathbb{R}}(T\boldsymbol{M}, \boldsymbol{E})).$$

We will continue to assume for the remainder of this discussion that all objects are finite dimensional unless otherwise noted.

3.3.3 Connection 1-forms

As an alternative to the Christoffel symbols, one can express covariant derivatives in local trivializations via matrix-valued 1-forms. If $\mathcal{U}_\alpha \subset M$ is an open subset and $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ is a trivialization, we can write any smooth section $s \in \Gamma(E)$ over $\mathcal{U}_\alpha$ via the smooth map $s_\alpha : \mathcal{U}_\alpha \to \mathbb{F}^m$ such that

$$\Phi_\alpha \circ s(x) = (x, s_\alpha(x)). \tag{3.14}$$

Then for $x \in \mathcal{U}_\alpha$ and $X \in T_x M$, the covariant derivative of s in the direction of X can always be written in the form

$$(\nabla_X s)_\alpha = ds_\alpha(X) + A_\alpha(X) s_\alpha(x), \tag{3.15}$$

where A_α is an m-by-m matrix-valued 1-form. The existence of such a 1-form is another easy consequence of Lemma 3.15; in fact, it's not hard to express A_α directly in terms of the Christoffel symbols:

Exercise 3.19. Choosing coordinates $(x^1, \dots, x^n)$, let $A_i = A_\alpha(\partial_i)$ and denote the entries of this m-by-m matrix by $(A_i)^a{}_b$. Show that $(A_i)^a{}_b = \Gamma^a_{ib}$.

We call A_α the *connection* 1-*form* for ∇ with respect to the trivialization Φ_α. This leads to yet another definition of connections that is somewhat untidy but popular in the physics world: a connection is a choice of m-by-m matrix-valued 1-forms A_α over $\mathcal{U}_\alpha$ corresponding to each local trivialization Φ_α and satisfying the appropriate transformation property with respect to change of trivialization (see the exercise below).

Exercise 3.20. If $g = g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F})$ is the transition map relating two trivializations Φ_α and Φ_β, prove the transformation formula

$$A_\alpha(X) = g^{-1} A_\beta(X) g + g^{-1} dg(X). \tag{3.16}$$

Physicists refer to (3.16) as a *gauge transformation*, alluding to the important role that connection 1-forms play in quantum field theory: in that context they are called *gauge fields*, and they serve to model elementary particles such as photons and other "gauge bosons" that mediate the fundamental forces of nature. The choice of the letter A to denote a connection form is in fact motivated by physics, where the *vector potential* of classical electromagnetic field theory (conventionally denoted by $\mathbf{A}$) can be interpreted as a connection form for a trivial Hermitian line bundle.

There is another reason to use connection 1-forms rather than Christoffel symbols when the vector bundle has extra structure. In this case it's appropriate to restrict attention to a particular class of connections, and it turns out that this restriction can be expressed elegantly via the connection forms.

Definition 3.21. Let $\pi : E \to M$ be a vector bundle with a G-structure, for some Lie group G. Then a connection ∇ on E is called *G-compatible* if all parallel transport isomorphisms respect the G-structure: this means that for any sufficiently short path $\gamma(t) \in M$, the maps $P^t_\gamma : E_{\gamma(0)} \to E_{\gamma(t)}$ can be written in a G-compatible trivialization as

$$P^t_\gamma : \mathbb{F}^m \to \mathbb{F}^m : \mathbf{v} \mapsto g(t)\mathbf{v}$$

for some smooth map $g(t) \in G$ with $g(0) = \mathbb{1}$.

The definition seems less abstract when we apply it to particular structures: e.g. for $G = \mathrm{O}(m)$ or $\mathrm{U}(m)$, the structure in question is a bundle metric, and ∇ is called a *metric connection* if all parallel transport maps are isometries. The terms *complex connection* and *symplectic linear connection* can be defined analogously.

The existence of G-compatible connections will follow from our discussion of principal bundles below. For now, we take existence as a given and examine the consequences for connection 1-forms. Suppose in particular that $\pi : E \to M$ is a vector bundle of rank m with a G-structure, for some Lie subgroup $G \subset \mathrm{GL}(m, \mathbb{F})$—as a concrete example to keep in mind, the reader may assume $G = \mathrm{O}(m)$ and the G-structure is a bundle metric. Assume $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ is a G-compatible trivialization (e.g. an orthonormal frame), ∇ is a G-compatible connection, and $\gamma(t)$ is a smooth path in M with $\gamma(0) = x \in \mathcal{U}_\alpha$. Let $s(t) \in E_{\gamma(t)}$ be a section along γ which is parallel, i.e. $\nabla_t s \equiv 0$. Then (3.15) gives

$$\dot{s}_\alpha(t) + A_\alpha(\dot{\gamma}(t))s(t) = 0. \tag{3.17}$$

Since the trivialization and connection are both G-compatible, the fact that $s(t)$ is parallel also implies we can write $s_\alpha(t) = g(t)s_\alpha(0)$ for some smooth path of matrices $g(t) \in G$ with $g(0) = \mathbb{1}$, thus $A_\alpha(\dot{\gamma}(0)) = -\dot{g}(0)$. This cannot be just any arbitrary m-by-m matrix: the tangent space $T_{\mathbb{1}}G$

is generally a proper subspace of the space of all matrices, called the *Lie algebra*

$$T_{\mathbb{1}}G = \mathfrak{g} \subset \mathbb{F}^{m\times m}$$

of the group G (see Appendix B). For example if $G = \mathrm{O}(m)$, then $\mathfrak{g} = \mathfrak{o}(m)$ is the space of antisymmetric matrices. This leads to a convenient characterization of G-compatible connections.

Proposition 3.22. *If $E \to M$ is a vector bundle with a G-structure and ∇ is a connection on E, then ∇ is G-compatible if and only if for every G-compatible trivialization Φ_α, the corresponding connection 1-form*

$$A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathbb{F}^{m\times m})$$

takes values in the Lie algebra $\mathfrak{g} \subset \mathbb{F}^{m\times m}$ of G.

Proof. The argument above proves that G-compatibility implies $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{g})$. Conversely if the latter is true and $v(t) \in E_{\gamma(t)}$ is any parallel section along a smooth path $\gamma(t) \in \mathcal{U}_\alpha$, (3.17) implies

$$\dot{v}_\alpha(t) = -A_\alpha(\dot{\gamma}(t))v_\alpha(t).$$

The result now follows from Exercise 3.23 below. □

Exercise 3.23. For any matrix Lie group $G \subset \mathrm{GL}(m, \mathbb{F})$ and a smooth path of matrices $\mathbf{A}(t) \in \mathfrak{g}$, show that the unique solution $\mathbf{\Phi}(t) \in \mathbb{F}^{m\times m}$ to the initial value problem

$$\begin{cases} \dot{\mathbf{\Phi}}(t) &= \mathbf{A}(t)\mathbf{\Phi}(t), \\ \mathbf{\Phi}(0) &= \mathbb{1} \end{cases}$$

satisfies $\mathbf{\Phi}(t) \in G$. *Hint:* show first that for any $\mathbf{A} \in \mathfrak{g}$ and $\mathbf{B} \in G$, $\mathbf{AB} \in T_{\mathbf{B}}G$; then $\mathbf{\Phi}(t)$ is an orbit of a time-dependent vector field on G.

3.3.4 Linearization of a section at a zero

The following observation is trivial but useful:

Proposition 3.24. *Suppose $\pi : E \to M$ is a smooth vector bundle and $s : M \to E$ is a smooth section with $s(x) = 0$. Then the linear map*

$$\nabla s(x) : T_xM \to E_x$$

is independent of the choice of connection.

The proof is easy if we view it in the right context: recall that every vector bundle has a preferred embedding $M \hookrightarrow E$, the *zero section*. One sees easily from the condition $H_{\lambda v}E = (m_\lambda)_* H_v E$ that horizontal subspaces in TE are always tangent to M—thus every linear connection looks identical along the zero section. Put another way, at any point $x \in M \subset E$, which we view as lying either in M or in the zero section of E, there is a natural isomorphism

$$T_x E = E_x \oplus T_x M,$$

and any connection map $K : TE \to E$ defines the projection to the first factor. Thus the expression $\nabla s(x) = K \circ ds(x)$ is invariantly defined as long as $s(x) = 0$.

We call $\nabla s(x) : T_x M \to E_x$ the *linearization* of s at $x \in s^{-1}(0)$, and since it doesn't depend on ∇, we may as well denote

$$ds(x) := \nabla s(x).$$

The definition of $ds(x)$ via a connection is often convenient, but not necessary, as the next result shows.

Proposition 3.25. *Suppose the section $s : M \to E$ has a zero at x, and we choose a trivialization on some neighborhood $x \in \mathcal{U} \subset M$ so as to identify sections with maps $f : \mathcal{U} \to \mathbb{F}^m$. Then the linearization at x is expressed in this trivialization as $df(x) : T_x M \to \mathbb{F}^m$, and the resulting map $T_x M \to E_x$ is independent of the chosen trivialization.*

Proof. Differentiation in a trivialization can be viewed simply as covariant differentiation with respect to a connection determined by the trivialization. The result then follows from Proposition 3.24. (Alternatively, one can prove this by a direct computation without mentioning connections at all. Try it if you have a moment to spare.) □

In the special case $E = TM$, there is a nice way to write the linearization without any arbitrary choices. A section in this case is a vector field $X \in \mathrm{Vec}(M)$, and it determines a flow $\varphi^t : M \to M$; these diffeomorphisms may not be globally defined if M is noncompact, but they are at least defined for t close to 0 and x in a neighborhood of any point x_0 with $X(x_0) = 0$. In particular, $\varphi^t(x_0) = x_0$ for all t, and there is a corresponding smooth family of linear maps

$$d\varphi^t(x_0) : T_{x_0} M \to T_{x_0} M.$$

These determine a differentiable path through the identity in the vector space $\mathrm{End}(T_{x_0} M)$, and as it turns out,

$$\left. \frac{d}{dt} d\varphi^t(x_0) \right|_{t=0} = dX(x_0). \tag{3.18}$$

Exercise 3.26. Use a coordinate system around x_0 to prove (3.18).

Another example of this construction is the Hessian of a smooth function $f : M \to \mathbb{R}$ at a critical point. In general, the first derivatives of f are easily characterized via the differential $df \in \Omega^1(M)$, but without a connection there is no such simple coordinate-invariant construction that describes its *second* derivatives. If a connection ∇ is chosen on T^*M, then this information is of course contained in the tensor field $\nabla df \in \Gamma(T^0_2 M)$. We observe now that for all *critical* points $x \in M$ of f, i.e. points where $df_x : T_xM \to \mathbb{R}$ is the zero map, the bilinear form

$$\nabla df_x : T_xM \times T_xM \to \mathbb{R}$$

doesn't depend on the choice of connection. We call this bilinear map the *Hessian* of f at x.

Proposition 3.27. *For any critical point x of $f \in C^\infty(M)$, the Hessian ∇df_x is symmetric, i.e. $\nabla df_x(X, Y) = \nabla df_x(Y, X)$ for all $X, Y \in T_xM$.*

Proof. Since the bilinear form doesn't depend on ∇, we can choose coordinates $(x^1, \dots, x^n)$ on a neighborhood $\mathcal{U}$ of x and define ∇ on $T^*M|_{\mathcal{U}}$ so that the sections dx^j are parallel. Then identifying T_xM with $\mathbb{R}^n$ via the coordinates, we have $\nabla f_x(\mathbf{v}, \mathbf{w}) = \mathbf{v}^{\mathrm{T}}\mathbf{H}\mathbf{w}$ where $\mathbf{H}$ is the symmetric n-by-n matrix

$$\mathbf{H} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1} & \cdots & \frac{\partial^2 f}{\partial x^n \partial x^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x^1 \partial x^n} & \cdots & \frac{\partial^2 f}{\partial x^n \partial x^n} \end{pmatrix}.$$

□

Remark 3.28. For every proof that uses coordinates there is a cleverer proof that avoids them: we'll see in Chapter 4 how to give a coordinate-free proof of Prop. 3.27 using symmetric connections.

To see why the linearization is useful, we need some basic facts about *transversality*; more details can be found in [Hir94]. Suppose M is a smooth manifold with two smooth submanifolds N_1 and N_2. We say that N_1 and N_2 are *transverse* in M, written $N_1 \pitchfork N_2$, if for every intersection point $x \in N_1 \cap N_2$,

$$T_xN_1 + T_xN_2 = T_xM.$$

The expression on the left means all vectors that can be written as $X + Y$ for $X \in T_xN_1$ and $Y \in T_xN_2$. At any point of transverse intersection, the subspace $T_xN_1 \cap T_xN_2 \subset T_xM$ has the smallest possible dimension, determined by the simple formula

$$\operatorname{codim}(T_xN_1 \cap T_xN_2) = \operatorname{codim} T_xN_1 + \operatorname{codim} T_xN_2,$$

where the *codimension* of a subspace $V \subset T_x M$ is defined as $\operatorname{codim} V = \dim T_x M - \dim V$.

Similarly, we define the codimension of a smooth submanifold $N \subset M$ by $\operatorname{codim} N = \dim M - \dim N$. The key fact about transversality is the following result, which can be proved via the implicit function theorem:

Proposition 3.29. *Suppose M is a manifold without boundary, containing two transverse submanifolds N_1 and N_2 without boundary. Then the intersection $N_1 \cap N_2$ is a smooth submanifold of M, with*

$$\operatorname{codim}(N_1 \cap N_2) = \operatorname{codim} N_1 + \operatorname{codim} N_2.$$

Thus transversely intersecting submanifolds induce a very nice structure on their intersection. Not much can be said in general about submanifolds that are not transverse, but the following basic result of differential topology permits us to avoid worrying about it much of the time.

Proposition 3.30. *Given any two smooth submanifolds $N_1 \subset M$ and $N_2 \subset M$ without boundary, one can move N_1 by an arbitrarily small perturbation so that $N_1 \pitchfork N_2$.*

One often abbreviates this by saying that *generic* submanifolds intersect transversely.[1] We refer to [Hir94] for the proof, but mention a special case in which the result is easy to visualize: suppose N_1 and N_2 are smooth curves in the plane $M = \mathbb{R}^2$. Then transversality can only fail if N_1 and N_2 have a point of tangent intersection. One can always change the direction of N_1 just slightly to kill the tangency.

Notice that transverse intersection points are impossible unless $\dim N_1 + \dim N_2 \geq \dim M$; in this case Proposition 3.30 says that one can perturb N_1 so that the two submanifolds have no intersection at all. This is the case for instance with a pair of smooth curves in $\mathbb{R}^3$. It is also the reason why airplanes generically do not crash into each other.

Now, suppose $\pi : E \to M$ is a smooth vector bundle of rank m over an n-dimensional manifold. Any smooth section $s : M \to E$ then defines a submanifold $s(M) \subset E$ of codimension m, and in particular there is the special submanifold $M \subset E$ defined by the zero section. Using this notation, we say that a section $s \in \Gamma(E)$ is *transverse to the zero section* if $s(M) \pitchfork M$.

Theorem 3.31. *A smooth section $s : M \to E$ is transverse to the zero section if and only if for every zero $x \in s^{-1}(0)$, the linearization $ds(x)$:*

[1] The word "generic" has a variety of precise meanings in different mathematical contexts, but it always refers to a situation in which one *expects* things to appear a certain way, and the alternative is somehow an exceptional case. For example, a generic pair of vectors in $\mathbb{R}^2$ is linearly independent, just as a generic matrix is invertible. Given a non-generic situation, one can always achieve the generic case by a small perturbation.

$T_xM \to E_x$ is a surjective map. In this case the zero set $s^{-1}(0) \subset M$ is a smooth submanifold, with dimension equal to $\dim_{\mathbb{R}} \ker ds(x)$ for any $x \in s^{-1}(0)$.

Proof. Identify M with the zero section so that we can treat it as a submanifold of E. Then if $s(x) = 0$, transversality is achieved if and only if

$$\operatorname{im} Ts|_{T_xM} + T_xM = T_xE.$$

Identifying T_xE with $E_x \oplus T_xM$ in the canonical way, it's equivalent that the projection $K : T_xE \to E_x$ should map $\operatorname{im} Ts|_{T_xM}$ onto E_x, which means $K \circ Ts|_{T_xM} = \nabla s(x)$ is surjective. The dimension formula is a simple exercise. □

Note that by Proposition 3.30, sections $s : M \to E$ are *generically* transverse to the zero section.

Remark 3.32. One can alternatively use a more straightforward implicit function theorem argument to show that $s^{-1}(0) \subset M$ is a smooth submanifold if and only if $ds(x)$ is surjective for all $x \in s^{-1}(0)$. A similar statement is true for smooth maps $f : \mathbb{R}^n \to \mathbb{R}^N$; this case can be reduced to that one by choosing local coordinates and trivializations.

3.4 Connections on principal bundles

Before plunging into the definitions for principal connections, let us provide some motivation. As we saw in Section 2.8, one can always use a principal bundle to encode the essential data of a vector bundle and any additional structure that's attached to it. Thus instead of dealing with the assorted variety of structures such as bundle metrics or symplectic structures that might be associated with a vector bundle, one can describe all of these in a *unified* way by identifying the structure group and defining the corresponding frame bundle. A connection on the frame bundle will induce a connection on the original bundle, with parallel transport that respects any special structure that may be present. Indeed, the same is true for more general fiber bundles with finite dimensional structure groups. Thus the construction of connections on principal bundles has implications that go well beyond the study of principal bundles themselves.

3.4.1 Definition

Let G be a Lie group, and $\pi : E \to M$ a principal G-bundle. Recall that the fibers of E have intrinsic structure in the form of a smooth fiber preserving right group action

$$E \times G \to E : (p, g) \mapsto pg,$$

which is free and transitive on each fiber. A connection on $E \to M$ is called a *principal connection* if the parallel transport diffeomorphisms $P^t_\gamma : E_{\gamma(0)} \to E_{\gamma(t)}$ are G-equivariant, that is,

$$P^t_\gamma(pg) = P^t_\gamma(p)g$$

for all $p \in E_{\gamma(0)}$ and $g \in G$.

As usual, this first definition is conceptually simple but hard to work with in practice, so we'll give some equivalent definitions. The first step is to determine the implications of G-equivariance for the horizontal subbundle $HE \subset TE$. Each $g \in G$ determines a fiber preserving diffeomorphism $R_g : E \to E : p \mapsto pg$. Choose a path $\gamma(t) \in M$ with $\gamma(0) = x$, $\dot{\gamma}(0) = X \in T_xM$, and let $p \in E_x$. Given a principal connection, the horizontal lift isomorphisms have the property

$$\begin{aligned}\mathrm{Hor}_{pg}(X) = \left.\frac{d}{dt} P^t_\gamma(pg)\right|_{t=0} &= \left.\frac{d}{dt} R_g \circ P^t_\gamma(p)\right|_{t=0} \\ &= (R_g)_* \left.\frac{d}{dt} P^t_\gamma(p)\right|_{t=0} = (R_g)_* \,\mathrm{Hor}_p(X).\end{aligned}$$

We conclude $H_{pg}E = (R_g)_* H_pE$. Conversely, any fiber bundle connection with this property defines G-equivariant parallel transport; one can prove this along the same lines as for vector bundles. The following is therefore an equivalent definition for principal connections.

Definition 3.33. A *connection* on the principal fiber bundle $\pi : E \to M$ with structure group G is a smooth distribution HE on the total space such that $HE \oplus VE = TE$ and for any $g \in G$ and $p \in E$,

$$H_{pg}E = (R_g)_* H_pE.$$

3.4.2 Global connection 1-forms

The connection map $K : TE \to VE$ for a principal G-bundle $E \to M$ can be put in a simplified form by noting that in this situation the vertical bundle $VE \to E$ is necessarily trivial; in fact, we will now show that it has a *preferred* trivialization. Recall that every Lie group G has an associated *Lie algebra* $\mathfrak{g}$, which is the tangent space T_eG with a bracket operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ induced by the group multiplication on G. There is also a so-called *exponential map*

$$\exp : \mathfrak{g} \to G$$

defined such that for each $X \in \mathfrak{g}$, $t \mapsto \exp(tX)$ is the unique Lie group homomorphism $\mathbb{R} \to G$ with $\frac{d}{dt} \exp(tX)\big|_{t=0} = X$. For matrix groups $G \subset$

$\mathrm{GL}(m, \mathbb{F})$ in particular, where $\mathfrak{g}$ is naturally a subspace of $\mathbb{F}^{m\times m}$, it turns out that $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ and $\exp(t\mathbf{A}) = e^{t\mathbf{A}}$ is defined by the familiar power series expansion. (These concepts are reviewed in Appendix B). Now if $E \to M$ is a principle G-bundle, the right action $E \times G \to E$ defines a natural bundle isomorphism

$$E \times \mathfrak{g} \to VE : (p, X) \mapsto \overline{X}(p) = \frac{d}{dt}\left(p\ \exp(tX)\right)\Big|_{t=0}. \tag{3.19}$$

The vertical vector field on E defined by $\overline{X}(p)$ is called the *fundamental vector field* determined by $X \in \mathfrak{g}$.

If $HE \subset TE$ is a principal connection, we can use the isomorphism (3.19) to rewrite the connection map $K : TE \to VE$ as a $\mathfrak{g}$-valued 1-form

$$A \in \Omega(E, \mathfrak{g}) := \Gamma(\mathrm{Hom}(TE, E \times \mathfrak{g})),$$

giving for each $p \in E$ a linear map $A_p : T_pE \to \mathfrak{g}$, such that $\ker A_p = H_pE$. Since K is a projection onto VE, A has the property $A(\overline{X}(p)) = X$ for all $p \in E$ and $X \in \mathfrak{g}$. Recall also that HE is required to satisfy $H_{pg}E = (R_g)_* H_pE$ for all $p \in E$ and $g \in G$. To express the consequence of this condition for A, we need the following fact about fundamental vector fields. Recall from Appendix B the definition of the *adjoint representation* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) : g \mapsto \mathrm{Ad}_g$, which for matrix groups takes the form

$$\mathrm{Ad}_{\mathbf{B}}(\mathbf{A}) = \mathbf{BAB}^{-1}.$$

Lemma 3.34. *If $g \in G$ and $X \in \mathfrak{g}$, then*

$$(R_g)_*\overline{X} = \overline{\mathrm{Ad}_{g^{-1}}(X)}.$$

Proof. For $p \in E$, compute

$$\begin{aligned}
((R_g)_*\overline{X})(pg) &= \frac{d}{dt} R_g(p\ \exp(tX))\Big|_{t=0} = \frac{d}{dt} p\ \exp(tX)g\Big|_{t=0} \\
&= \frac{d}{dt} pg\left(g^{-1}\exp(tX)g\right)\Big|_{t=0} = \frac{d}{dt} pg\ \exp\left(t\,\mathrm{Ad}_{g^{-1}}(X)\right)\Big|_{t=0} \\
&= \overline{\mathrm{Ad}_{g^{-1}}(X)}(pg).
\end{aligned}$$

□

Any vector $\xi \in T_pE$ can be written uniquely as $\xi = \xi_h + \overline{X}(p)$ where $\xi_h \in H_pE$ and $X \in \mathfrak{g}$. Then $A(\xi) = X$, and $H_{pg}E = (R_g)_*H_pE$ implies

$$\begin{aligned}
R_g^*A(\xi) &= A((R_g)_*(\xi_h + \overline{X}(p))) = A(\overline{\mathrm{Ad}_{g^{-1}}(X)}(pg)) = \mathrm{Ad}_{g^{-1}}(X) \\
&= \mathrm{Ad}_{g^{-1}} \circ A(\xi).
\end{aligned}$$

Conversely, it's not hard to show (Exercise 3.36 below) that any $\mathfrak{g}$-valued 1-form satisfying these conditions defines a principal connection by $HE = \ker A$. We therefore have a useful new definition for principal connections.

Definition 3.35. A *connection* on the principal fiber bundle $\pi : E \to M$ with structure group G is a smooth $\mathfrak{g}$-valued 1-form $A \in \Omega(E, \mathfrak{g})$ such that:

(i) $A(\overline{X}(p)) = X$ for all $X \in \mathfrak{g}$ and $p \in E$,

(ii) $R_g^* A = \mathrm{Ad}_{g^{-1}} \circ A$ for all $g \in G$.

Exercise 3.36. Show that if $A \in \Omega(E, \mathfrak{g})$ satisfies the conditions in Definition 3.35, then the distribution $HE = \ker \mathfrak{g}$ satisfies the conditions in Definition 3.33.

Defining connections in terms of global 1-forms has several advantages, the first of which is that proving existence is now a simple exercise with partitions of unity.

Theorem 3.37. *Every principal fiber bundle admits a connection.*

Proof. Assume $\pi : E \to M$ is a principal G-bundle and $\{(\mathcal{U}_\alpha, \Phi_\alpha)\}$ is a system of local trivializations. Since M is paracompact, we can replace $\{\mathcal{U}_\alpha\}$ with a locally finite refinement and choose a smooth partition of unity $\{\varphi_\alpha\}$. Each trivialization $\Phi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times G$ defines an obvious notion of G-equivariant parallel transport within $\mathcal{U}_\alpha$, and thus a connection $A_\alpha \in \Omega(\mathcal{U}_\alpha, \mathfrak{g})$ over $\mathcal{U}_\alpha$. We use these to define a global $\mathfrak{g}$-valued 1-form

$$A = \sum_\alpha (\varphi_\alpha \circ \pi) A_\alpha \in \Omega^1(E, \mathfrak{g}).$$

We claim that A is a connection. For $X \in \mathfrak{g}$, $x \in M$ and $p \in E_x$, we have

$$A(\overline{X}(p)) = \sum_\alpha \varphi_\alpha(x) A_\alpha(\overline{X}(p)) = \sum_\alpha \varphi_\alpha(x) X = X.$$

Likewise for $g \in G$,

$$\begin{aligned} R_g^* A = \sum_\alpha (\varphi_\alpha \circ \pi) \cdot R_g^* A_\alpha &= \sum_\alpha (\varphi_\alpha \circ \pi) \cdot \mathrm{Ad}_{g^{-1}} A_\alpha \\ &= \mathrm{Ad}_{g^{-1}} \sum_\alpha (\varphi_\alpha \circ \pi) A_\alpha = \mathrm{Ad}_{g^{-1}} A, \end{aligned}$$

proving the claim. □

3.4.3 Frame bundles and linear connections

Theorem 3.37 is more than an existence result for principal connections: it also implies the existence of compatible connections on any fiber bundle with a finite dimensional structure group. We now prove this in particular for linear connections on vector bundles.

Recall from Example 2.81 in Chapter 2 that for every vector bundle $E \to M$ with structure group G, there is an associated frame bundle $F^G E \to M$, a principal G-bundle whose fibers are spaces of preferred bases for the fibers of E. Each element of the fiber $F^G E_x$ thus defines a unique isomorphism $\mathbb{F}^m \to E_x$, so there is a smooth, fiber preserving inclusion map

$$\Psi : F^G E \to \mathrm{Hom}(M \times \mathbb{F}^m, E)$$

taking each $p \in F^G E_x$ to the corresponding isomorphism $\Psi(p) : \mathbb{F}^m \to E_x$. Treating any $g \in G$ as a linear map on $\mathbb{F}^m$, we have

$$\Psi(pg) = \Psi(p) \circ g.$$

There is also a a "vertical tangent map"

$$V\Psi : VF^G E \to \mathrm{Hom}(M \times \mathbb{F}^m, E)$$

such that for any smooth path $p(t) \in F^G E_x$,

$$V\Psi(\dot{p}(t)) = \frac{d}{dt}\Psi(p(t)),$$

where the right hand side is interpreted as the derivative of a smooth path in the vector space $\mathrm{Hom}(\mathbb{F}^m, E_x)$.

Theorem 3.38. *Suppose $\pi : E \to M$ is a smooth vector bundle, G is a Lie group and $E \to M$ is equipped with a G-structure. Then the set of G-compatible linear connections on E is in one-to-one correspondence with the set of principal connections on the frame bundle $F^G E \to M$.*

In particular, given a connection on $F^G E$ there is a unique connection on E such that for every smooth path $\gamma(t)$ in M, the parallel sections of E along γ are of the form $v(t) = \Psi(s(t))\mathbf{v}$, where $s(t)$ is a parallel section of $F^G E$ along γ and $\mathbf{v} \in \mathbb{F}^m$ is constant. This connection on E is G-compatible, and for any smooth section $s(t) \in F^G E_{\gamma(t)}$ and smooth map $\mathbf{v}(t) \in \mathbb{F}^m$, the section $\Psi(s)\mathbf{v} \in E_{\gamma(t)}$ satisfies

$$\nabla_t(\Psi(s)\mathbf{v}) = V\Psi(\nabla_t s)\mathbf{v} + \Psi(s)\partial_t \mathbf{v}. \tag{3.20}$$

Proof. If a connection on $F^G E$ is given, then for a smooth path $\gamma(t) \in M$ with $\gamma(0) = x$ and $\dot{\gamma}(t) = X \in T_x M$, we pick any frame $p \in F^G E_x$ and define a horizontal lift map by

$$\mathrm{Hor}_{\Psi(p)\mathbf{v}} : T_x M \to T_{\Psi(p)\mathbf{v}} E : X \mapsto \left.\frac{d}{dt}\Psi(P^t_\gamma(p))\mathbf{v}\right|_{t=0}$$

for every $\mathbf{v} \in \mathbb{F}^m$. Any other choice of $p \in F^G E_x$ gives the same result due to the G-equivariance of P^t_γ. The images of all the horizontal lift maps define a distribution HE, which is clearly a fiber bundle connection on E. The fact that it is G-compatible follows from the observation that each $p \in F^G E_x$ defines a G-compatible trivialization of E along γ by

$$E_{\gamma(t)} \to \{\gamma(t)\} \times \mathbb{F}^m : v \mapsto \left(\gamma(t), \left[\Psi\left(P_\gamma^t(p)\right)\right]^{-1}(v)\right);$$

in this trivialization, parallel transport in E along γ is the identity. Note that since G acts linearly on $\mathbb{F}^m$, the resulting connection on E is automatically linear.

Conversely if E has a G-compatible connection with parallel transport $Q_\gamma^t : E_{\gamma(0)} \to E_{\gamma(t)}$ along γ, there is a unique principal connection on $F^G E$ such that for each $p \in FE_x$, the expression

$$\Psi(p(t))\mathbf{v} = Q_\gamma^t \circ \Psi(p)\mathbf{v}$$

extends p to a parallel section $p(t)$ of $F^G E$ along γ.

Equation (3.20) now follows from the definition of covariant differentiation along a path: we have

$$\begin{aligned}
\nabla_t(\Psi(s)\mathbf{v})|_{t=0} &= \left.\frac{d}{dt}(Q_\gamma^t)^{-1}\left(\Psi\left(s(t)\right)\mathbf{v}(t)\right)\right|_{t=0} \\
&= \left.\frac{d}{dt}\left[\Psi\left([P_\gamma^t]^{-1}(s(t))\right)\mathbf{v}(t)\right]\right|_{t=0} \\
&= V\Psi\left(\left.\frac{d}{dt}(P_\gamma^t)^{-1}(s(t))\right|_{t=0}\right)\mathbf{v}(0) + \Psi(s(0))\left.\frac{d}{dt}\mathbf{v}(t)\right|_{t=0} \\
&= V\Psi(\nabla_t s)|_{t=0} \cdot \mathbf{v}(0) + \Psi(s(0))\dot{\mathbf{v}}(0).
\end{aligned}$$

□

Corollary 3.39. *Every smooth vector bundle with a G-structure admits a G-compatible connection.*

This proves the existence of metric connections, complex connections and linear symplectic connections on every bundle with the corresponding structure. The beauty of the principal bundle formalism is that all of these existence results follow from a single construction. We obtain also a convenient relation between the global connection 1-form $A \in \Omega^1(F^G E, \mathfrak{g})$ and the local connection forms $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{g})$ corresponding to trivializations.

Proposition 3.40. *Suppose $A \in \Omega^1(F^G E, \mathfrak{g})$ is a connection on $F^G E$. Let $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ denote any local trivialization of E, with corresponding local connection 1-form $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{g})$, and denote by $s_\alpha : \mathcal{U}_\alpha \to F^G E$ the unique local section of $F^G E$ such that*

$$\Phi_\alpha^{-1}(x, \mathbf{v}) = \Psi(s_\alpha(x))\mathbf{v}.$$

Then

$$A_\alpha = s_\alpha^* A$$

Proof. Let $x \in \mathcal{U}_\alpha$ and $X \in T_x M$. The form A_α is defined such that for any smooth map $\mathbf{v} : \mathcal{U}_\alpha \to \mathbb{F}^m$,

$$\nabla_X(\Psi(s_\alpha)\mathbf{v}) = \Psi(s_\alpha)\left(d\mathbf{v}(X) + A_\alpha(X)\mathbf{v}\right),$$

whereas (3.20) gives

$$\nabla_X(\Psi(s_\alpha)\mathbf{v}) = V\Psi(\nabla_X s_\alpha)\mathbf{v} + \Psi(s_\alpha)d\mathbf{v}(X),$$

implying $V\Psi(\nabla_X s_\alpha)\mathbf{v} = \Psi(s_\alpha)A_\alpha(X)\mathbf{v}$. Recalling from (3.19) the definition of the fundamental vector field, we have

$$\begin{aligned}\nabla_X s_\alpha &= K(Ts_\alpha(X)) = \overline{A(Ts_\alpha(X))}(s_\alpha(x)) = \overline{s_\alpha^* A(X)}(s_\alpha(x)) \\ &= \frac{d}{dt}\left[s_\alpha(x)\ \exp(ts_\alpha^* A(X))\right]\Big|_{t=0},\end{aligned}$$

and thus

$$\begin{aligned}V\Psi(\nabla_X s_\alpha)\mathbf{v} &= \frac{d}{dt}\Psi\left(s_\alpha(x)\ \exp(ts_\alpha^* A(X))\right)\mathbf{v}\Big|_{t=0} \\ &= \Psi(s_\alpha(x))\,\frac{d}{dt}\exp(ts_\alpha^* A(X))\mathbf{v}\Big|_{t=0} = \Psi(s_\alpha(x)) \circ s_\alpha^* A(X)\mathbf{v}.\end{aligned}$$

We conclude $s_\alpha^* A(X) = A_\alpha(X)$. □

Remark 3.41. One can define frame bundles not just for vector bundles but for any smooth fiber bundle $E \to M$ with a finite dimensional structure group G: the frame bundle is then a principal G-bundle whose fibers consist of preferred diffeomorphisms between fibers of E and the standard fiber F. With this notion, the existence result above generalizes nicely to a construction of G-compatible connections on any such fiber bundle. The only limitation is that this argument applies only to fiber bundles with finite dimensional structure groups—extending it beyond this setting would require considerably more analytical effort. Fortunately one can often prove existence by more direct means in particular cases of interest, e.g. symplectic fibrations (cf. [MS98]).

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