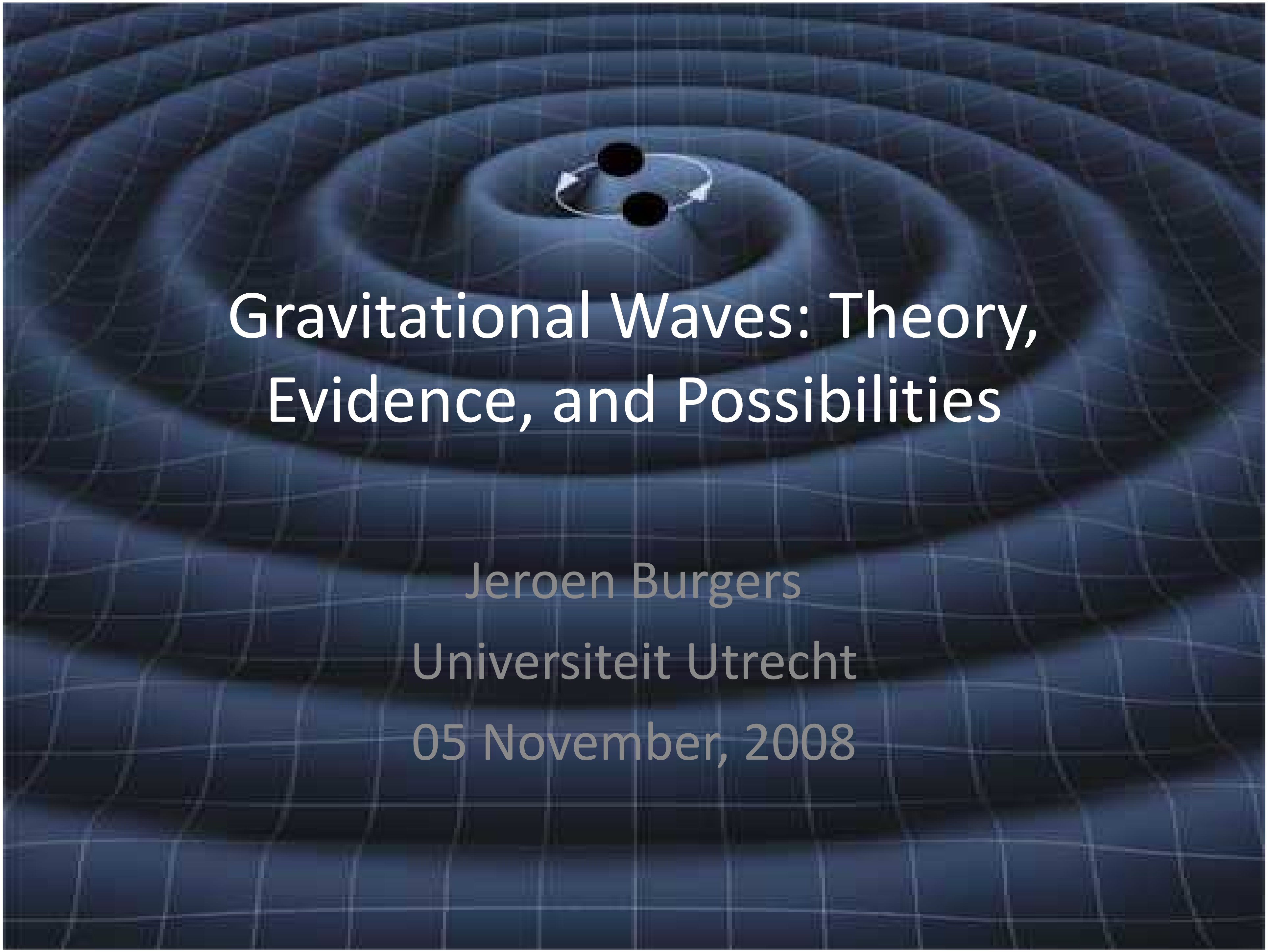


Gravitational Waves: Theory, Evidence, and Possibilities

Jeroen Burgers
Universiteit Utrecht
05 November, 2008

Contents

- Linearized Gravity
- Fixing the Gauge
- Physical Effects
- Leading Order Contribution
- General Linearized Gravity
- A Simple Application
- Indirect Evidence
- Direct Evidence
- Benefits of Gravitational Radiation

Linearized Gravity

- Gravitational waves (GW's) follow from perturbing the background metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$||h_{\mu\nu}|| << 1$$

- Result: linearized gravity

$$\{-,+,+,+\}$$

Linearized Gravity

- Introducing all the relevant quantities:
  - Christoffel Connections

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \\ &= \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right)\end{aligned}$$

  - Riemann Tensor

$$\begin{aligned}R^{\rho}_{\sigma\mu\nu} &= \partial_\mu \Gamma^{\rho}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} \\ &= \frac{1}{2}\left(\partial_\mu\partial_\sigma h^{\rho}_{\nu} + \partial_\nu\partial^\rho h_{\sigma\mu} - \partial_\mu\partial^\rho h_{\sigma\nu} - \partial_\nu\partial_\sigma h^{\rho}_{\mu}\right)\end{aligned}$$

  - Ricci Tensor

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \frac{1}{2}\left(\partial_\rho\partial_\nu h^{\rho}_{\mu} + \partial^\rho\partial_\mu h_{\rho\nu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right)$$

Linearized Gravity

- Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R$$
$$= \frac{1}{2}\left(\partial_\rho\partial_\nu h^\rho_\mu + \partial^\rho\partial_\mu h_{\rho\nu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial_\rho\partial^\sigma h^\rho_\sigma + \eta_{\mu\nu}\Box h\right)$$

- Tedious, introduce *trace-reversed perturbation*

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$$

$$\bar{h}^\mu_\mu = -h^\mu_\mu$$

- Reduced Expression

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\nu\bar{h}^\rho_\mu + \partial^\rho\partial_\mu\bar{h}_{\nu\rho} - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial^\sigma\bar{h}^\rho_\sigma\right)$$

Fixing the Gauge

- Can be simplified even more by picking a gauge.
  - Consider a general infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} + \xi^{\mu}$$

  - The metric transforms as

$$h'_{\mu\nu} = h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}$$

  - Implies for the trace-reversed metric

$$\begin{aligned} \bar{h}'_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h' \\ &= \bar{h}_{\mu\nu} - 2\partial_{(\nu}\xi_{\mu)} + \eta_{\mu\nu}\partial^{\rho}\xi_{\rho} \end{aligned}$$

Fixing the Gauge

- These infinitesimal gauge transformations allow us to impose the *de Donder* (Harmonic) gauge

$$\partial^\mu \bar{h}_{\mu\nu} = 0$$

- Applying to the Einstein Tensor

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} \Rightarrow \Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

Fixing the Gauge

- Interested in perturbations far from the source in the vacuum

$$\Box \bar{h}_{\mu\nu} = 0$$

- Wave equation! GW's travel at the speed of light. Metric can be found using Transverse-Traceless (TT) gauge

Fixing the Gauge

- We have the gauge transformation

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \xi_{\mu\nu}$$

$$\xi_{\mu\nu} = \eta_{\mu\nu}\partial_\rho \xi^\rho - \xi_{\mu,\nu} - \xi_{\nu,\mu}$$

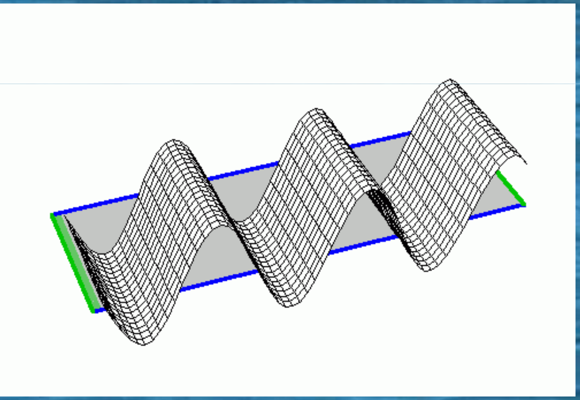
- Transverse Traceless (TT) Gauge. This transformation can be fixed such that

$$h^{00} = 0, \quad h^{0i} = 0, \quad \partial_i h^{ij} = 0, \quad h^{ii} = 0$$

Fixing the Gauge

- Consider a plane wave propagating along the z-axis. Using the TT gauge conditions, the metric is given by

$$h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\left(\left[\omega\left(t - \frac{z}{c}\right)\right]\right)$$



- Result: two polarization states.

Physical Effects

- Effect on point particles? Consider the geodesic equation for a particle at rest:

$$\begin{aligned}\frac{d^2x^i}{d\tau^2} &= -\left(\Gamma^i_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right)\\ &= -\left(\Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}\right) \Rightarrow (\Gamma^i_{00})^{TT} = 0\end{aligned}$$

- In the TT gauge, a particle at rest remains at rest. Coordinate themselves stretch.

Physical Effects

- We consider the effect on two particles separated on the x-axis. The full metric is

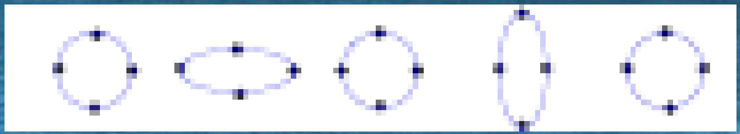
$$\begin{aligned} ds^2 &= -c^2dt^2 + dz^2 + dy^2\left[1 - h_+ \cos\left[\omega\left(t - \frac{z}{c}\right)\right]\right] \\ &+ dx^2\left[1 + h_+ \cos\left[\omega\left(t - \frac{z}{c}\right)\right]\right] + 2dxdyh_\times \cos\left[\omega\left(t - \frac{z}{c}\right)\right] \end{aligned}$$

- Thus, the proper distance changes in the x-direction in an oscillatory manner

$$\begin{aligned} S_x &= \int_0^{L_x} dx\sqrt{1 + h_+ \cos\left[\omega\left(t - \frac{z}{c}\right)\right]} \\ &\approx L_x\left(1 + \frac{h_+}{2}\cos\left[\omega t\right]\right) \end{aligned}$$

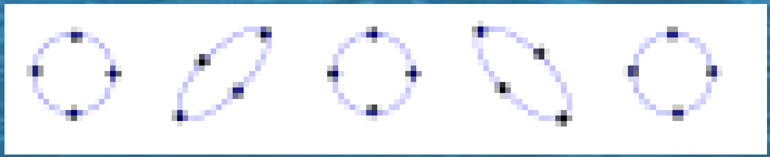
Physical Effects

- What about a ring of particles? Consider each polarization separately.
  - h+



$$\delta x(t) = \frac{h_+}{2} x_0 \sin[\omega t], \quad \delta y(t) = -\frac{h_+}{2} y_0 \sin[\omega t]$$

  - hx



$$\delta x(t) = \frac{h_\times}{2} y_0 \sin[\omega t], \quad \delta y(t) = \frac{h_\times}{2} x_0 \sin[\omega t]$$

Leading Order Contribution

- Finding the leading order contribution to the metric perturbations

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \quad \partial_\mu \bar{h}^{\mu\nu} = 0, \quad \partial_\mu T^{\mu\nu} = 0$$

- Method of Green's functions

$$G(x - x') = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|} \delta\left(t - \frac{|\vec{x} - \vec{x}'|}{c} - t'\right)$$

$$\Rightarrow \quad \bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x' \, G(x - x') T_{\mu\nu}(x')$$

Leading Order Contribution

- We introduce the projection and Lambda operators, which bring any metric into the TT gauge

$$P_{ij}(\vec{n}) = \delta_{ij} - n_i n_j$$

$$\Lambda_{ij,kl}(\vec{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$$

$$h_{ij}^{TT} = \Lambda_{ij,kl}h_{kl}$$

Leading Order Contribution

- The TT metric is then given by

$$\bar{h}_{ij}^{TT}(t,\vec{x}) = \Lambda_{ij,kl}(\vec{n})\frac{4G}{c^4}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}T_{kl}\left(t-\frac{|\vec{x}-\vec{x}'|}{c};\vec{x}'\right)$$

- We approximate the distance *r* to the source to be much larger than the size *d* of the source itself

$$|\vec{x}-\vec{x}'| = r - \vec{x}'\cdot\vec{n} + O(d^2/r)$$

$$\bar{h}_{ij}^{TT}(t,\vec{x}) = \frac{4G}{c^4}\frac{1}{r}\Lambda_{ij,kl}(\vec{n})\int_{|x'|<d} d^3x' T_{kl}\left(t-\frac{r}{c}+\frac{\vec{x}'\cdot\vec{n}}{c};\vec{x}'\right)$$

Leading Order Contribution

- Performing a Fourier decomposition of the stress-energy tensor

$$T_{kl}\left(t-\frac{r}{c}+\frac{\vec{x}'\cdot\vec{n}}{c};\vec{x}'\right) = \int \frac{d^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega,\vec{k}) \times \exp\left[-i\omega\left(t-\frac{r}{c}+\frac{\vec{x}'\cdot\vec{n}}{c}\right)+i\vec{k}\cdot\vec{x}'\right]$$

- Now assume a slowly moving source, such that

$$\omega\vec{x}'\cdot\vec{n}\sim\omega d/c\sim v/c<<1$$

- This approximation permits a Taylor expansion of the exponential

$$\begin{aligned}\bar{h}_{ij}^{TT}(t,\vec{x}) \approx\ & \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\vec{n})\Big[\int d^3x T^{kl}(t,\vec{x}) \\ & + \frac{1}{c}n_m\frac{d}{dt}\int d^3x T^{kl}(t,\vec{x})x^m \\ & + \frac{1}{2c^2}n_m n_p\frac{d^2}{dt^2}\int d^3x T^{kl}(t,\vec{x})x^m x^p+\cdots\Big]_{|t-r/c}\end{aligned}$$

Leading Order Contribution

- For convenience, we introduce the momenta of mass density

$$
\begin{aligned}
M &= \frac{1}{c^2}\int d^3x\, T^{00}(t,\vec{x}) \\
M^i &= \frac{1}{c^2}\int d^3x\, T^{00}(t,\vec{x})x^i \\
M^{ij} &= \frac{1}{c^2}\int d^3x\, T^{00}(t,\vec{x})x^i x^j
\end{aligned}
$$

- Consider the conservation law for the stress-energy tensor

$$\partial_\mu T^{\mu\nu} = 0 \qquad \nu = 0 \Rightarrow \partial_0 T^{00} + \partial_i T^{i0} = 0$$

- Leads to conservation of mass and momentum

$$\dot{M} = 0, \;\; \ddot{M}^i = 0$$

Leading Order Contribution

- The third momentum of mass density turns out to be very useful!

$$c\dot{M}^{ij} = \int_V d^3x \, x^i x^j \partial_0 T^{00} = -\int_V d^3x \, x^i x^j \partial_k T^{0k}$$
$$= \int_V d^3x \, (x^j T^{0i} + x^i T^{0j})$$

$$\ddot{M}^{ij} = 2\int_V d^3x \, T^{ij}$$

- Using this relation, the metric can be rewritten in a simpler form:

$$h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\vec{n})\ddot{M}^{kl}(t - \frac{r}{c})$$

- This is the leading order contribution to the metric.

Non-Linearized Gravity

- What if self-gravitation is not negligible?

$$\sqrt{-g}g^{\mu\nu} = n^{\mu\nu} - H^{\mu\nu}$$

- Wave metric now satisfies

$$\partial_\mu H^{\mu\nu} = 0$$

- Thus the derived equations stay the same, with the addition of a new tensor

$$\Box H^{\mu\nu} = -\frac{16\pi G}{c^4}\left[(-g)T_{\mu\nu} + \tau_{\mu\nu}\right]$$

General Linearized Gravity

- We can also choose a general background metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

- Need to distinguish between background and perturbation:
  - Case 1: The background has a typical scale L and the perturbation has a typical wavelength which is much smaller than L, such that it can be treated as a small ripple on a smooth background.

  - Case 2: The background has frequencies up to F, while those of the perturbation are much larger than F. As a result, the background can be treated as static.

General Linearized Gravity

- Result of this analysis

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}\left(\bar{T}_{\mu\nu} + t_{\mu\nu}\right)$$

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)$$

- The computed effective stress energy tensor is given by

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right\rangle$$

- Its corresponding energy density component

$$t_{00} = \frac{c^2}{32\pi G}\left\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\right\rangle = \frac{c^2}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2\right\rangle$$

General Linearized Gravity

- Thus, gravitational wave energy flux per unit area is given by

$$\frac{dE}{dtdA} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle$$

- Converting to solid angles

$$\frac{dP}{d\Omega} = \frac{r^2c^3}{32\pi G}\left\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT} \right\rangle = \frac{G}{8\pi c^5}\Lambda_{kl,mp}(\vec{n})\left\langle \frac{d}{dt}\left(\ddot{Q}_{kl}\right)\frac{d}{dt}\left(\ddot{Q}_{mp}\right) \right\rangle$$

- Where we have introduced the traceless quadrupole tensor

$$Q_{ij} = M_{ij} - \frac{1}{3}\delta_{ij}M_{kk}$$

General Linearized Gravity

- Integrating we find the total power radiated

$$P = \frac{G}{5c^5}\left\langle \frac{d}{dt}\left(\ddot{Q}_{ij}\right)\frac{d}{dt}\left(\ddot{Q}_{ij}\right)\right\rangle$$

- This is known as the *Einstein Quadrupole Formula*.

General Linearized Gravity

- Waves do not just carry away energy but also angular and linear momentum

$$\frac{dL^i}{dt} = \frac{2G}{5c^5}\epsilon^{ijk}\left\langle \ddot{Q}_{jl}\frac{d}{dt}\ddot{Q}_{lk}\right\rangle$$

$$\frac{dP^i}{dt} = -\frac{G}{8\pi c^5}\int d\omega \left(\frac{d}{dt}\ddot{Q}^{TT}_{jk}\right)\left(\partial^i \ddot{Q}^{TT}_{jk}\right)$$

- Now we can finally start calculating some results!

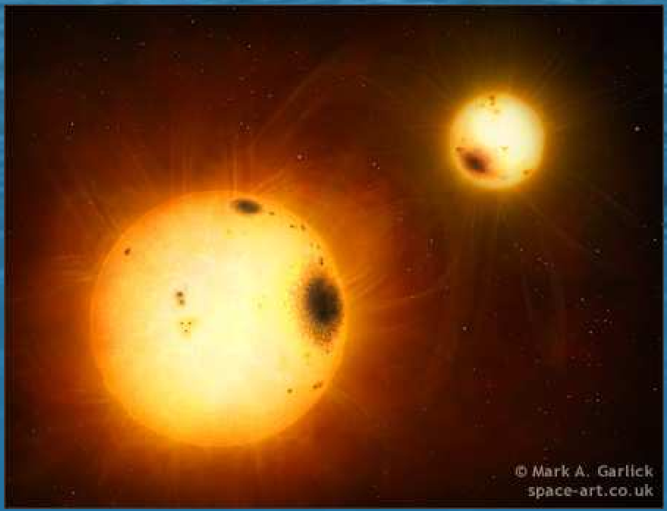
A Simple Application

- We consider a simple example - a binary, circular star system with masses *M* and *m*. In the center of mass frame:

$$\mu = \frac{Mm}{M+m}, \quad X(t) = R\cos(\omega t),$$
$$Y(t) = R\sin(\omega t), \quad Z(t) = 0$$

- The (non-zero) moments of mass density are then

$$M_{11} = \frac{1}{2}\mu R^2(1+\cos(2\omega t)),$$
$$M_{22} = \frac{1}{2}\mu R^2(1-\cos(2\omega t)),$$
$$M_{12} = \frac{1}{2}\mu R^2\sin(2\omega t)$$



A Simple Application

- Power radiated by such a system

$$P = \frac{32}{5}\frac{G\mu^2R^4\omega^6}{c^5}$$

- For the Sun-Jupiter binary, we find:

$$m_J = 2\times10^{27}\ \mathrm{kg},\ \ \mathrm{m_S} = 2\times10^{30}\ \mathrm{kg}$$
$$R = 7.8\times10^{13}\ \mathrm{cm},\ \ \omega = 1.68\times10^{-7}\ \mathrm{Hz}$$

$$P_{gw} = 5\times10^3\ \mathrm{J/s}$$

- Compare this to the amount of electromagnetic radiation emitted by the Sun

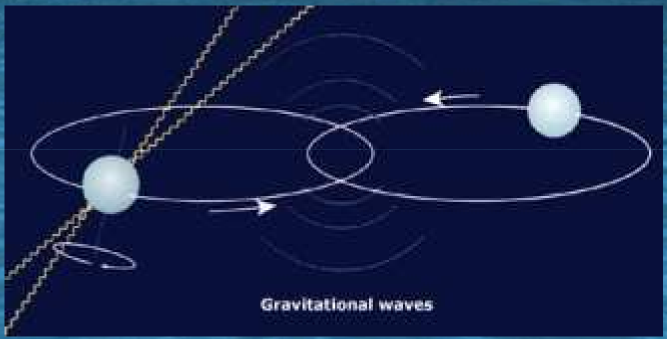
$$P_{em} = 3.9\times10^{26}\ \mathrm{J/s}$$

- And to give an idea of the magnitude of the polarizations

$$h_+(t) = \frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2\frac{(1+\cos^2\theta)}{2}\cos(2\omega t)$$
$$h_\times(t) = \frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2\cos\theta\sin(2\omega t)$$

Indirect Evidence

- In 1974, Hulse and Taylor discovered PSR B1913+16, now more commonly known as the Hulse-Taylor Binary Pulsar



- Pulsed radio signals every 59 milliseconds
- Variation over a period of every 7.75 hours led to conclusion that it is in a binary system

Indirect Evidence

- Observation of decrease in the orbital period
- According to gravitational wave theory, for an eccentric orbit the power radiated should be

$$P = \frac{32}{5}\frac{G^4\mu^2M^2}{a^5c^5}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

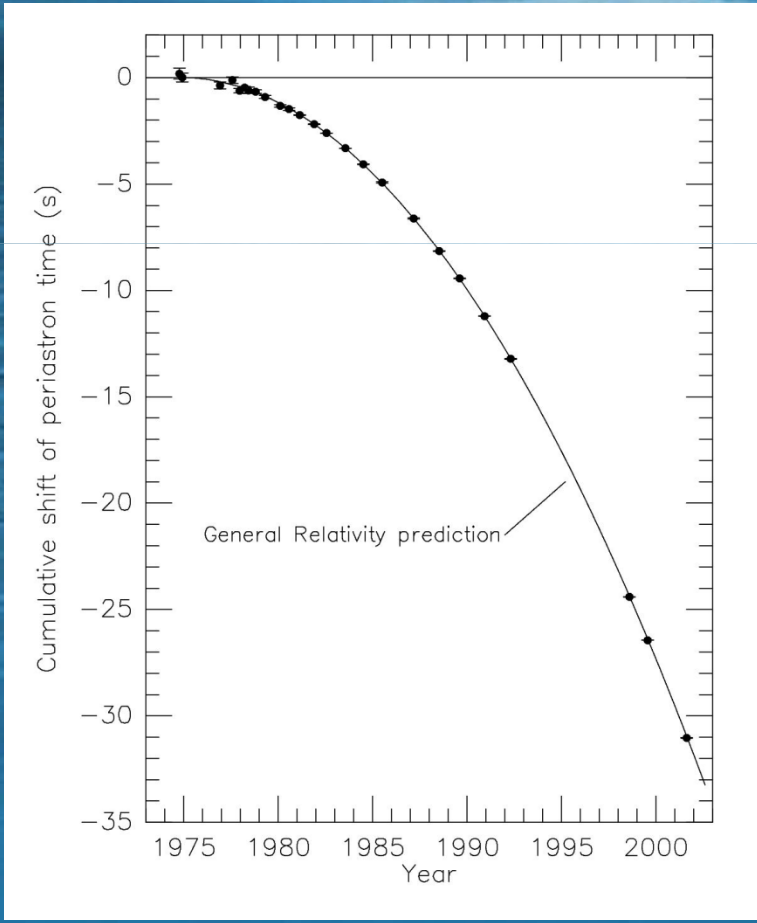
- For the Hulse-Taylor binary, we find

$$\begin{aligned} a &= 1.95\times10^{11}\ \mathrm{cm},\ \ \mathrm{m}_1 = 1.441\mathrm{M}_{\mathrm{sun}}, \\ m_2 &= 1.383M_{\mathrm{sun}},\ \ e = 0.617 \end{aligned}$$

$$P = 7.35\times10^{24}\ \mathrm{J/s}$$

Indirect Evidence

- Agreement between theory and experiment?



Good enough for the 1993 Nobel Prize at least

Direct Evidence

- As of yet, gravitational waves have not been detected directly.

- Categorize waves into four different frequency bands
  - High frequency $1 \text{ Hz} \leq f \leq 10^4 \text{ Hz}$
  - Low frequency $10^{-5} \text{ Hz} \leq f \leq 1 \text{ Hz}$
  - Very low frequency $10^{-9} \text{ Hz} \leq f \leq 10^{-7} \text{ Hz}$
  - Ultra low frequency $10^{-18} \text{ Hz} \leq f \leq 10^{-13} \text{ Hz}$

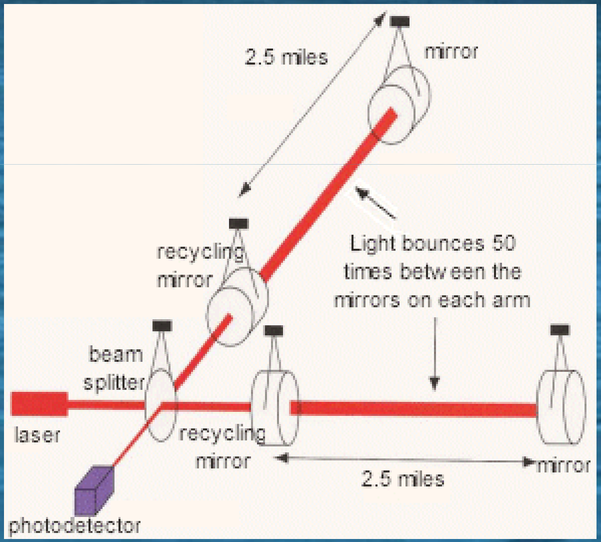
Direct Evidence

- High frequency $1\ \mathrm{Hz} \leq \mathrm{f} \leq 10^4\ \mathrm{Hz}$
  - Target band of new generation detectors such as LIGO
  - Lower limit is due to difficulty associated with mechanical coupling of detector to ground vibrations at low frequencies
  - Upper limit due to the (supposed) non-existence of a source with low mass and yet highly compact
  - Interferometers: LIGO, Virgo, GEO600, TAMA300, and ACIGA

Direct Evidence

- How do these interferometers work?

- Possible sources:
  - Coalescing Compact Binaries
  - Stellar Core Collapse
  - Periodic Emitters
  - Stochastic Backgrounds…



more on that later!

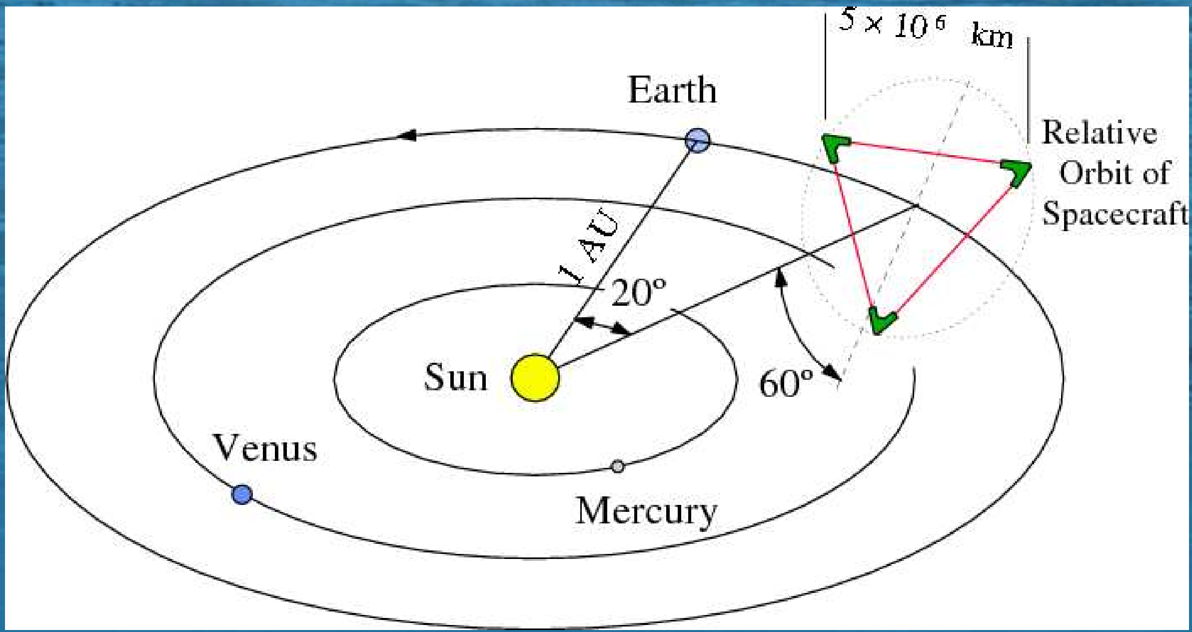
Direct Evidence

- Low Frequency Band $10^{-5}\,\mathrm{Hz} \leq \mathrm{f} \leq 1\,\mathrm{Hz}$
  - Cannot be measured with ground-based detectors
  - Many interesting sources can be investigated the in the low frequency bandwidth however
  - Examples:
    - Binary white dwarves
    - Coalescing binary systems with black holes
    - Stochastic backgrounds…

Direct Evidence

- There is a way around our terrestrial limitations! Put the detector in a nice, quiet place called *space*

- LISA (2017?)



Benefits of Gravitational Radiation

- A new form of radiation!
- Another test of the theory of general relativity
- Does not strongly interact with matter
- GW astronomy is a nearly all-sky study
- Gravitons in a GW burst are phase-coherent
- Strain *h* goes as 1/r

More (not so useful) Benefits

- Solution to the global energy crisis?

- An end to the McCulture?

Discussion

- ***How solid is the evidence supporting the existence of gravitational waves?***