My Lunch with Arnol'd

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In Hungary I teach Civil Engineering. I lean more towards the mathematical side of the subject than to designing buildings. Just after the political changes swept over the country in the late 1980s I got a Fulbright Fellowship to visit America, to an engineering department known to have people with mathematical taste like mine. I had a good year writing papers with various American professors. One of them was Andy Ruina, with whom I became friends and had an infinite number of conversations on not quite as many topics. One recurrent theme was Andy's friend Jim Papadopoulos, a guy with academic taste but not an academic job. Through Andy, I came to respect the unseen Jim.

One day Andy told me that Jim had a simple conjecture, but that Jim was too busy with his day job, designing machines to refill laser toner cartridges, to work on trying to prove it. Jim offered through Andy, as a gift of sorts, that I could work on the problem.

Jim imagined drawing a closed curve on a thick piece of plywood. A convex curve, meaning that it had no indented places. Now cut along that line with a jigsaw and balance the plywood piece, on edge, on a flat table. Gently keep it on edge so it doesn't fall flat on the table. In mathematical language, think of this as a two-dimensional (2D) problem. This plywood is stable only in certain positions. For example a square piece of plywood is stable on all four edges. In the positions where one diagonal or the other is vertical, the plywood is in equilibrium, but it is an unstable equilibrium. A tiny push and it will fall towards lying on one of the edges. An ellipse is in stable equilibrium when horizontal and resting on one of the two flatter parts. And the same ellipse is in unstable equilibrium when balanced on either end, like an upright egg. Jim conjectured that no matter what convex shape you draw and cut out, it has at least two orientations where it is stable. The ellipse has two such positions, a triangle has three (the three flat edges), a square has four, a regular polygon has as many stable equilibrium positions as it has edges. And a circle is a degenerate special case that is in equilibrium in every orientation (none of which is stable or unstable). Jim's conjecture was that *every* shape, but for a circle, has at least as many stable positions as an ellipse has, two.

Jim's plywood conjecture was a simple idea, and it was true for every shape we could think of. Of course it is not true if you are allowed to add weights. For example, you can put a big weight in a plywood ellipse near one of the sharp ends, so the only stable configuration is standing upright, like a child's toy called the "comeback kid". We didn't allow that. We only allowed homogeneous shapes, uniform plywood.

After some days of thought and talk with Andy, Jim, and others, we found a proof that every convex piece of plywood has at least two orientations where it will stand stably. Then we generalized the idea to include things made from wire. We published the results in the respectable but not widely read *Journal of Elasticity*.

What kept bugging us was the 3D generalization. Imagine something made of solidified clay. Was it true that you could always find at least two orientations for such a thing where it would sit stably on a table? We couldn't prove this, and for good reason. Finally I found a counter-example; I found a shape that could balance stably on a table in just one position. Take a long solid cylinder and diagonally chop off one end, then at the opposite angle chop off the other end. This truncated cylinder is happy lying on the table with its long side down, but in no other position. Just one stable equilibrium. We never published this, and I stopped thinking about balancing plywood shapes, wire loops, and clay solids.

About five years later there was the International Congress on Industrial and Applied Mathematics in Hamburg. This was to be the biggest mathematics meeting ever, with over 2000 people attending. Coming from a second-world country I needed, applied for, and got a little first-world money so I could attend. The meeting had over 40 parallel sessions. At any one time of the day I had a choice of over 40 different talks I could listen to. My own talk was on something I thought was profound at that time. But it was put in the wrong session. To an outsider, math might seem like math. But either the subject is broad or mathematicians are narrow; the number of talks that any single conference attendee could hope to understand was small. Although my audience sat politely through my carefully practiced 15-minute presentation, I don't think any of the few who understood my English understood a word of my mathematics. Mine seems not to have been the only misplaced talk, I didn't understand any of the talks I went to, either. Besides thousands of these incomprehensible 15-minute talks, there were three simultaneous 45-minute invited talks each day.

But most centrally, there was one plenary talk with no simultaneous sessions. All 2000 mathematicians could attend without conflict. This plenary lecture was to be presented by no less a figure than Vladimir Igorevich Arnol'd, the man who solved Hilbert's thirteenth problem when he was a teenager and the author of countless famous articles, reviews, books, and theorems.

Like everyone else, I felt obligated to go, despite (again like everyone else) having little hope of understanding anything of this great man's work. There was a steady murmur in the room as Arnold began to speak; people chatting to their friends whom they understood rather than listening to Arnold whom they had no hope of understanding. When a talk is over my head I either switch off completely, as I did for most of the conference talks, or I try to catch a detail here or there that might fit together loosely in my mind somehow. I did the latter until my breath was taken away. Arnold's talk made excursions into various topics that I don't know about, like differential geometry and optics. But each topic ended with something about the number four. He said these topics were examples of a theorem created by the great nineteenth-century mathematician Jacobi. He said Jacobi's theorem had many applications, and that always something had to be bigger or equal to four. He covered one topic or another that would be familiar to each person in the audience, always coming back to the number four. After everyone in the audience had seen the number four appear in some problem that he or she knew something about, the murmurs of distracted conversation quieted. The giant auditorium became almost silent, with people practically holding their breath in attentiveness. Four in this problem, four in that, four in some problem or other that everyone could understand. Four, four, four.

My respect for Arnold grew. Being a brilliant mathematician is one thing. Riveting the attention of 2000 mathematicians, most of whom can't understand each other, is another. Although I didn't understand the lecture, I felt exhilarated and happy.

As I left the auditorium it suddenly struck me that Jim's plywood and wire problem might be related to Jacobi's theorem. We had proved that at least two stable equilibria existed, but this implies that there are at least four equilibria, two stable and two unstable. Like the ellipse. Arnold's four. I was so impressed with myself that I stopped dead for a minute, blocking the exit.

I had to tell this to Arnold. Maybe the number four was a coincidence, maybe not. He would know. But of course Arnold was mobbed after the talk. I realized that getting face to face with the great man might be impossible. But almost immediately I noticed a big poster. The conference organizers were advertising special lunches. For an exorbitant fee one could buy a ticket to eat with a math celebrity. Although my budget was tight and my mathematics is not at the level of Arnold, I could calculate that if I reduced my eating from two hotdogs a day to one I could afford a lunch ticket with the great Professor.

The lunch was a disaster, both from my point of view and Arnold's. The organizers had tried to maximize their profit rather than the ticket-buyers' pleasure. At the big round table with Arnold were ten eager young mathematicians. Each was carrying one or two "highly important" scientific papers which were full of "highly relevant" results they wanted to share with Arnold. He could not eat as they held out their papers and made claims about their great original contributions. And unless I was willing to butt into this noisy whining, as each of the people was doing to the others, I could not speak. I sat and tried to look attentive at the pathetic scene.

At the end of the meal Arnold finally asked me, "And what is *your* paper about?"

I said, "nothing."

"Surely you have something to ask or say," he said.

But I was depressed by the fray and said no, I had just wanted to listen. The big meeting went on day after day. I ate one hotdog a day and I went to a hundred fifteen-minute talks that I didn't understand.

On the last day I packed my suitcase and headed for the airport. The main lobby of the conference center was deserted, maintenance people were taking down posters, the buffet was closed, people were fading out. As I strolled across the big hall I noticed, next to a young Asian man, leaning on a counter near the closed buffet, Professor V. I. Arnold. The young Asian man was talking excitedly in the tone I had noted at the disastrous lunch. As I walked closer, Arnold raised his voice slightly.

"As I told you already several times, there is nothing new in what you are telling me. I published this in 1980. Look it up. I do not want to discuss this further; moreover, I have an appointment with the gentleman carrying the suitcase over there. Good-bye."

The disappointed young mathematician got up to leave and Arnold turned to me. "You wanted to talk to me, right?" Stunned that he even remembered me, but aware of the part I suddenly was supposed to play, I pretended that the discussion was expected. "You sat at the lunch table, right? You must have had a reason. What is it about? Tell me fast. I have to catch my train."

We sat down, I collected my thoughts and explained about the plywood and the wire and how they gave the number two, which really meant four. He stared off without saying a word. After five minutes I asked him if he wanted to know how we proved that the plywood had at least four equilibria. He waved me away impatiently. "Of course I know how you proved it" and then he breezily outlined the proof in a few phrases. "That is not what I am thinking about. The question is whether your result follows from the Jacobi theorem or not."

He stared off again. I reminded him of his train but he waved me away again. Looking at his enormous concentration, and not knowing what I should be thinking about, the minutes went by slowly. Finally he said, “I think the Jacobi theorem and your problem are related, but yours is certainly not an example of the other. I think there is a third theorem that includes both Jacobi’s theorem and your problem. I could tell better if I knew about the 3D version of your problem.”

I proudly described the counter-example, the single stable equilibrium of the chopped-off cylinder, but he cut me off:

“You realize of course that this is not a counter-example! The main point of your 2D result was NOT to show that there are two or more stable equilibria, but to show that there are FOUR or more equilibria altogether.” This was not the main point of our 2D result in my mind, or at least hadn’t been. But now I realized that there was a higher level of thought going on here. Four and not two. “And your cylinder has four equilibria, three of which are unstable.”

In a moment’s pondering I realized he was right. The cylinder could also balance unstably when rotated 180 degrees on its axis and also on its two ends. Four. I was stunned. “A counter-example still may exist. Send me a letter when you find a body with less than four equilibria in the three-dimensional case,” he said, “I have to catch my train. Good-bye, young man, and good luck to you!”

I returned to Hungary and my life of teaching and pretty little irrelevant problems, each important in my mind for a few months or years. It is possible that, besides the proofreader at the *Journal of Elasticity*, no-one’s eyes have ever passed over every word in our paper on plywood and wire. Ten years later Arnold’s conjecture turned out to be correct: the 3D counterexample not only existed but appeared to me as a mathematically most exciting object (see the following paper). I never saw Arnold again. Besides the number four, and four again, I still have no idea what the Jacobi theorem is about. So I will never understand the generalization of Jacobi’s theorem that V. I. Arnold imagined to encompass also our balancing plywood and wire, cooked up there in the huge convention hall in Hamburg, Germany, sitting next to me at the deserted buffet.

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