MÉMOIRE SUR LES PROBABILITÉS*

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I.

I intend to treat in this Memoir two important points in the analysis of chances which do not seem yet to have been sufficiently deeply studied: the first has for object the manner of calculating the probability of events composed of simple events of which one does not know the respective probabilities; the object of the second is the influence of past events on the probability of future events, and the law according to which, in its expansion, shows us the causes which have produced them. These two objects, which have much analogy between them, depend upon a very delicate metaphysics, and the solution of the problems which are relative to them require some new artifices of analysis; they form a new branch of the theory of probabilities, of which the usage is indispensable when we wish to apply this theory to civil life. I give, relative to the first, a general method to determine the probability of any event, when we know only the law of possibility of the simple events, and, in the case where this law is unknown, I determine that of which we must make usage. The consideration of the second object leads me to speak of births: as this matter is one of the most interesting in which we are able to apply the Calculus of probabilities, I manage so to treat with all care owing to its importance, by determining what is, in this case, the influence of the observed events on those which must take place, and how, by its multiplying, they uncover for us the true ratio of the possibilities of the births of a boy and of a girl. By generalizing next these researches, I arrive to a method to determine, not only the possibilities of simple events, but yet the probability of any future event, when the observed event is very compound, whatever be moreover its nature. I give, on this occasion, the solution of some interesting problems in the natural history of man, such as the one of the more or less facility of the births of boys relatively to those of the girls in different climates: it is here chiefly that it is necessary to have a rigorous method in order to distinguish, among the observed phenomenon, those which are able to depend on chance, from those which depend on the particular causes, and in order to determine with what probability these last indicate the existence of these causes. The principal difficulty that we encounter in these researches depends on the integration of certain differential functions which

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have for factors some quantities raised to some very great powers, and of which it is necessary to have the integrals approximated by some convergent series: I dare flatter myself that the analysis of which I am to avail myself for this object may merit the attention of geometers. Finally I end this Memoir with some reflections in which I present that which the calculus of probabilities has seemed to furnish me illumination on the mean that we must choose among the results of many observations.

II.

In the analysis of chance, we intend to know the probability of composite events, following any law, of simple events of which the possibilities are given; these are able to be determined in these three ways: 1° *a priori*, when, by the like nature of the events, we see that they are possibles in a given ratio; it is in the same way that, in the game of *heads* and *tails*, if the piece that we cast into the air is homogeneous and if its two faces are entirely similar, we judge *heads* and *tails* equally possible; 2° *a posteriori*, by repeating a great number of times the experience which can bring about the event of which there is question, and by examining how many times it has happened; 3° finally, by the consideration of the grounds which can resolve for us to say on the existence of this event; if, for example, the respective skills of two players A and B are unknown, as we have no reason to suppose A stronger than B, we conclude from it that the probability of A to win a game is $\frac{1}{2}$. The first of these ways gives the absolute probability of the events; the second makes it known very nearly, as we will show in the following, and the third gives only their possibility relative to the state of our knowledge.

Each event being determined by virtue of the general laws of this universe, it is probable only relatively to us, and, for this reason, the distinction of its absolute possibility and of its relative possibility can seem imaginary; but we must observe that, among the circumstances which compete in the production of the events, there are some variables at each instant, such as the movement that the hand imprints on the dice, and it is the reunion of these circumstances which we name *chance*: it is of others of them which are constant, such as the ability of the players, the inclination of the dice to fall on one of their faces rather than on the others, etc.; these form the *absolute possibility* of the events, and knowledge of them more or less extensive forms their *relative possibility*; alone, they do not suffice to produce them; it is more necessary that they be joined to the variable circumstances of which I speak; they serve thus only to augment the probability of the events, without determining necessarily their existence.

The researches that we have made until now on the analysis of chances suppose the knowledge of the absolute possibility of the events, and, with the exception of some remarks that I have given in the Volumes VI and VII of the *Mémoires des Savants étranges*, I do not know that one has considered the case where we have only their relative possibility. This case contains a great number of interesting questions, and the greater part of the problems on the games correspond to them; we can therefore believe that if geometers had not paid particular attention, it comes from this that they have regarded it as susceptible to the same methods as the ones where we know the absolute possibility of the events; however, the essential difference of these possibilities cannot fail to influence on the results of the calculation, such that we will be exposed often to some considerable errors in employing them in the same manner: it is this of which it is easy to convince oneself by the following example.

We suppose that two players A and B, of whom the respective skills are unknown, play at any game, and we propose to determine the probability that A will win the first n games.

If the question were only of a single game, it is clear that A or B must necessarily win it, these two events are equally probable, in such a way that the probability of the first is $\frac{1}{2}$; whence, by following the ordinary rule of the analysis of chances, we conclude that the probability of A winning the first n games is $\frac{1}{2^n}$. This consequence will be exact if the probability $\frac{1}{2}$ were based on an absolute equality between the possibilities of the two events of which there is question; but there is equality only relatively to the ignorance which we have of the skills of two players, and this equality does not prevent that the one is able to be much stronger than the other. We suppose consequently that $\frac{1+\alpha}{2}$ represents the probability of the strongest player to win a game, and $\frac{1-\alpha}{2}$ that of the weakest; by naming P the probability that A will win the first n games, we will have

$$P = \frac{1}{2^n}(1+\alpha)^n \quad \text{or} \quad P = \frac{1}{2^n}(1-\alpha)^n,$$

according as A will be the strongest or the weakest: now, as we have no reason to suppose the one rather than the other, it is clear that, in order to have a true value of P, we must take the mean of the sum of the two preceding values, which gives

$$P = \frac{1}{2^{n+1}}[(1+\alpha)^n + (1-\alpha)^n].$$

By expanding this expression, we have

$$P = \frac{1}{2^n}\left[1 + \frac{n(n-1)}{1.2}\alpha^2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\alpha^4 + \cdots\right].$$

This value of P being greater than $\frac{1}{2^n}$, when n is greater than unity, we see that the inequality which can exist between the skills of the two players favors the one who wagers 1 against $2^n - 1$ that A will win the first n games, provided that we not know on which side is found the greatest skill. This remark, that I have already made elsewhere, is, if I do not delude myself, very useful in the analysis of chances, not only in that it indicates the necessity to have regard to the unknown inequality of the skills of the players, but still in that we can often determine if this inequality is favorable or contrary to the one who wagers according to the ordinary Calculus of probabilities.

III.

We consider again two players A and B, each with a given number of tokens, and playing together in a way that, at each trial, the one who loses gives a token to his adversary; we suppose that the game must end only when there remain no more tokens to one of the players, and we determine, in this case, their respective probabilities to win this game.

For this, we name generally p the skill of A, $1-p$ that of B and y_x the probability of A winning the game, when he has x tokens; it can happen at the following trial that he wins a token from B, and in this case his probability is changed to y_{x+1}; it can happen that he gives one of them to B, that which reduces his probability to y_{x-1}: now the probability of the first of these two events is p, and that of the second is $1-p$; we will have therefore the equation in finite differences

$$y_x = py_{x+1} + (1-p)y_{x-1}.$$

In order to integrate it, let $y_x = Ca^x$, we will have

$$a = pa^2 + 1 - p;$$

the two roots of this equation are $a = 1$ and $a = \frac{1-p}{p}$; hence, if C and C' represent two arbitrary constants, the complete expression of y_x will be

$$y_x = C + C'\left(\frac{1-p}{p}\right)^x.$$

In order to determine these two constants, we will observe: 1° that, x being null, we have $y_x = 0$, and that, x being equal to the total number of tokens of A and of B, we have $y_x = 1$; let n be this number, m the number of tokens of A at the beginning of the game, and consequently $n-m$ that of the tokens of B, we will have

$$0 = C + C',$$
$$1 = C + C'\left(\frac{1-p}{p}\right)^n,$$

whence we deduce

$$C = \frac{1}{1-\left(\frac{1-p}{p}\right)^n},$$
$$C' = -\frac{1}{1-\left(\frac{1-p}{p}\right)^n},$$

hence,

$$y_x = \frac{1-\left(\frac{1-p}{p}\right)^x}{1-\left(\frac{1-p}{p}\right)^n}.$$

We will have the probability y_m of A to win the game, by changing in this expression x to m, that which gives

$$y_m = \frac{1-\left(\frac{1-p}{p}\right)^m}{1-\left(\frac{1-p}{p}\right)^n};$$

and, by changing m into $n-m$, p into $1-p$, and reciprocally, we will have the probability of B to win the game, and we will find $1-y_m$ for this probability; it is this of which it is easy to assure ourselves moreover by considering that, A or B must necessarily win the game, the sum of their probabilities must be equal to unity.

Now, if we suppose the skills of two players equal, and, consequently, $p = \frac{1}{2}$, the preceding expression of y_m becomes $\frac{0}{0}$, which shows nothing; but, by differentiating the numerator and the denominator in this expression with respect to p, we find that in this case $y_m = \frac{m}{n}$, so that the probabilities of the two players A and B are in ratio of the number of their tokens: their respective wagers must therefore be in the same ratio. We examine presently the change that any inequality between their skills must cause in their lot.

Let $\frac{1+\alpha}{2}$ be the greatest and $\frac{1-\alpha}{2}$ the least ; we will change successively, in the expression of y_m, p to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$; we will have thus two values which will take place according as A will be the strongest or the weakest; the true expression of y_m will be therefore equal to the half of the sum of these two values; whence we deduce

$$y_m = \frac{1}{2}\frac{[(1+\alpha)^{n-m} + (1-\alpha)^{n-m}][(1+\alpha)^m - (1-\alpha)^m]}{(1+\alpha)^n - (1-\alpha)^n};$$

we can put this expression under this form

$$y_m = \frac{1}{2} - \frac{1}{2}(1-\alpha^2)^m \frac{[(1+\alpha)^{n-2m} + (1-\alpha)^{n-2m}]}{(1+\alpha)^n - (1-\alpha)^n}.$$

In the case of $\alpha = 0$, we just saw that $y_m = \frac{m}{n}$; so that, then,

$$(1-\alpha^2)^m \frac{[(1+\alpha)^{n-2m} + (1-\alpha)^{n-2m}]}{(1+\alpha)^n - (1-\alpha)^n} = \frac{n-2m}{n};$$

now, if we suppose m less than $\frac{n}{2}$, it is clear that, augmenting α, the fraction $\frac{(1+\alpha)^{n-2m}+(1-\alpha)^{n-2m}}{(1+\alpha)^n-(1-\alpha)^n}$ diminishes, at the same time the factor $(1-\alpha^2)^m$; we will have therefore, under the assumption of α greater than zero,

$$(1-\alpha^2)^m \frac{[(1+\alpha)^{n-2m} + (1-\alpha)^{n-2m}]}{(1+\alpha)^n - (1-\alpha)^n} = \frac{n-2m}{n} - 2h,$$

h being necessarily positive. Hence,

$$y_m = \frac{m}{n} + h;$$

whence it follows that the inequality of the skills of A and of B is favorable to the one of the two players who has the least number of tokens.

α being the same, if m and n increase in conserving always the same ratio, it is clear that

$$(1-\alpha^2)^m \frac{[(1+\alpha)^{n-2m} + (1-\alpha)^{n-2m}]}{(1+\alpha)^n - (1-\alpha)^n}$$

will become smaller, and that we can so make n and m grow, that this quantity is smaller than any given size; therefore, if the two players agree to double, to triple, etc. their tokens, their lot, which, in the case where the skills are equal, will not change for them, will become very different if there is any inequality between their skills; the probability of the one who has the least number of tokens will increase more and more, up to the point of differing infinitesimally little from $\frac{1}{2}$, and consequently from the probability of his adversary.

IV.

In general, if, in any problem relative to the two players A and B, we represent by $\frac{1+\alpha}{2}$ the skill of the strongest, and by $\frac{1-\alpha}{2}$ that of the weakest, the lot P of player A supposed the strongest will be expressed by a function of α, which, reduced to a series, will have the following form:

$$P = a + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \cdots$$

By changing α to $-\alpha$, we will have, for the expression of P, in the case where the player A is the weakest,

$$P = a - a_1\alpha + a_2\alpha^2 - a_3\alpha^3 + \cdots$$

We will have therefore the true value of P by taking the mean of the sum of the two preceding series, that which gives

$$P = a + a_2\alpha^2 + a_4\alpha^4 + \cdots$$

When α is very small, we can be satisfied with the first two terms of this series, and we have clearly

$$P = a + a_2\alpha^2;$$

we will know therefore then, by the sign of a_2, if P is greater or lesser than in the case where the skills are equal; it will be greater if a_2 is positive, and lesser if it is negative.

Since there remains in the value of P only the even powers of α, there results that the case of $\alpha = 0$ indicates always a maximum or a minimum for this value; but it is possible that it be susceptible to many maxima or minima, and it is this which will take place if the differential of P, taken with respect to α and equated to zero, gives for α one or many positive values, contained between the limits in which α is able to bounded; in this case, we will seek if the assumption of $\alpha = 0$ gives the greatest of all these maxima, or the least of all these minima; if this is, we may be assured that the lot P of A is or is not more advantageous than when the skills are equal; but, if it is not, it will be impossible to pronounce on this object, at least to know the law of possibility of the respective skills.

V.

It is easy to extend the preceding remarks to any number of players; we suppose, for example, i players A, B, C, D,... and that we propose to determine the probability P that the r players A, B, C, ...will win the first n games. It is clear that, if their skills were equal, the probability that each of the players to win a game or, what comes to the same, their respective skill will be $\frac{1}{i}$, so that the sought probability P will be $\left(\frac{r}{i}\right)^n$; but, if there exists any inequality among the skills of the players, by naming $\frac{1+\alpha}{i}$ the greatest, $\frac{1+\alpha'}{i}$ the second in order of magnitude, $\frac{1+\alpha''}{i}$ the third, and so on, we will have first

$$\alpha + \alpha' + \alpha'' + \cdots = 0,$$

because the sum of all these skills must be equal to unity.

If we name next s, s', s'', ... the different sums that we can form by adding a number r of the preceding skills, we will have as many corresponding values of P, which will be $P = s^n$, $P = s'^n$, $P = s''^n$, ...; the number of these values is equal to that of the combinations of i quantities, taken r by r, and, consequently, equal to $\frac{i(i-1)\cdots(i-r+1)}{1.2.3\ldots r}$; we will have therefore the true value of P, by dividing by this number the sum of the preceding values, that which gives

$$P = \frac{1.2.3\ldots r}{i(i-1)(i-2)\cdots(i-r+1)}(s^n + s'^n + s''^n + \cdots).$$

It is easy to see that each skill is found repeated in the sum $s + s' + s'' + s''' + \ldots$ as many times as we can combine $i-1$ quantities $r-1$ by $r-1$; whence it follows that this sum is independent of $\alpha, \alpha', \alpha'', \cdots$ and equal to

$$\frac{(i-1)(i-2)\cdots(i-r+1)}{1.2.3\ldots(r-1)}.$$

Now we will prove easily that, in this case, the sum

$$s^n + s'^n + s''^n + \cdots$$

is the least possible when $s^n = s'^n = s''^n = \cdots$, that which supposes $\alpha = \alpha' = \alpha'' = \cdots = 0$; therefore the value of P is the least when the skills of the players are equal, so that the inequality of these skills favor the one who wagers that the first n games will be won by the r players A, B, C,. . ..

It is clear that we can make some analogous remarks on the games in which one makes use of polyhedra, such as the game of dice; because, with whatever care that we had formed these polyhedra, there is encountered necessarily among their different faces some inequalities which result from the heterogeneity of the matter which one uses and from the inevitable faults in their construction. In general, these remarks hold for all events of which the possibility is unknown and can be varied within certain limits; and, if in the following we consider particularly the events of the game between many players of whom the skills are unknown, it is only to render more clear to us, by fixing the ideas on a determined object.

VI.

It is infinitely less probable that the skills of two players A and B be perfectly equal; but, at the same time as we do not know on which side is found the greatest or the least skill, we do not know equally the quantity of their difference; thus, all that which we can conclude from the preceding theory, is that the lot of such and such player is more favorable than according to the ordinary Calculus of probabilities, without that we are in a state to assign by how much it is increased.

Now, if we would know the limit and the law of possibility of the values of α, nothing would be more easy than to solve this problem exactly; because, if we name q this limit and if we represent by $\psi(\alpha)$ the probability of α, we see first that, α having necessarily to fall between 0 and q, the function $\psi(\alpha)$ must be such that we have

$$\int d\alpha\psi(\alpha) = 1,$$

the integral being taken from $\alpha = 0$ to $\alpha = q$. We will multiply therefore by $d\alpha\psi(\alpha)$ the probabilities determined by that which precedes, and, by integrating these products from $\alpha = 0$ to $\alpha = q$, we will have the sought probabilities; we will find in this manner, for the value of P in article II,

$$P = \int \frac{d\alpha\psi(\alpha)}{2^{n+1}}[(1+\alpha)^n + (1-\alpha)^n].$$

If, for example, $\psi(\alpha)$ is equal to a constant l, so that all the values of α are equally possible, the equation $\int d\alpha\psi(\alpha) = 1$ will give $l = \frac{1}{q}$, and we will have

$$P = \frac{1}{(n+1)q.2^{n+1}}[(1+q)^{n+1} + (1-q)^{n+1}].$$

The quantity α is a function of the ratio of the absolute skills of the two players; therefore instead of supposing the law of its possibility immediately known, it is much more natural to deduce it from that which represents the possibility of the absolute skill of any player. For this, we compare the skills of all the players to that of a unique player, who we will take for unity of skill; and, by representing by the abscissa x all these ratios, we imagine, raised on each point of the abscissa, some ordinates y proportional to the supposed infinite number of all the players of whom the skill is x: we will have thus a curve contained between the limits h and h', h being the least skill and h' the greatest; and it is clear that the ratio of the ordinate y to the sum of all the ordinates, or, what comes to the same, to the entire area of the curve, will express the probability that the skill of any player is x. This put, in order to conclude from it the law of possibility of the values of α, let $y = \phi(x)$, and we name a the integral $\int dx\phi(x)$, taken from $x = h$ to $x = h'$; let, moreover, x be the skill of the one of the two players A and B who is the weakest, and $x + u$ that of the strongest player; we will have

$$\frac{x}{x+u} = \frac{1-\alpha}{1+\alpha},$$

that which gives

$$x + u = \frac{1+\alpha}{1-\alpha}x.$$

Now the probability that the skill of one of the players being x, that of the other will be $x + u$, is equal to the double of the product of the probabilities of x and of $x + u$, and consequently equal to

$$\frac{2\phi(x)\phi(x+u)}{a^2} = \frac{2\phi(x)\phi\left(\frac{1+\alpha}{1-\alpha}x\right)}{a^2};$$

we will have therefore

$$2\int dx\,\frac{\phi(x)\phi\left(\frac{1+\alpha}{1-\alpha}x\right)}{a^2}$$

for the entire probability of α, the integral being taken from $x = h$ to $x = \frac{1-\alpha}{1+\alpha}h'$. As for the limit q of α, we will observe that, h being the least skill and h' the greatest, we have

$$\frac{h'}{h} = \frac{1+q}{1-q},$$

whence we conclude

$$q = \frac{h'-h}{h'+h}.$$

When the function $\phi(x)$ is unknown, it is impossible to know exactly the lot of the two players A and B, and one is reduced to choosing the most likely functions. We will now occupy ourselves on this object in the following; but before we proceed to exhibit a general method to determine the respective lot of any number of players, when we know concerning their skills only the law of their possibility: this matter presents some rather considerable difficulties of analysis, of which the solution is contained in that of the following problem.

VII.

PROBLEM. — *Let there be n variable and positive quantities $t,\, t_1,\, t_{2,} \ldots, t_{n-1}$ of which the sum is s and of which the law of possibility is known; we propose to find the sum of the products of each value that can accept a given function $\psi(t,\, t_1,\, t_2, \ldots)$ of these variables, multiplied by the probability corresponding to this value.*

Solution. — We suppose, for more generality, that the functions which express the possibility of the variables $t,\, t_1,\, t_{2,} \ldots$are discontinuous, and we represent by q the least value of t; by $\phi(t)$ the possibility of t, from $t = q$ to $t = q'$; by $\phi'(t) + \phi(t)$ its possibility, from $t = q'$ to $t = q''$; by $\phi''(t)+\phi'(t)+\phi(t)$ this possibility, from $t = q''$ to $t = q'''$, and thus in sequence to $t = \infty$. We designate next the same quantities relative to the variables $t_1,\, t_2,\, t_3, \ldots$ by the same letters, by writing at the base the numbers $1,\, 2,\, 3, \ldots$, so that $q_1,\, q_2,\, q_3, \ldots$ express the least values of $t_1,\, t_2,\, t_3, \ldots$, that $\phi_1(t_{1)}$ expresses the possibility of t_1, from $t = q_1$ to $t = q_1'$, and thus the rest; in this manner of representing the possibilities of the variables, it is clear that the function $\phi(t)$ holds from $t = q$ to $t = \infty$, that the function $\phi'(t)$ holds from $t = q'$ to $t = \infty$, thus in sequence. In order to recognize the values of $t,\, t_1,\, t_{2,} \ldots$ when these functions begin to hold, we will multiply $\phi(t)$ by l^q; $\phi'(t)$ by $l^{q'}$; $\phi_1(t_1)$ by $l^{q_1}, \ldots$; the exponents of the powers of l which multiply each function will indicate then these values; it will suffice next to suppose $l = 1$ in the last result of the calculation: it is by these very simple artifices that we owe the facility with which we are going to resolve the proposed problem.

The probability of the function $\psi\ (t,\, t_1,\, t_2, \ldots)$ is evidently equal to the product of the probabilities of $t,\, t_1,\, t_{2,} \ldots$ so that, if one substitutes for t its value $s - t_1 - t_2 - \ldots$ which gives the equation

$$t + t_1 + t_2 + \cdots = s,$$

the product of the proposed function by its probability will be

$$\text{(A)} \qquad \left\{\begin{array}{l} \psi(s-\,t_1-\,t_2-\ldots,t_1,\,t_2,\,\ldots) \\ \quad\times\,[l^q\phi(s-\,t_1-\,t_2-\ldots)+l^{q'}\phi'(s-\,t_1-\,t_2-\ldots)+\cdots] \\ \quad\times\,[l^{q_1}\phi_1(t_1)+l^{q'_1}\phi'_1(t_1)+\cdots] \\ \quad\times\,[l^{q_2}\phi_2(t_2)+l^{q'_2}\phi'_2(t_2)+\cdots] \\ \quad\times\cdots \end{array}\right.$$

We will have therefore the sum of all these products: 1 ° by multiplying the preceding quantity by dt_1, and by integrating it over all the values of which t_1 is susceptible; 2 ° by multiplying this integral by dt_2 and by integrating over all the values of which t_2 is susceptible, and thus in sequence to the last variable t_{n-1}; but these successive integrations require some particular attentions.

We consider any term of the quantity (A), such as

$$l^{q^{(i)}+q_1^{(i')}+q_2^{(i'')}+\cdots}\psi(s-\,t_1-\,t_2-\ldots)\phi^{(i)}(s-\,t_1-\,t_2-\ldots)\phi_1^{(i')}(\,t_1)\phi_2^{(i'')}(\,t_2)\cdots$$

In multiplying it by dt, it is necessary to integrate over all the possible values of t_1: now it is clear that the function $\phi^{(i)}(s-\,t_1-\,t_2-\ldots)$ holds only when t or $s-\,t_1-\,t_2-\ldots$ is equal or greater than $q^{(i)}$; the greatest value that t_1 can accept is therefore $s-\,q^{(i)}-\,t_2-\,t_3-\ldots$ Moreover, $\phi_1^{(i')}(t_1)$ holding only when t_1 is equal or greater than $q_1^{(i')}$, this quantity is the least value that t_1 can receive; it is necessary therefore to take the integral in question from $t_1=q_1^{(i')}$ to $t_1=s-\,q^{(i)}-\,t_2-t_3-\ldots$ or, what comes to the same, from $t_1-q_1^{(i')}=0$ to $t_1-\,q^{(i')}=s-\,q^{(i)}-\,q_1^{(i')}-t_2-\ldots$

We will find in the same manner that, by multiplying this new integral by dt_1, it will be necessary to integrate from $t_2=q_2^{(i'')}=0$ to

$$t_2-q_2^{(i'')}=s-\,q^{(i)}-\,q_1^{(i')}-q_2^{(i'')}-t_3-\ldots$$

By continuing to operate thus, we will arrive at a function of

$$s-\,q^{(i)}-\,q_1^{(i')}-q_2^{(i'')}-\ldots$$

in which there will remain none of the variables $t,\,t_1,\,t_2,\ldots$. This function must be rejected if $s-\,q^{(i)}-q_1^{(i')}-\ldots$ is negative; because it is clear that, in this case, the system of functions $\phi^{(i)}(t),\,\phi_1^{(i')}(\,t_1),\,\phi_2^{(i'')}(\,t_2),\,\ldots$ cannot be employed; in reality, the least values of $t_1,\,t_2,\ldots$ being, by the nature of these functions, equal to $q_1^{(i')},\,q_2^{(i'')},\ldots$ the greatest value that t can accept is $s-\,q_1^{(i')}-q_2^{(i'')}-\ldots$; hence, the greatest value of $t-\,q^{(i)}$ is $s-\,q^{(i)}-\,q_1^{(i')}-q_2^{(i'')}-\ldots$. Now the function $\phi^{(i)}(t)$ is not able to be employed when $t-\,q^{(i)}$ is positive.

Instead of rejecting the function in question, it is equal to suppose then all the terms of this function $s-\,q^{(i)}-\,q_1^{(i')}-\ldots$ constantly equal to zero; because, by considering, for example, only the three variables $t,\ t_1,\ t_2$,the last integral relative to dt_2 must be taken from $t_2 - q_2^{(i'')} = 0$ to

$$t_2 - q_2^{(i'')} = s - \ q^{(i)} - \ q_1^{(i')} - q_2^{(i'')},$$

it is clear that this integral will be null all the time that we will assume

$$s - \ q^{(i)} - \ q_1^{(i')} - q_2^{(i'')} = 0.$$

There results from what we just said a very simple method to solve the problem proposed.

Let us substitute: 1 ° in place of t, $q+u$ in $\phi(t)$, $q'+u$ in $\phi'(t)$, $q''+u$ in $\phi''(t), \ldots$; 2 ° in place of t_1, $q_1 + u_1$ in $\phi_1(t_1)$, $q_1' + u_1$ in $\phi_1'(t_1), \ldots$; 3 ° in place of t_2, $q_2 + u_2$ in $\phi_2(t_2), \ldots$, and thus in sequence, the quantities

$$\begin{aligned} & l^q \phi(t) + l^{q'} \phi'(t) + \cdots , \\ & l^{q_1} \phi(t_1) + l^{q_1'} \phi_1'(t_1) + \cdots , \\ & \cdots , \end{aligned}$$

which represent the probabilities of $t,\ t_1, \ \ldots$, will be changed: the first, into a function of u; the second, into a function of $u_1, \ldots$. We will designate these functions by $\Pi(u),\ \Pi_1(u_1),\ \Pi_2(u_2), \ldots$

Let us change next, in $\psi(t,\ t_1,\ t_2, \ldots)$, t to $k+u$, t_1 to $k_1+u_1, \ldots$, we will have a function of $u,\ u_1,\ u_2, \ldots$ that we will represent by $\Gamma(u,\ u_1,\ u_2, \ldots)$; this put, we will take the integral

$$\int du_1 \Gamma(s - u_1 - u_2 - \cdots ,\ u_1,\ u_2,\ \cdots) \Pi(s - u_1 - u_2 - \cdots) \Pi_1(u_1) \Pi_2(u_2)\ \ldots$$

from $u_1 = 0$ to $u_1 = s - u_2 - u_3 - \cdots$

We will multiply this first integral by du_2, and we will integrate from $u_2 = 0$ to $u_2 = s - u_3 - \cdots$; we will multiply this second integral by du_3, and we will integrate from $u_3 = 0$ to $u_3 = s - u_4 - \cdots$. By continuing thus, we will arrive at a function of s alone, which we will designate by $\daleth(s)$, and this function will be the sum demanded of all the values of $\psi(t,\ t_1,\ t_2, \ldots)$, multiplied by their respective probabilities; but, for this, it is necessary to take care to change, in any term multiplied by $l^{q^{(i)} + q_1^{(i')} + q_2^{(i'')} + \cdots}$, k into $q^{(i)}$, k_1 into $q_1^{(i')}$, k_2 into $q_2^{(i'')}, \ldots$; to diminish s by the exponent of l and, consequently, to write, in place of s, $s - \ q^{(i)} - \ q_1^{(i')} - q_2^{(i'')} - \cdots$; to make this last quantity equal to zero every time that it will be negative; finally, to suppose $l = 1$.

If $\Gamma(u,\ u_1,\ u_2, \ldots)$, $\Pi(u)$, $\Pi_1(u_1)$, $\Pi_2(u_2), \ldots$ are some rational and entire functions of the variables $u,\ u_1,\ u_2, \ldots$, of exponentials, of sines and cosines, all these successive integrations will be possible, because it is in the nature of these quantities to reproduce by the integrations only the quantities of the same kind; in the other cases, these integrations would not be possible, but the preceding method reduces the problem then to the quadrature of curves.

VIII.

The case of the rational and entire functions offers some simplifications which are not unuseful to exhibit. For this, let $u^i\, u_1^{i'},\, u_2^{i''},\, \ldots$ be any product of the variables $u,\, u_1,\, u_2, \ldots$; if, after having substituted for u its value $s - u_1 - u_2 - \cdots$, we multiply it by du_1, it is easy to assure ourselves that the integral

$$\int du_1 (s - u_1 - u_2 - \cdots)^i u_1^{i'} u_2^{i''} \cdots$$

taken from $u_1 = 0$ to $u_1 = s - u_2 - \cdots$ is

$$\frac{1.2.3 \ldots i.1.2.3 \ldots i'}{1.2.3.4 \ldots (i + i' + 1)} (s - u_2 - u_3 - \cdots)^{i+i'+1} u_2^{i''} \cdots ;$$

by multiplying this integral by du_2 and by integrating it from $u_2 = 0$ to $u_2 = s - u_3 - \cdots$, we will have similarly

$$\frac{1.2.3 \ldots i.1.2.3 \ldots i'.1.2.3 \ldots i''}{1.2.3.4 \ldots (i + i' + i'' + 2)} (s - u_3 - \cdots)^{i+i'+i''+2} \ldots$$

and thus in sequence; therefore, if we suppose

$$\begin{aligned} \Pi(u) &= A + Bu + Cu^2 + \cdots , \\ \Pi_1(u_1) &= A_1 + B_1 u_1 + C_1 u_1^2 + \cdots , \\ \Pi_2(u_2) &= A_2 + B_2 u_2 + C_2 u_2^2 + \cdots , \\ &\cdots \end{aligned}$$

and if we designate by $Hu^i u_1^{i'} u_2^{i''}$ any term of

$$\Gamma(u,\, u_1,\, u_2, \ldots),$$

the part corresponding to ⅂(s) will be

$$\text{(B)} \qquad \left\{ \begin{aligned} &1.2.3 \ldots i.1.2.3 \ldots i'.1.2.3 \ldots i'' \ldots H s^{n+i+i'+i''+\cdots-1} \\ &\quad \times [A + (i+1)Bs + (i+1)(i+2)Cs^2 + \cdots] \\ &\quad \times [A_1 + (i'+1)B_1 s + (i'+1)(i'+2)C_1 s^2 + \cdots] \\ &\quad \times [A_2 + (i''+1)B_2 s + (i''+1)(i''+2)C_2 s^2 + \cdots] \\ &\quad \times \cdots , \end{aligned} \right.$$

provided that, in the development of this quantity, in place of any one power c of s, we write $\frac{s^c}{1.2.3 \ldots c}$.

We will have next the corresponding part of the entire sum of the values of $\psi(t,\, t_1,\, t_2, \ldots)$, multiplied by their respective probabilities by changing any term, such as $H\lambda l^\mu s^c$, into $H\lambda(s - \mu)^c$, and by substituting into H, in place of k, the part of the exponent μ which is relative to t; in place of k_1, the part relative to t_1, and thus the rest.

If, in formula (B), we suppose $H = 1$ and $0 = i = i' = i'' = \cdots$, we will have the sum of the values of unity, multiplied by their respective probabilities: now it is clear that this sum, being nothing but the sum of all the combinations in which the equation

$$t + t_1 + t_2 + \cdots = s$$

holds, multiplied by their probabilities, expresses consequently the possibility of this equation itself. If, in the preceding hypothesis, we suppose moreover that the law of possibility is the same for the first r variables $t,\ t_1,\ \ldots,\ t_{r-1}$, and that for the last $n - r$ it is again the same, but other than for the first, we will have

$$\begin{aligned} A &= A_1 = \cdots = A_{r-1}, \\ B &= B_1 = \cdots = B_{r-1}, \\ &\cdots \\ A_r &= A_{r+1} = \cdots = A_{n-1}, \\ B_r &= B_{r+1} = \cdots = B_{n-1}, \\ &\cdots \end{aligned}$$

and formula (B) will be changed into this

$$\text{(C)} \qquad \begin{cases} s^{n-1} \times (A + Bs + 2Cs^2 + \cdots)^{n-r} \\ \qquad\quad \times (A_r + B_r s + 2C_r s^2 + \cdots)^r; \end{cases}$$

this formula will serve to determine the probability that the sum of the errors of any number of observations of which the law of facility is known will be contained in the given limits, this which can be useful in many circumstances, and particularly when the question is to predict the result of any number of observations. As this problem is besides the most simple to which we can apply the preceding method, it is quite proper to clarify it, and, in this view, we are going to consider the following examples.

IX.

We suppose $n - 1$ observations of which the errors are able to extend from $-h$ to $+g$ and that, by naming z the error of the first, its facility is expressed by $a + bz + cz^2$; we suppose next that this facility is the same for the errors $z_1,\ z_2,\ \ldots,\ z_{n-2}$ of the other observations, and we seek the probability that the sum of the errors of these observations will be contained between the limits p and $p + e$.

If we make

$$z = t - h, \quad z_1 = t_1 - h, \quad \ldots, \quad z_{n-2} = t_{n-2} - h,$$

it is clear that $t,\ t_1,\ t_2,\ \ldots$ will be positive and may be extended from zero to $h + g$; moreover, we will have

$$z + z_1 + z_2 + \cdots + z_{n-2} = t + t_1 + t_2 + \cdots + t_{n-2} - (n-1)h.$$

Therefore, the greatest value of the sum $z + z_1 + \cdots + z_{n-1}$ being, by assumption, equal to $p+e$, and the least being equal to p, the greatest value of $t+t_1+t_2+\cdots+t_{n-2}$ will be $(n-1)h+p+e$, and the least will be $(n-1)h+p$; by making thus

$$(n-1)h+p+e = s \quad \text{and} \quad t+t_1+t_2+\cdots+t_{n-2} = s - t_{n-1},$$

t_{n-1} will always be positive and may be extended from zero to e. This put, if we apply to this case the formulas of the two preceding articles, we will have

$$q = 0, \quad q' = f + g;$$

besides, the law of facility of error z being $a + bz + cz^2$, we will conclude from it the law of facility of t, by changing z to $t-h$; let

$$a' = a - bh + ch^2, \quad b' = b - 2ch,$$

we will have

$$a' + b't + ct^2$$

for this facility: this will be therefore the function $\phi(t)$; but, since, from $t = h+g$ to $t = \infty$, the facility of the values of t is null by hypothesis, we will have

$$\phi'(t) + \phi(t) = 0,$$

that which gives

$$\phi'(t) = -(a' + b't + ct^2);$$

therefore, if we make

$$\begin{aligned} a'' &= a' + b'(h+g) = c(h+g)^2, \\ b'' &= b' + 2c(h+g), \end{aligned}$$

the quantity that we have named $\Pi(u)$ in article VII will be here

$$a' + b'u + cu^2 - l^{h+g}(a'' + b''u + cu^2),$$

and we will have

$$\Pi_1(u_1), \quad \Pi_2(u_2), \quad \cdots, \quad \Pi_{n-2}(u_{n-2}),$$

by changing, in this quantity, u successively into $u_1, u_2, \ldots, u_{n-2}$.

As for the variable t_{n-1}, we will observe that the possibility of the equation

$$z + z_1 + \cdots + z_{n-2} = \mu$$

being, whatever be μ, equal to the product of the possibilities of $z, z_1, \ldots, z_{n-2}$ the possibility of the equation

$$t + t_1 + t_2 + \cdots + t_{n-2} = s - t_{n-1}$$

will be equal to the product of the possibilities of $t,\ t_1,\ \ldots,\ t_{n-2}$; but this same possibility is evidently equal to the product of the possibilities of $t,\ t_1,\ \ldots,\ t_{n-1}$. The law of possibility of t_{n-1} is therefore constant and equal to unity, and, since this variable must extend only from $t_{n-1} = 0$ to $t_{n-1} = e$, we will have

$$q_{n-1} = 0, \quad q'_{n-1} = e, \quad \phi_{n-1}(t_{n-1}) = 1, \quad \phi'_{n-1}(t_{n-1}) + \phi_{n-1}(t_{n-1}) = 0,$$

hence

$$\phi'_{n-1}(t_{n-1}) = -1,$$

whence it is easy to conclude

$$\Pi_{n-1}(u_{n-1}) = 1 - l^e;$$

formula (C) of the preceding article will be changed consequently into this

$$s^{n-1}[a' + b's + 2cs^2 - l^{h+g}(a'' + b''s + 2cs^2)]^{n-1}(1 - l^e).$$

Let

$$\begin{aligned}
&(a' + b's + 2cs^2)^{n-1} &&= a^{(1)} + b^{(1)}s + c^{(1)}s + \cdots,\\
&(a' + b's + 2cs^2)^{n-2}(a'' + b''s + 2cs^2) &&= a^{(2)} + b^{(2)}s + c^{(2)}s + \cdots,\\
&(a' + b's + 2cs^2)^{n-3}(a'' + b''s + 2cs^2)^2 &&= a^{(3)} + b^{(3)}s + c^{(3)}s + \cdots,\\
&\cdots &&\cdots
\end{aligned}$$

and this last formula will take the following form

$$\begin{aligned}
&a^{(1)}s^{n-1} + b^{(1)}s^n + c^{(1)}s^{n+1} + \cdots\\
&- l^e(a^{(1)}s^{n-1} + b^{(1)}s^n + c^{(1)}s^{n+1} + \cdots)\\
&- (n-1)l^{h+g}(a^{(2)}s^{n-1} + b^{(2)}s^n + c^{(2)}s^{n+1} + \cdots)\\
&+ (n-1)l^{h+g+e}(a^{(2)}s^{n-1} + b^{(2)}s^n + c^{(2)}s^{n+1} + \cdots)\\
&+ \frac{(n-1)(n-2)}{1.2}l^{2h+2g}(a^{(3)}s^{n-1} + \cdots)\\
&- \frac{(n-1)(n-2)}{1.2}l^{2h+2g+e}(a^{(3)}s^{n-1} + \cdots)\\
&- \cdots;
\end{aligned}$$

we will conclude the sought probability by changing in it any term such as $\lambda l^\mu s^c$ into

$\frac{\lambda(s-\mu)^c}{1.2.3...c}$, which gives, for this probability, the following expression

$$\frac{1}{1.2.3\ldots(n-1)}\left\{\begin{array}{l} a^{(1)}[s^{n-1}-(s-e)^{n-1}]+\dfrac{b^{(1)}}{n}[s^n-(s-e)^n] \\ \quad +\dfrac{c^{(1)}}{n(n+1)}[s^{n+1}-(s-e)^{n+1}]+\cdots \\ \quad -(n-1)\left\{a^{(2)}\left[(s-h-g)^{n-1}-(s-h-g-e)^{n-1}\right]\right. \\ \quad \left. +\dfrac{b^{(2)}}{n}\left[(s-h-g)^n-(s-h-g-e)^n\right]+\cdots\right\} \\ \quad +\dfrac{(n-1)(n-2)}{1.2}\left\{a^{(3)}\left[(s-2h-2g)^{n-1}\right.\right. \\ -\left.\left.(s-2h-2g-e)^{n-1}\right]+\cdots\right\} \\ \quad -\cdots \end{array}\right\},$$

by observing to reject the terms multiplied by $(s-\mu)^c$, in which μ is greater than s. We can, by means of this formula, resolve a problem that I have proposed to myself elsewhere, on the inclination of the orbits of the comets; by assuming all the inclinations to the ecliptic equally possible, the question is to determine the probability that the mean inclination of the orbits of $n-1$ comets will be contained within the limits θ and θ' or, what comes to the same, that the sum of their inclinations will be contained within the limits $(n-1)\theta$ and $(n-1)\theta'$. By naming $t,\ t_1,\ \ldots,\ t_{n-2}$ these inclinations, as they can be extended from zero to 90°, we will have

$$f=0 \quad \text{and} \quad g=90^\circ \text{ or } \frac{\pi}{2},$$

π expressing the ratio of the semi-circumference to the radius; moreover, their possibility in this interval being constant, the function $a'+b't+ct^2$ is reduced to the constant a', whence it is easy to conclude

$$a^{(1)}=a'^{n-1}=a^{(2)}=a^{(3)}=\cdots, \quad 0=b^{(1)}=b^{(2)}=\cdots, \quad 0=c^{(1)}=c^{(2)}=\cdots$$

Moreover, the value of t being necessarily contained in the limits 0 and $\frac{\pi}{2}$, $\int a'dt=1$, the integral being taken for the entire extent of these limits, whence we deduce $a'=\frac{2}{\pi}$, the preceding formula will give thus for the demanded probability

$$\frac{2^{n-1}}{1.2.3\ldots(n-1)\pi^{n-1}}\left\{\begin{array}{l} s^{n-1}-(s-e)^{n-1}-(n-1)\left[\left(s-\dfrac{\pi}{2}\right)^n-\left(s-e-\dfrac{\pi}{2}\right)^{n-1}\right] \\ \quad +\dfrac{(n-1)(n-2)}{1.2}[(s-\pi)^{n-1}-(s-e-\pi)^{n-1}] \\ \quad -\dfrac{(n-1)(n-2)(n-3)}{1.2.3}\left[\left(s-\dfrac{3}{2}\pi\right)^{n-1}-\left(s-e-\dfrac{3}{2}\pi\right)^{n-1}\right]+\cdots\} \end{array}\right\},$$

whence we must observe that $s=(n-1)\theta'$ and $e=(n-1)(\theta'-\theta)$.

X.

$z, z_1, z_2, \dots$ representing always the errors of $n-1$ observations, we suppose that the law of facility, as much of the positive error z as of the negative error $-z$, let $h-z$, and let h and $-h$ be the limits of this error; we suppose, moreover, that this law be the same for the errors $z_1, z_2, \dots, z_{n-2}$ of the other observations, and let us seek the probability that the sum of the errors will be contained within the limits p and $p+e$.

If we make $z = t-h$, $z_1 = t_1 - h$, $\dots$, it is clear that t, t_1, $\dots$ will always be positive and may be extended from zero to $2h$; the law of facility of t, from $t=0$ to $t=h$, will be expressed by t; this same law, from $t=h$ to $t=2h$, will be $2h-t$; it will be null from $t=2h$ to $t=\infty$. We will have thus in this case

$$\begin{aligned} q = 0, \quad q' = h, \quad q'' = 2h,\\ \phi(t) = t,\\ \phi'(t) + \phi(t) = 2h - t,\\ \phi''(t) + \phi'(t) + \phi(t) = 0,\end{aligned}$$

whence we deduce

$$\begin{aligned}\phi'(t) &= 2h - 2t,\\ \phi''(t) &= t - 2h.\end{aligned}$$

The function that we have designated by $\Pi(u)$ in article VII will be therefore $u(1-l^h)^2$, and we will have the functions

$$\Pi_1(u_1),\ \dots,\ \Pi_{n-2}(u_{n-2}),$$

by changing in it u successively to $u_1,\ u_2,\ \dots,\ u_{n-2}$.

Presently, we have

$$z + z_1 + \cdots + z_{n-2} = t + t_1 + \cdots + t_{n-2} - (n-1)h;$$

therefore the sum of the errors $z, z_1, \dots$ must, by hypothesis, be contained within the limits p and $p+e$, the sum of the values of t, t_1, t_2 $\dots$ will be contained within the limits $(n-1)h+p+e$ and $(n-1)h+p$, so that, if we make

$$(n-1)h + p + e = s \quad \text{and} \quad t + t_1 + t_2 + \cdots + t_{n-2} = s - t_{n-1},$$

t_{n-1} may be extended from zero to e, and we will prove, as in the preceding example, that its facility must be supposed constant and equal to unity within this interval, and that it must be supposed null from $t_{n-1} = e$ to $t_{n-1} = \infty$; whence we will conclude, as in that same example,

$$\Pi_{n-1}(u_{n-1}) = 1 - l^e.$$

Formula (C) of article VIII will become thus

$$s^{2n-2}(1-l^h)^{2n-2}(1-l^e),$$

and we will have the sought probability by changing in the expansion of this quantity any term such as $\lambda l^\mu s^{2n-2}$ into $\frac{\lambda(s-\mu)^{2n-2}}{1.2.3\ldots(2n-2)}$, that which gives for the expression of this probability

$$\frac{1}{1.2.3\ldots(2n-2)}\left\{\begin{aligned}&s^{2n-2}-(s-e)^{2n-2}\\&\quad-(2n-2)\left[(s-h)^{2n-2}-(s-h-e)^{2n-2}\right]\\&\quad+\frac{(2n-2)(2n-3)}{1.2}\left[(s-2h)^{2n-2}-(s-2h-e)^{2n-2}\right]\\&\quad-\cdots\end{aligned}\right\},$$

by taking care to reject the terms multiplied by $(s-\mu)^{2n-2}$ when $s-\mu$ is negative.

I must observe here that Mr. de la Grange has already resolved the problem where we have proposed to find the probability that the sum of the errors of many observations will be contained within some given limits, when the law of facility of these errors is expressed by a rational and entire function of these errors, of exponentials, of sines and of cosines (*see* Volume V of the *Mémoires de Turin*, p. 221); his method is very ingenious and worthy of its illustrious author; but the preceding has, if I do not deceive myself, the advantage to be more direct and more general, in that it reduces the solution of the problem to the quadrature of curves, whatever be the law of facility of the errors of the observations.

XI.

We see now the usage that we can make of the preceding theory in the solution of the problems relative to a number $n-1$ of players of whom we know only the possibility of the skills. Let

$t,\ t_1,\ \ldots,\ t_{n-2}$ be the absolute skills of the players;
$h,\ h_1,\ h_2,\ \ldots$ the least values of $t,\ t_1,\ \ldots$;
$h',\ h'_1,\ h'_2,\ \ldots$ the greatest values;
if we make

$$h'+h'_1+h'_2+\cdots=s$$

and

$$t+t_1+\cdots+t_{n-2}=s-t_{n-1},$$

the variable t_{n-1} can extend from zero to

$$h'-h+h'_1-h_1+\cdots\ ;$$

the law of its possibility must be supposed constant and equal to unity in this interval, and null upwards to $t_{n-1}=\infty$; moreover, it is clear that the respective skills of the players will be

$$\frac{t}{s-t_{n-1}},\quad\frac{t_1}{s-t_{n-1}},\quad\frac{t_2}{s-t_{n-1}},\quad\ldots$$

We will seek therefore, by the known methods of the analysis of chances, the solution of the proposed problem, by starting from these respective skills, and we will arrive at a result which will be a function of

$$\frac{t}{h'+h'_1+\cdots-t_{n-1}},\quad\frac{t_1}{h'+h'_1+\cdots-t_{n-1}},\quad\ldots$$

By substituting, in place of s, its value, this function will be the one that we have designated by $\psi(t,\, t_1,\, t_2, \ldots)$ in the problem of article VII; the question will be no more afterwards but to seek by the method of this problem the sum of all the values of which this function is susceptible, multiplied by their probabilities, and this sum will be the demanded result: there remains no more, as we see, in this kind of problems, but the inevitable difficulties of the analysis, difficulties which become much less if we suppose that the law of possibility of the skills is the same for all the players.

XII.

This law can be known only by a long sequence of observations, and most often circumstances do not permit making them; we can make up for this ignorance only by the choice of the most likely functions; the analysis of chances, which is in itself only the art to estimate the likelihoods, must therefore guide us in this choice: we examine that which it can furnish us to shed light on this object.

We observe first that, if it is difficult to know by observation the law of facility of the skills of the players, it is much easier to know the limits of them; for we suppose that we have observed the greatest inequality in these skills, and that we have found that the ratio of the skills of the strongest player to the weakest is m, by naming h the least skill of the players and h' the greatest, we will have

$$\frac{h'}{h} = m;$$

now, if we name 1 the mean skill and x the excess of h' over this skill, we will have

$$1 + x = h', \quad 1 - x = h;$$

therefore

$$\frac{1+x}{1-x} = m.$$

whence we deduce

$$x = \frac{m-1}{m+1},$$

hence

$$h = \frac{2}{m+1} \quad \text{and} \quad h' = \frac{2m}{m+1}.$$

Now, the law of possibility of the skills being null beyond the limits $h$and h', it is very likely that it increases from these limits to the middle of the interval which separates them and that it is the same on each side of this middle. Here is therefore a condition to which we must subject the function of which we will make a choice; but this remains yet very indeterminate, and, since, among those which are able to satisfy the preceding condition, we have no reason to prefer one of them, it is necessary to take a mean function among all these functions: the question is thus reduced to determine this mean function.

For this, let $2a$ be the interval contained between the two limits and x the distance from the middle of this interval to any point taken on either side of this middle; if we raise at this point an ordinate y, which represents the probability of x, we will have a curve contained between the two limits, and, the value of x must necessarily fall in this interval, the area of this curve will be equal to unity, so that, from the middle to one of the limits, this area will be $\frac{1}{2}$; we can therefore imagine this quantity $\frac{1}{2}$ divided into an infinite number of equal parts distributed above the different points of the interval a; by the condition of the problem, this repartition must be such that it has as much less of these parts above each point as it is extended further from the mean; all the combinations in which this exists are equally admissible, and we will have the mean ordinate which results from it for the abscissa x, by taking the sum of all the ordinates y relative to each combination and by dividing it by the number of these combinations.

We suppose first the number of the points of the interval a finite and equal to n, and we name s the infinite number of parts which it is necessary to distribute above these points, by observing the preceding condition; let, moreover, z be the ordinate relative to the n^{th} point; $z+z_1$ the ordinate relative to the $(n-1)^{\text{st}}$ point; $z+z_1+z_2$ the ordinate relative to the $(n-2)^{\text{nd}}$ point, and thus in sequence, so that the ordinate relative to the first point or to the point in the middle of the interval $2a$ is $z+z_1+\cdots+z_{n-1}$: it is clear that $z,\ z_1,\ \cdots,\ z_{n-1}$ will be necessarily positive and that we will have

$$nz+(n-1)z_1+(n-2)z_2+\cdots+z_{n-1}=s.$$

Let

$$ns=t,\quad (n-1)z_1=t_1,\quad (n-2)z_2=t_2,\quad \cdots,\cdots z_{n-1}=t_{n-1},$$

the preceding equation will become

$$t+\ t_1+t_2+\cdots+\ t_{n-1}=s;$$

the variables $t,\ t_1,t_2,\ldots$ can be extended from zero to s, and the ordinate relative to the r^{th} point will be

$$\frac{t}{n}+\frac{t_1}{n-1}+\frac{t_2}{n-2}+\cdots+\frac{t_{n-r}}{r}.$$

It is necessary consequently to determine the sum of all the variations which can accept this quantity and to divide it by the number of these variations: now it is clear that this problem returns to the one of article VII; that the quantity which we have named $\psi(t,\ t_1,\ t_2,\ldots)$ is here

$$\frac{t}{n}+\frac{t_1}{n-1}+\cdots+\frac{t_{n-r}}{r};$$

that the quantities q and q' are here 0 and s, and that the law of facility of the variations of t must be supposed equal to a constant b and the same as for $t_1,\ t_2,\ \ldots$. We will have therefore, in the present case,

$$\Gamma(u,u_1,u_2,\ldots)=\frac{k}{n}+\frac{k_1}{n-1}+\cdots+\frac{k_{n-r}}{r}+\frac{u}{n}+\frac{u_1}{n-1}+\cdots+\frac{u_{n-r}}{r},$$
$$\Pi(u)=\Pi_1(u_1)=\Pi_2(u_2)=\cdots=b(1-l^s);$$

but, as it is necessary to distinguish the limits 0 and s, which belong to the variables $t,\ t_1,\ \ldots,t_{n-r}$,in order to assign to $k,\ k_1,\ldots,k_{n-r}$ the values which suit them, we will represent by c', s', c'', s'', c''', s''', ... these limits. This put, formula (B) of article VIII will give for $\daleth(s)$

$$\left(\frac{\frac{k}{n} + \frac{k_1}{n-1} + \frac{k_2}{n-2} + \cdots + \frac{k_{n-r}}{r}}{1.2.3\ldots(n-1)} s^{n-1} + \frac{\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{r}}{1.2.3\ldots n} s^n \right)$$
$$\times\, b^n (1 - l^s)^r (l^{c'} - l^{s'})(l^{c''} - l^{s''}) \cdots$$

It is necessary next, in the development of this quantity, to substitute for k the part of the exponent of l which depends on c' and on s'; for k_1, the part of that exponent which depends on c'' and s'', etc.; to diminish s by the entire exponent of l, and to reject that term every time that this exponent, thus diminished, will be negative; finally to suppose

$$0 = c' = c'' = c''' = \cdots, \quad s = s' = s'' = \cdots \quad \text{and} \quad l = 1.$$

The preceding quantity reduces thus to this very simple formula

$$\frac{b^n s^n}{1.2.3\ldots n} \left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{r} \right);$$

by dividing this quantity by the number of all the combinations, which cannot be a function of n, we will have, for the mean ordinate corresponding to the r^{th} point,

$$N \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{r} \right),$$

N being a function of n.

We suppose now that the numbers n and r become infinite, that the r^{th} point corresponds to the abscissa x and the n^{th} point to the abscissa a, we will have, as we know,

$$\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{r} = \log n - \log r = \log \frac{n}{r} = \log \frac{a}{x};$$

therefore the mean ordinate y which corresponds to the abscissa x is $N \log \frac{a}{x}$; we will determine N, by observing that we must have $\int N dx \log \frac{a}{x} = \frac{1}{2}$, the integral being taken from $x = 0$ to $x = a$, that which gives

$$N = \frac{1}{2a},$$

hence

$$y = \frac{1}{2a} \log \frac{a}{x}.$$

It is necessary to observe that this equation must be supposed the same, x being positive or negative, which reverts to supposing here the logarithms of positive quantities equal to the logarithms of negative quantities, that is to say, $\log \mu = \log(-\mu)$.

XIII.

Such is the equation of which it is necessary to make use when we have, relative to the possibility of the values of x, no other givens, except that it is as much less as those values are greater: now it is that which takes place in a great number of circumstances. We suppose, for example, that the question concerns the true instant of a phenomenon observed by many observers; each of them can easily fix the greatest error of which his observation is susceptible, either to *plus*, or to *minus*, by taking for this limit the half of the greatest interval that it can suppose among two similar observations, without rejecting them as wrong; this interval is that which we have named $2a$; it depends on the skill of the observer, of the goodness of his instruments and on the precision by which the observation of which there is question is susceptible, and it must be assumed likewise for all the observers, if we have no reason to prefer, under this point of view, one observation to another. Now, it is natural to think that the same errors, to the plus and to the minus, are equally probable and that their facility is as much less as they are greater; if we have nothing other given, relatively to their facility, we revert evidently to the case of the preceding problem; it is necessary therefore to suppose then the possibility, so much of the positive error x, as of the negative error $-x$, equal to $\frac{1}{2a}\log\frac{a}{x}$; and it is this law of possibility from which it is necessary to start, in the research of the mean that we must choose among the results of many observations.

When there is question of the skills of the players, we have (art. XII) $2a = h' - h$; the skill t of any player is equal to $1 \pm x$: the possibility of t, from $t = h$ to $t = h'$, will be therefore represented by

$$\frac{1}{h'-h}\log\frac{h'-h}{2-2t},$$

provided that we make the logarithms of the negative quantities equal to the logarithms of the positive quantities. By applying in this case the formulas of article VII, we will have

$$q = h, \quad q' = h',$$

$$\phi(t) = \frac{1}{h'-h}\log\frac{h'-h}{2-2t},$$

$$\phi'(t) + \phi(t) = 0 \quad \text{or} \quad \phi'(t) = \frac{1}{h-h'}\log\frac{h'-h}{2-2t};$$

we must suppose moreover this law of possibility the same for the skills of all the players: we have thus all the data necessary to the solution of the problems which we are able to propose relatively to any number of players; and, by applying to these data the analysis of article VII, we will arrive at the sole result which agrees with the state of ignorance in which we suppose ourselves relatively to the facility of the skills of the players.

XIV.

The preceding theory supposes that we have no reason to attribute to one of the players more skill than to the others, which is true when the game commences; but, in measure as the games succeed each other and as the events of the game are multiplied, we obtain new light on their respective forces, so that they will be exactly known if the number of games was infinite, as we will demonstrate in the following: the skills of the players and, more generally, the different causes of the events are thus linked to their existence by some laws which are very important to know well, and, under this point of view, we can not doubt that the past events have an influence on the probability of future events. We examine this influence and the manner in which we must take account of it.

For this, we name E the event already past; e the future event on which we propose to calculate the probability P; $E+e$ an event composed of the event E happening first and the event e happening next. Suppose we determine by the preceding theory and without regard to the past events the probability of the event E and that of the event $E+e$; let us name V the first of these probabilities and v the second, it is clear that this last probability v will be equal to the probability of the event E, multiplied by the sought probability P, that, E having already taken place, the event e will succeed it; we will have thus $PV = v$, which gives

$$P = \frac{v}{V}.$$

The preceding method is applied therefore equally in the case where we have regard to the past events, and there remains for them only a more compound calculation.

When the possibility of the events is known *a priori* and by the same nature of the causes which produce them, as the possibility to bring a given face from a die of which the material is homogeneous and of which the faces are perfectly equal, the probability v of the event $E+e$ is determined by calculating separately the probabilities of E and of e, and by multiplying them by one another, so that the value of P is equal to the probability of e. It follows thence that the past events have then no influence on the probability of future events; we can be assured moreover, by observing that, whatever be the events already arrived, their absolute probability remains always the same, this which renders the consideration of the past entirely useless when this possibility is exactly known; but it is not thus when it is not; because it is clear that the past events must render more or less probable the different values which we may assume to them, in the same proportion to which they themselves are more or less favorable. This remark leads us naturally to determine the probability of the causes taken from the events.

XV.

We suppose a given event can be produced only by the n causes $A,\ A',\ \ldots, A^{(n-1)}$; let x be the probability that results from them for the existence of A; x' that for the existence of A'; x'' that for the existence of A'',etc. If we name $a,\ a', a'', \ldots$ the probabilities that the causes A, A', A'', …, being supposed to exist, will produce the event in question, it is clear that the probability of a second event similar to the first will be equal to the product of a by the probability x of the cause of A, plus to the product of a' by the probability x' of the cause of A', plus etc.; whence it follows that we will have

$$ax + a'x' + a''x'' + \cdots$$

for this probability; we will find, in the same manner,

$$a^2x + a'^{\,2}x' + a''^2x'' + \cdots$$

for the probability of two consecutive events similar to the first;

$$a^3x + a'^{\,3}x' + a''^3x'' + \cdots$$

for the probability of three consecutive similar events, and thus in sequence. We will have, by the preceding article, these same probabilities, by seeking *a priori* the probabilities of two, of three, of four, etc. consecutive events, and by dividing them by the probability of the first; now the probability of one event is a or a', or a'', etc. according as the cause A or the cause A', or etc. exists; that which gives

$$\frac{1}{n}(a + a' + a'' + \cdots)$$

for this probability. Similarly, the probabilities of two, of three, etc. similar events are

$$\frac{1}{n}(a^2 + a'^2 + a''^2 + \cdots), \quad \frac{1}{n}(a^3 + a'^3 + a''^3 + \cdots), \quad \ldots;$$

therefore the probabilities that a first event having already taken place, there will be following one or two, etc. similar events, are

$$\frac{a^2 + a'^2 + a''^2 + \cdots}{a + a' + a'' + \cdots}, \quad \frac{a^3 + a'^3 + a''^3 + \cdots}{a + a' + a'' + \cdots}, \quad \ldots$$

By equating these probabilities to the preceding, we will have

$$\begin{aligned} ax + a'x' + a''x'' + \cdots &= \frac{a^2 + a'^2 + a''^2 + \cdots}{a + a' + a'' + \cdots} \\ a^2x + a'^{\,2}x' + a''^2x'' + \cdots &= \frac{a^3 + a'^3 + a''^3 + \cdots}{a + a' + a'' + \cdots} \end{aligned}$$

we will form $n - 1$ similar equations, and, by combining them with the equation

$$x + x' + x'' + \cdots = 1,$$

which results from the assumption that the event can be produced only by the n causes $A,\ A', A'', \ldots$, we will have in all n equations of the first degree, which will serve to determine $x,\ x', x'', \ldots$; now it is clear that we will satisfy it by making

$$\begin{aligned} x &= \frac{a}{a + a' + a'' + \cdots}, \\ x' &= \frac{a'}{a + a' + a'' + \cdots}, \\ &\cdots; \end{aligned}$$

whence it follows that, in order to have the probability of the existence of any cause $A^{(r)}$ resulting from a given event, it is necessary to determine the probability $a^{(r)}$ that this cause having taken place will produce that event, and to divide that probability by the sum of the similar probabilities $a, a', a'', \ldots$ relative to all the causes which can produce it.

XVI.

In order to apply this theory and in order to make sense by a quite simple example the influence of past events on the probability of those which follow, we consider two players A and B of whom the skills are unknown; it is infinitely less likely that they will be perfectly equal. Let therefore $\frac{1+\alpha}{2}$ be the greatest and $\frac{1-\alpha}{2}$ the least; if we seek the probability P that A will win the first two games, we will have, by article II,

$$P = \frac{1+\alpha^2}{4},$$

so that there is advantage to wager 1 against 3 that this will take place; but, if we seek the probability that B having already won the first game, A will win the two following, it is clear that the preceding value of P is too considerable, because there is reason to believe that the skill of B is the greatest. In reality, if we consider each skill as a particular cause of the event, the probability that the skill of B is $\frac{1+\alpha}{2}$ will be, by the preceding article, equal to the probability that B having this skill will win the first game, divided by the sum of the probabilities that he will win it by having successively the skills $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$; whence we deduce $\frac{1+\alpha}{2}$ for this probability.

In order to determine, in this case, the value of P, we will observe that the event that we have named E in article XIV is here the gain of the first game by B, and that the event e is the gain of the two following games by A; the probability V of the event E is therefore $\frac{1+\alpha}{2}$ or $\frac{1-\alpha}{2}$, according as the greatest or the least skill belongs to B, which gives, by taking the half of the sum of these two values, $V = \frac{1}{2}$; similarly, the probability v of the event $E+e$ is equal to $\frac{1-\alpha}{2}\left(\frac{1+\alpha}{2}\right)^2$ or to $\frac{1+\alpha}{2}\left(\frac{1-\alpha}{2}\right)^2$, hence

$$v = \frac{1-\alpha^2}{8},$$

therefore

$$P = \frac{v}{V} = \frac{1-\alpha^2}{4};$$

there is therefore disadvantage to wager 1 against 3 that A will win the two games following, so that the inequality of the skills which, in the first case, favor that one who wagers consistently with the ordinary Calculus of probabilities, to him is unfavorable in this one here.

We will find in the same manner that, B having already won the first game, the probability P that A will win the n following is

$$P = \frac{1-\alpha^n}{2^{n+1}}\left[(1+\alpha)^{n-1} + (1-\alpha)^{n-1}\right].$$

If α is considerably small, we will have very nearly

$$P = \frac{1}{2^n}\left\{1 + \alpha^2\left[\frac{(n-1)(n-2)}{1.2} - 1\right]\right\};$$

now, every time that n will surpass 3, this quantity will be greater than the probability $\frac{1}{2^n}$ that the assumption of equal skills gives; whence there results that, in this case, although it be probable that A is the weakest player, however the probability that he will win the n following games is greater than if we would assume A and B of equal forces.

XVII.

When we have nothing given *a priori* on the possibility of an event, it is necessary to assume all the possibilities, from zero to unity, equally probable; thus, observation can alone instruct us on the ratio of the births of boys and of girls, we must, considering the thing only in itself and setting aside the events, to assume the law of possibility of the births of a boy or of a girl constant from zero to unity, and to start from this hypothesis into the different problems that we can propose on this object.

We suppose, for example, that we have observed that, out of $p+q$ infants, there is born p boys and q girls, and that we seek the probability P that, out of $m+n$ infants who must be born, there will be m boys and n girls; if we name x the probability that an infant who must be born will be a boy, and $1-x$ that it will be a girl, by designating

$$\frac{1.2.3\ldots(p+q)}{1.2.3\ldots p.1.2.3\ldots q}$$

by λ, we will have

$$\lambda x^p(1-x)^q$$

for the probability that, out of $p+q$ infants, there will be p boys and q girls; this event is the one which we have named E in article XIV. Similarly, if we designate by γ the product

$$\frac{1.2.3\ldots(m+n)}{1.2.3\ldots m.1.2.3\ldots n},$$

we will have

$$\gamma\lambda x^{p+m}(1-x)^{q+n}$$

for the probability that, out of $p+q$ infants who will be born first, there will be p boys and q girls, and that, out of $m+n$ infants who will be born next, there will be m boys and n girls; this event is the one that we have named $E+e$ in the article cited. Now, x being susceptible to all the values from $x=0$ to $x=1$, and all these values being *a priori* equally probable, it is necessary, in order to have the true probability of E, to multiply $\lambda x^p(1-x)^q$ by $a\,dx$, a being constant, and to take the integral $\lambda\int ax^p(1-x)^q dx$ (from $x=0$ to $x=1$); the value of a will be determined by observing that, x owing necessarily to fall between 0 and 1, we have

$$\int a\,dx = 1,$$

the integral being taken from $x=0$ to $x=1$, which gives $a=1$. We will have similarly

$$\lambda\gamma\int x^{p+m}(1-x)^{q+n}dx$$

for the entire probability of the event $E + e$; therefore the sought probability P, that, out of $m + n$ infants who must be born, there will be m boys and n girls, will be, by article XIV,

$$P = \frac{\gamma\lambda \int x^{p+m}(1-x)^{q+n} dx}{\int x^p (1-x)^q dx}$$

the integrals of the numerator and of the denominator being taken from $x = 0$ to $x = 1$. This condition gives

$$\int x^p (1-x)^q dx = \frac{1.2.3 \ldots q}{(p+1)(p+2)\cdots(p+q+1)},$$
$$\int x^{p+m} (1-x)^{q+n} dx = \frac{1.2.3 \ldots (q+n)}{(p+m+1)(p+m+2)\cdots(p+q+m+n+1)},$$

that which changes the expression of P into this here

$$(\theta) \qquad P = \gamma \frac{(q+1)(q+2)\cdots(q+n)(p+1)(p+2)\cdots(p+m)}{(p+q+2)(p+q+3)\cdots(p+q+m+n+1)}.$$

Now we have, as we know,

$$\log(1.2.3 \ldots u) = \frac{1}{2}\log 2\pi + \left(u + \frac{1}{2}\right)\log u - u + \frac{1}{12u} - \frac{1}{360u^2} + \cdots$$

that which gives very nearly, when u is large,

$$1.2.3 \ldots u = \sqrt{2\pi} u^{u+\frac{1}{2}} e^{-u},$$

π being the ratio of the semi-circumference to the radius and e the number of which the hyperbolic logarithm is unity; therefore, if we suppose p and q very large numbers, we will have

$$(q+1)(q+2)\cdots(q+n) = \frac{1.2.3 \ldots (q+n)}{1.2.3 \ldots q} = \frac{(q+n)^{q+n+\frac{1}{2}}}{q^{q+\frac{1}{2}}} e^{-n},$$
$$(p+1)(p+2)\cdots(p+m) = \frac{(p+m)^{p+m+\frac{1}{2}}}{p^{p+\frac{1}{2}}} e^{-m},$$
$$(p+q+1)\cdots(p+q+m+n) = \frac{(p+q+m+n)^{p+q+m+n+\frac{1}{2}}}{(p+q)^{p+q+\frac{1}{2}}} e^{-m-n}.$$

By substituting these values into the expression of P, and by observing that we have very nearly

$$\frac{p+q+1}{p+q+m+n+1} = \frac{p+q}{p+q+m+n},$$

it will become

$$(\varpi) \qquad P = \gamma \frac{(q+n)^{q+n+\frac{1}{2}} (p+q)^{p+q+\frac{3}{2}} (p+m)^{p+m+\frac{1}{2}}}{p^{p+\frac{1}{2}} q^{q+\frac{1}{2}} (p+q+m+n)^{p+q+m+n+\frac{3}{2}}}$$

If μ and s are very small numbers with respect to p and to q, we have

$$\begin{aligned}\log(p+\mu)^{p+s} =&(p+s)\left[\log p+\log\left(1+\frac{\mu}{p}\right)\right]\\ =&(p+s)\left(\frac{\mu}{p}+\log p\right)=\mu+(p+s)\log p;\end{aligned}$$

therefore

$$(p+\mu)^{p+s}=p^{p+s}e^{\mu};$$

hence, if m and n are very small relatively to p and q, we have

$$\begin{aligned}(q+n)^{q+n+\frac{1}{2}}&=e^{n}q^{q+n+\frac{1}{2}}\\ (p+m)^{p+m+\frac{1}{2}}&=e^{m}p^{p+m+\frac{1}{2}}\\ (p+q+m+n)^{p+q+m+n+\frac{3}{2}}&=e^{m+n}(p+q)^{p+q+\frac{3}{2}}\end{aligned}$$

whence we deduce

$$P=\gamma\frac{p^{m}q^{n}}{(p+q)^{m+n}}.$$

XVIII.

This value of P is the same as that to which we could arrive by supposing the possibilities of the births of the boys and girls in the ratio of p to q; whence it is natural to conclude that these possibilities are very nearly in the same ratio, and that thus the true possibility of the birth of a boy is very near $\frac{p}{p+q}$; except that, absolutely speaking, it cannot have a value quite different, but the expression $\frac{p}{p+q}$ and those which are quite adjacent to it are incomparably more probable than the others, and we can announce thus the preceding conclusion:

If we designate by θ a very small quantity and by P the probability that the possibility of the birth of a boy is contained within the limits $\frac{p}{p+q}-\theta$ and $\frac{p}{p+q}+\theta$, the value of P will differ so much the less from certitude or from unity as p and q will be greater numbers, and we can so increase p and q that the difference from P to unity is less than any given quantity, as small as θ is besides.

We see thence how the events, in their multiplying, indicate to us in a manner more and more probable their respective possibility; but, as the preceding theorem is true only in the infinite and as the value of P differs always a little from unity when p and q are finite numbers, it is interesting to know this difference, and for this we are going to give the expression of P by a very convergent series that we will see reduces itself to unity, when p and q are infinite, and which will furnish us, in this manner, a direct and rigorous demonstration of the theorem in question.

Let x be the possibility of the birth of a boy and $1-x$ that of the birth of a girl; the probability that, out of $p+q$ infants, there will be p boys and q girls, will be, as one has seen in the preceding article, equal to $\lambda x^{p}(1-x)^{q}$; now, if we regard x as a particular case of this event, $\frac{\int x^{p}(1-x)^{q}dx}{\int x^{p}(1-x)^{q}dx}$ will be, by article XV, the probability of this cause, provided that the integral of the denominator is taken from $x=0$ to $x=1$; therefore the probability P, that x will be contained in the given limits, will be $\frac{\int x^p(1-x)^q\,dx}{\int x^p(1-x)^q\,dx}$, provided that the integral of the numerator is taken only in the extent of these limits; the question is thus reduced to determine, in this last case, the value of $\int x^p(1-x)^q dx$, when p and q are very great numbers.

Let $y = x^p(1-x)^q$, we will have

$$y\,dx = \frac{x(1-x)}{p-(p+q)x}dy,$$

and if we make $p = \frac{1}{\alpha}$, $q = \frac{\mu}{\alpha}$, α being an extremely small fraction, since p and q are quite considerable, we will have

$$y\,dx = \alpha z\,dy,$$

z being equal to $\frac{x(1-x)}{1-(1+\mu)x}$; thence we will deduce, whatever be z,

$$(\lambda) \qquad \int y\,dx = C + \alpha yz\left\{1 - \alpha\frac{dz}{dx} + \alpha^2\frac{d(zdz)}{dx^2} - \alpha^3\frac{d[zd(zdz)]}{dx^3} + \cdots\right\},$$

C being an arbitrary constant which depends on the value of $\int y\,dx$, at the origin of the integral. This series, which is of great use in these researches, is demonstrated easily by observing:

1° That

$$\int y\,dx = \int \alpha z\,dy = \alpha yz - \alpha\int y\,dz;$$

2° That the equation

$$y\,dx = \alpha z dy \quad \text{gives} \quad y = \alpha z\,\frac{dy}{dx},$$

and that thus

$$\int y\,dx = \alpha\int\frac{zdz}{dx}\,dy = \alpha y\frac{zdz}{dx} - \alpha\int y\frac{d(zdz)}{dx};$$

3° That

$$\int y\frac{d(zdz)}{dx} = \alpha\int z\frac{d(zdz)}{dx}\,dy = \alpha y\frac{d(zdz)}{dx^2} - \alpha\int y\frac{d[zd(zdz)]}{dx^3}.$$

and thus in sequence.

The preceding series ceases to be convergent when the denominator of z is very small on the order of α, and it is this which takes place when x differs from $\frac{1}{1+\mu}$ only by a quantity of this order; it is necessary therefore to employ this series only in the case where this difference is very great with respect to α. But this is not yet sufficient: each differentiation augmenting by one unit the powers of the denominators of z and of its differentials, it is clear that the term of the series multiplied by α^i has for denominator that of z, raised to the power $2i-1$; therefore, for the convergence of this series, it is necessary that α be much less, not only than the denominator of z, but even than the square of this denominator.

It follows thence that the series (λ) will give, by a rapid approximation, the integral $\int y\,dx$ taken from $x = 0$ to $x = \frac{1}{1+\mu} - \theta$, provided that α be much smaller than θ^2; and if we observe that we have $y = 0$ and $z = 0$ when $x = 0$, we will find, for the value of $\int y\,dx$, in this case,

$$\int y\,dx = \frac{\alpha\mu^{q+1}[1-(1+\mu)\theta]^{p+1}\left(r + \frac{1+\mu}{\mu}\theta\right)^{q+1}}{(1+\mu)^{p+q+3}\theta}\left\{1 - \frac{\alpha[\mu+(1+\mu)^2\theta]}{(1+\mu)^3\theta^2} + \cdots\right\}$$

This series has the advantage of giving the limits between which the value of $\int y\,dx$ is tightened; in fact, this value is less than the first term and greater than the sum of the first two terms. In order to demonstrate it, we will give to z this form

$$z = -\frac{\mu}{(1+\mu)^2} + \frac{x}{1+\mu} + \frac{\mu}{(1+\mu)^2[1-(1+\mu)x]},$$

and we will have

$$dz = \frac{dx}{1+\mu} + \frac{\mu dx}{(1+\mu)[1-(1+\mu)x]^2}.$$

We see in the same way z and dz increasing in measure as x increases from $x = 0$ to $x = \frac{1}{1+\mu}$; the quantities z, $\frac{dz}{dx}$ and $\frac{d(zdz)}{dx^2}$ are therefore always positives in this interval, in the same way that the integrals $\int y\;dx$ and $\int y\;\frac{d(zdz)}{dx}$; now we have, by that which precedes,

$$\int y\;dx = \alpha yz - \alpha\int y\;dx.$$

Hence $\int y\;dx$ is less than αyz; similarly

$$\int y\;dx = \alpha yz\frac{dz}{dx} - \alpha\int y\;\frac{d(zdz)}{dx},$$

and, consequently, $\int y\;dx$ is less than $\alpha yz\frac{dz}{dx}$; therefore $\int y\;dx$ is less than αyz and greater than $\alpha yz\left(1 - \alpha\frac{dz}{dx}\right)$. This remark can be useful when, without seeking the exact value of $\int y\;dx$, we wish to be assured if it is greater or lesser than a given quantity.

The series (λ) will give again the integral $\int y\;dx$, from $x = \frac{1}{1+\mu} + \theta$ to $x = 1$, and if we consider that, x being 1, we have $y = 0$ and $z = 0$, we will see easily that the value of $\int y\;dx$, in this last case, is the same value of $\int y\;dx$ in the first case, taken in lesser, and in which one changes θ to $-\theta$; therefore, if we name k the entire integral $\int y\;dx$, taken from $x = 0$ to $x = 1$, we will have, to the quantities near the order α^3, for this same integral, taken from $x = \frac{1}{1+\mu} - \theta$ to $x = \frac{1}{1+\mu} + \theta$, or, what comes to the same, from $x = \frac{p}{p+q} - \theta$ to $x = \frac{p}{p+q} + \theta$.,

$$k - \frac{\alpha\mu^{q+1}\left\{1 - \frac{\alpha[\mu+(1+\mu)^2\theta^2}{(1+\mu)^3\theta^2}\right\}}{(1+\mu)^{p+q+3}\theta} \times \left\{\begin{array}{l} [1-(1+\mu)\theta]^{p+1}\left(1+\dfrac{1+\mu}{\mu}\theta\right)^{q+1} \\ +[1+(1+\mu)\theta]^{p+1}\left(1-\dfrac{1+\mu}{\mu}\theta\right)^{q+1} \end{array}\right\},$$

that which gives

$$P = 1 - \frac{\alpha\mu^{q+1}\left\{1 - \frac{\alpha[\mu+(1+\mu)^2\theta^2}{(1+\mu)^3\theta^2}\right\}}{(1+\mu)^{p+q+3}\theta k} \times \left\{\begin{array}{l} [1-(1+\mu)\theta]^{p+1}\left(1+\dfrac{1+\mu}{\mu}\theta\right)^{q+1} \\ +[1+(1+\mu)\theta]^{p+1}\left(1-\dfrac{1+\mu}{\mu}\theta\right)^{q+1} \end{array}\right\},$$

There is no longer a concern now but to have the value of k; now we have, by the preceding article,

$$k = \frac{1.2.3\ldots p.1.2.3\ldots q}{1.2.3\ldots(p+q+1)}$$

and, whatever be u,

$$1.2.3\ldots u = \sqrt{2\pi}u^{u+\frac{1}{2}}e^{-u}\left(1 + \frac{1}{12u} + \cdots\right),$$

whence it is easy to conclude, by making $p = \frac{1}{\alpha}$ and $q = \frac{\mu}{\alpha}$,

$$k = \frac{\sqrt{2\pi\alpha}\mu^{q+\frac{1}{2}}}{(1+\mu)^{p+q+\frac{3}{2}}}\left\{1 + \alpha\frac{[(1+\mu)^2 - 13\mu]}{12\mu(1+\mu)} + \cdots\right\}.$$

We will have therefore, by neglecting the quantities of order $\alpha^{\frac{5}{2}}$,

$$P = 1 - \frac{\alpha^{\frac{1}{2}}\mu^{\frac{1}{2}}}{\sqrt{2\pi}(1+\mu)^{\frac{3}{2}}\theta}\left\{1 - \alpha\frac{[12\mu^2 + (1+\mu)^2(1+\mu+\mu^2)\theta^2]}{12\mu(1+\mu)^3\theta^2}\right\} \times \left\{\begin{array}{l} [1-(1+\mu)\theta]^{p+1}\left(1+\dfrac{1+\mu}{\mu}\theta\right)^{q+1} \\ +[1+(1+\mu)\theta]^{p+1}\left(1-\dfrac{1+\mu}{\mu}\theta\right)^{q+1} \end{array}\right\},$$

out of which we must observe that the quantity

$$[1-(1+\mu)\theta]^p\left(1+\frac{1+\mu}{\mu}\theta\right)^q$$

is at its maximum when $\theta = 0$; whence it follows that the greatest value of the factor

$$[1-(1+\mu)\theta]^{p+1}\left(1+\frac{1+\mu}{\mu}\theta\right)^{q+1} + [1+(1+\mu)\theta]^{p+1}\left(1-\frac{1+\mu}{\mu}\theta\right)^{q+1}$$

is very nearly of 2, and that it is much less if θ be in the least greater than zero.

In the present question, this factor is always extremely small; in order to show it, we will put the quantity

$$[1-(1+\mu)\theta]^{p+1}\left(1+\frac{1+\mu}{\mu}\theta\right)^{q+1}$$

under this form

$$\left[1+\frac{1-\mu^2}{\mu}\theta - \frac{(1+\mu)^2}{\mu}\theta^2\right][1-(1+\mu)\theta]^{p}\left(1+\frac{1+\mu}{\mu}\theta\right)^{q},$$

and we will observe that, θ being quite small, we have, by some convergent series,

$$\log[1-(1+\mu)\theta] = -(1+\mu)\theta - \frac{1}{2}(1+\mu)^2\theta^2 - \frac{1}{3}(1+\mu)^3\theta^3 - \cdots,$$

$$\log\left(1+\frac{1+\mu}{\mu}\theta\right) = \frac{1+\mu}{\mu}\theta - \frac{1}{2}\left(\frac{1+\mu}{\mu}\right)^2\theta^2 + \frac{1}{3}\left(\frac{1+\mu}{\mu}\right)^3\theta^3 - \cdots,$$

whence, by substituting in place of p, $\frac{1}{\alpha}$, and in place of q, $\frac{\mu}{\alpha}$, we deduce

$$\log[1-(1+\mu)\theta]^{p}\left(1+\frac{1+\mu}{\mu}\theta\right)^{q} = \frac{(1+\mu)^3}{2\mu}\frac{\theta^2}{\alpha} - \frac{(\mu-1)(1+\mu)^4}{3\mu^2}\frac{\theta^3}{\alpha} - \cdots$$

hence

$$[1-(1+\mu)\theta]^{p}\left(1+\frac{1+\mu}{\mu}\theta\right)^{q} = e^{-\frac{(1+\mu)^3}{2\mu}\frac{\theta^2}{\alpha} - \frac{(\mu-1)(1+\mu)^4}{3\mu^2}\frac{\theta^3}{\alpha} - \cdots}$$

θ^2 being, as we have supposed, much greater than α, and e, the hyperbolic logarithm of unity, being greater than 2, it is clear that the second member of this equation is very small and decreases very rapidly when α decreases; whence it follows that the quantity

$$[1-(1+\mu)\theta]^{p+1}\left(1+\frac{1+\mu}{\mu}\theta\right)^{q+1}$$

is likewise very small, that which is equally true of the quantity

$$[1+(1+\mu)\theta]^{p+1}\left(1-\frac{1+\mu}{\mu}\theta\right)^{q+1},$$

in which is changed the preceding by making θ negative.

We see in this way, θ remaining the same, however small it be besides, the difference of P from unity becomes so much less as α decreases, not only because the factor $\alpha^{\frac{1}{2}}$ which multiplies this factor decreases, but again because the factor

$$\begin{gathered}[1-(1+\mu)\theta]^{p+1}\left(1+\frac{1+\mu}{\mu}\theta\right)^{q+1} \\ +[1+(1+\mu)\theta]^{p+1}\left(1-\frac{1+\mu}{\mu}\theta\right)^{q+1}\end{gathered}$$

is very small and decreases with a great rapidity; and it is clear that we can so increase p and q, and, consequently, decrease α, that this difference from P to unity is less than any given quantity, which is the theorem of which we have spoken at the beginning of this article.

XIX.

One of the principal advantages of the preceding theory is to furnish a direct and general solution of an interesting problem, of which the object is more or less of facility of the births of boys and of girls in different climates. We have observed that in Paris and in London there are born constantly each year more boys than girls, and, although the difference be not very considerable, it will be rather extraordinary that this was due to chance, and it is much more natural to think that, in France and in England, nature favors more the birth of boys than that of girls. In truth, the births observed during four or five years in some small villages of France seem to indicate there a lesser facility for the birth of boys than for that of girls; but it is very possible that, out of a small number of births, such as four or five hundred, there were more girls than boys, although the facility of the birth of those is greater; it is necessary to make use in this delicate research much greater numbers, seeing especially the small difference which exists between the facility of births of boys and of girls, and it is only when we will be quite assured that the observed number of births in any indicated place, with a very great probability, that the births of boys are less possible there than those of girls, that it will be permitted to seek the cause of this phenomenon. The method of the preceding article gives a quite simple method in order to obtain this probability when we have a sufficient number of births; we are going to apply to this what has been observed in Paris, and to determine how probable it is that the births of boys in this great city are more possible than those of girls.

For this, we will make use of the births which have taken place from 1745 to 1770, and of which we can see the list in our *Mémoires* for the year 1771, page 857. By collecting all these births, we find that, in the space of these twenty-six years, there are born in Paris 251527 boys and 241945 girls, this which gives very nearly $\frac{105}{101}$ for the ratio of the births of boys to those of girls. This put, the probability that the possibility of the birth of a boy is equal or less than $\frac{1}{2}$ is, by the preceding article, equal to $\frac{\int y\,dx}{k}$, the integral $\int y\,dx$ being taken from $x=0$ to $x=\frac{1}{2}$; moreover this integral, taken from $x = 0$ to $x = \frac{1}{1+\mu} - \theta$ and divided by k, is, by the same article, equal to

$$\frac{\alpha^{\frac{1}{2}}\mu^{\frac{1}{2}}}{\sqrt{2\pi}(1+\mu)^{\frac{3}{2}}\theta}[1-(1+\mu)\theta]^{p+1}\left(1+\frac{1+\mu}{\mu}\theta\right)^{q+1} \times\left[1-\alpha\frac{12\mu^2+(1+\mu)^2(1+\mu+\mu^2)\theta^2}{12\mu(1+\mu)^3\theta^2}+\alpha^2\cdots\right].$$

By supposing therefore $\frac{1}{1+\mu} - \theta = \frac{1}{2}$ and, consequently, $\theta = \frac{1-\mu}{2(1+\mu)}$, we have, for the expression of the probability that x is equal or less than $\frac{1}{2}$,

$$\frac{\sqrt{\frac{2\alpha\mu}{(1+\mu)\pi}}}{1-\mu}\left(\frac{1+\mu}{2}\right)^{p+1}\left(\frac{1+\mu}{2\mu}\right)^{q+1} \times\left[1-\alpha\frac{48\mu^2+3\mu(1-\mu)^2+(1-\mu)^4}{12\mu(1+\mu)(1-\mu)^2}+\alpha^2\cdots\right]$$

In this present case,

$$\begin{aligned}p &= 251527,\\ q &= 241945,\\ \mu &= \frac{q}{p} = 0,9619047,\\ \alpha &= \frac{1}{p} = \frac{1}{251527},\end{aligned}$$

that which gives very nearly $\theta^2 = 24$, so that the series

$$1-\alpha\frac{[48\mu^2+3\mu(1-\mu)^2+(1-\mu)^4]}{12\mu(1+\mu)(1-\mu)^2}+\alpha^2\cdots$$

is very convergent, and we find, by the calculation, that the second term is around $\frac{1}{200}$; we can thus stick to the first term of it: now we have, by logarithms from Tables,

$$\log\frac{\sqrt{\frac{2\alpha\mu}{(1+\mu)\pi}}}{1-\mu} = \overline{2},4660039,$$

the number $\overline{2}$ indicating a negative characteristic; we have next, by carrying the precision to twelve decimals,

$$\begin{aligned}\log p &= 5,400584610947,\\ \log q &= 5,383716651469,\\ \log(p+q) &= 5,693262515480,\\ \log 2 &= 0,301029995664,\end{aligned}$$

whence we deduce

$$\log\left(\frac{p+q}{p}\right)^{p+1} = 73616,6879714,$$
$$\log\left(\frac{p+q}{q}\right)^{q+1} = 74893,3836139,$$
$$\log 2^{p+q+2} = 148550,4760803;$$

hence

$$\log\left(\frac{1+\mu}{2}\right)^{p+1}\left(\frac{1+\mu}{2\mu}\right)^{q+1}\frac{\sqrt{\frac{2\alpha\mu}{(1+\mu)\pi}}}{1-\mu} = \overline{42},0615089.$$

By passing again from logarithms to numbers, we will have, for the probability that x is equal or less than $\frac{1}{2}$, a fraction of which the numerator is little different from unity and equal to $1,1521$,and of which the denominator is the seventh power of one million; this fraction is even a little too great, and, as it is of excessive smallness, we can regard as certain as any other moral truth, that the difference observed in Paris between the births of boys and those of girls is due to a greater possibility in the births of boys. We see, in the remainder, that the smallness of the preceding fraction comes principally from the factor

$$\left(\frac{1+\mu}{2}\right)^{p+1}\left(\frac{1+\mu}{2\mu}\right)^{q+1},$$

that which confirms what we have said in the preceding article on the convergence of the value of P towards unity.

We have observed that, in the interval of the eighty-five years elapsed from 1664 to 1757, there are born, in London, 737629 boys and 698958 girls, which gives around $\frac{19}{18}$ for the ratio of births of boys to those of girls; this ratio being greater than the one of 105 to 101 which has place in Paris, and the number of births observed in London being very considerable, we would find for this city a greater probability that the births of boys are more possible than those of girls; but, when the probabilities differ likewise little from unity, they can be counted equal and confused with certitude.

XX.

The constancy with which the births of boys in Paris have surpassed each year over those of the girls, from 1745 to 1770, is yet one of those phenomena that we cannot attribute to chance. We determine its probability by starting from the previous data: for this, let $2a$ be the mean number of births of boys and of girls in the space of one year; we suppose, moreover, that out of this number there are m boys and, consequently, $2a-m$ girls: formula (θ) of article XVII will give, for the probability P of this event,

$$P = \frac{1.2.3\dots 2a}{1.2.3\dots(p+q+2a+1)}\frac{1.2.3\dots(p+q+1)}{1.2.3\dots p.1.2.3\dots q}$$
$$\times\frac{1.2.3\dots(q+2a-m)}{1.2.3\dots(2a-m)}\frac{1.2.3\dots(p+m)}{1.2.3\dots m};$$

we will have therefore the probability that the births of boys will not prevail over those of girls, by taking the sum of all the values of P, from $m = 0$ to $m = a$. Let

$$\frac{1.2.3\ldots(q+2a-m)1.2.3\ldots(p+m)}{1.2.3\ldots(2a-m)1.2.3\ldots m} = y_m,$$

and we seek the finite integral $\sum y_m$, from $m = 0$ to $m = a$, the characteristic Σ serving to designate the finite integral; we have clearly

$$y_m = \frac{(m+1)(q+2a-m)}{(2a-m)(p+m+1)} y_{m+1};$$

therefore

$$y_m \left[1 - \frac{(m+1)(q+2a-m)}{(2a-m)(p+m+1)}\right] = \frac{(m+1)(q+2a-m)}{(2a-m)(p+m+1)} \Delta y_m$$

or

$$y_m = \frac{(m+1)(q+2a-m)}{2a-p-m(p+q)} \Delta y_m,$$

the characteristic Δ being that of finite differences. We suppose generally

$$y_m = z_m \Delta y_m,$$

we will have, by integrating,

$$\sum y_m = y_m z_{m-1} - \sum (y_m \Delta z_{m-1});$$

now, if we substitute for y_m its value $z_m \Delta y_m$, we have

$$\begin{aligned}\sum (y_m \Delta z_{m-1}) &= \sum (z_m \Delta z_{m-1} \Delta y_m) \\ &= y_m z_{m-1} \Delta y_{m-2} - \sum [y_m \Delta (z_{m-1} \Delta z_{m-2})].\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum [y_m \Delta z_{m-1} \Delta z_{m-2})] \\ &\qquad = y_m z_{m-1} \Delta (z_{m-2} \Delta z_{m-3} - \sum \{y_m \Delta [z_{m-1} \Delta (z_{m-2} \Delta z_{m-3})]\},\end{aligned}$$

and thus in sequence; we will have therefore

$$(\gamma) \qquad \sum y_m = C + y_m z_{m-1} \left\{ \begin{array}{l} 1 - \Delta z_{m-2} + \Delta (z_{m-2} \Delta z_{m-3}) \\ \quad - \Delta [z_{m-2} \Delta (z_{m-3} \Delta z_{m-4})] + \cdots \end{array} \right\}$$

C being an arbitrary constant. This series is, in finite differences, that which is the series (λ) of article XVIII, in the infinitely small differences: in order to determine in which case it is convergent, we will observe that, if the dimension of z_{m-1}, in p, q, a, and m, is r, that of Δz_{m-2} will be $r-1$, that of $\Delta (z_{m-2} \Delta z_{m-3})$ will be $2r-2$, and thus for the rest: now the convergence of the series requires that these dimensions decrease, that which supposes that r is less than unity. In the question presented, where

$$z_{m-1} = \frac{m(q+2a+1-m)}{2ap+q-m(p+q)},$$

the dimensions of the numerator and of the denominator are equal to 2, and consequently $r = 0$; the series will therefore converge, provided that the denominator is not extremely small, that is that $\frac{m-1}{2a-m}$ differs sensibly from $\frac{p}{q}$: now this takes place, when m is equal or less than a, p being supposed greater than q.

We can put the quantity $\frac{m(q+2a+1-m)}{2ap+q-m(p+q)}$ under this form

$$E + Fm + \frac{G}{2ap+q-m(p+q)},$$

by making

$$E = \frac{-q(p+q) - 2aq - p}{(p+q)^2},$$
$$F = \frac{1}{p+q},$$
$$G = \frac{(2ap+q)[q(p+q) + 2aq + p]}{(p+q)^2};$$

we will have then

$$\Delta z_{m-1} = F + \frac{G(p+q)}{[2ap+q-m(p+q)][2ap+p+2q-m(p+q)]}.$$

Now, F and G being positive, it is clear that Δz_{m-2} is always positive as long as $\frac{m-1}{2a-m}$ is less than $\frac{p}{q}$; we see moreover that, in this case $z_{m-1}\Delta z_{m-2}$ always increases, so that $\Delta(z_{m-1}\Delta z_{m-2})$ is again a positive quantity; therefore $\sum y_m$ being equal to

$$y_m z_{m-1} - \sum(y_m \Delta z_{m-1}),$$

is less than $H + y_m z_{m-1}$, H being arbitrary. Similarly, $\sum(\Delta y_m z_{m-1})$ being equal to

$$y_m z_{m-1}\Delta(z_{m-2}) - \sum[y_m \Delta[(z_{m-1}\Delta z_{m-2})]$$

is less than $H' + y_m z_{m-1}\Delta z_{m-2}$, H' being a new arbitrary; therefore the integral $\sum y_m$ is less than $C + y_m z_{m-1}$ and greater than

$$C + y_m z_{m-1}(1 - \Delta z_{m-1}).$$

If we determine, by means of formula (γ), the integral $\sum y_m$ from $m = 0$ to $m = a$, the constant C will be null; if we assume next that there are born 20000 infants each year, which gives $a = 10000$, we will find, by employing for p and q the values of the preceding article relative to Paris,

$$z_{a-1} = 26,22,$$
$$z_{a-2} = 26,09.$$

Hence,

$$\Delta z_{a-1} = 0,13;$$

we will have thus

$$\sum y_m < 26,22 y_a$$

and

$$\sum y_m > 26,22 y_a (1 - 0,13).$$

By making therefore

$$\sum y_m = 26,22 y_a,$$

this value of $\sum y_m$ will surpass only by about $\frac{1}{10}$ the true value; it follows thence that, if we name P the probability that out of 20000 infants there will be as many boys as of girls, the probability that the number of boys will not surpass that of the girls will be a little smaller than $26,22P$.

We will determine the value of P by formula (ϖ) of article XVII; for this, we will suppose $m = n = a$, and we will put it under this form

$$P = \frac{\gamma p^a q^a}{(p+q)^{2a}} \frac{(p+q)^{\frac{3}{2}} \sqrt{(p+a)(q+a)}}{\sqrt{pq}(p+q+2a)^{\frac{3}{2}}} \left[1 + \frac{a(p-q)}{q(p+q+2a)}\right]^{q+a} \left[1 - \frac{a(p-q)}{q(p+q+2a)}\right]^{p-a}$$

We will observe next that

$$\gamma = \frac{1.2.3\ldots 2a}{(1.2.3\ldots a)^2},$$

whence we deduce, by article XVII,

$$\gamma = \frac{2^{2a}}{\sqrt{2\pi}};$$

we have besides

$$\log \left[1 + \frac{a(p-q)}{q(p+q+2a)}\right]^{q+a} = (q+a)\left[\frac{a(p-q)}{q(p+q+2a)} - \frac{1}{2}\frac{a^2(p-q)^2}{q^2(p+q+2a)^2} + \cdots\right],$$

$$\log \left[1 - \frac{a(p-q)}{q(p+q+2a)}\right]^{p+a} = (p+a)\left[\frac{-a(p-q)}{q(p+q+2a)} - \frac{1}{2}\frac{a^2(p-q)^2}{q^2(p+q+2a)^2} + \cdots\right].$$

a being considerably small with respect to p and p differing little from q, these series are very convergent, and we can keep to the first two terms of them; by adding therefore these logarithms, we will have

$$\begin{aligned}&\log \left[1 + \frac{a(p-q)}{q(p+q+2a)}\right]^{q+a} \left[1 - \frac{a(p-q)}{q(p+q+2a)}\right]^{p+a} \\ &= a^2(p-q)^2 \left[\frac{1}{pq(p+q+2a)} - \frac{1}{2}\frac{p^2q + q^2p + a(p^2+q^2)}{p^2q^2(p+q+2a)^2}\right].\end{aligned}$$

We can assume very nearly $a(p^2+q^2) = 2apq$, which reduces the second member of the preceding equation to $\frac{a^2(p-q)^2}{2pq(p+q+2a)}$; this logarithm is hyperbolic, and, in order to convert by logarithm of the Tables, it is necessary, as we know, to multiply by

$0,43429448$. By applying numbers to these formulas, we will find that the tabular logarithm of

$$\left[1+\frac{a(p-q)}{q(p+q+2a)}\right]^{q+a}\left[1-\frac{a(p-q)}{q(p+q+2a)}\right]^{p+a}$$

is $0,0638041$; we have then, by carrying the precision to ten decimals,

$$\begin{aligned}\log 2 &= 0,3010299957,\\ \log p &= 5,4005846109,\\ \log q &= 5,3837166515,\\ \log(p+q) &= 5,6932625156,\end{aligned}$$

which gives

$$\log\frac{p^a q^a}{\left(\frac{p+q}{2}\right)^{2a}} = \overline{2},3622260;$$

moreover,

$$\log\sqrt{a\pi} = 2,2485750,$$

$$\log\frac{(p+q)^{\frac{3}{2}}\sqrt{(p+a)(q+a)}}{\sqrt{pq}(p+q+2a)^{\frac{3}{2}}} = \overline{1},9913791:$$

we will have therefore

$$\log P = \overline{4},1688342,$$

whence we deduce

$$26,22P = 0,0038678 = \frac{1}{259}.$$

The probability that, in one year, the births of boys will not be by a greater number in Paris than those of girls, is therefore less than $\frac{1}{259}$; now, by supposing it equal to this fraction, we have, very nearly, the number of years in which we can wager one against one that it will not happen, by multiplying its denominator 259 by the hyperbolic logarithm of 2, that is by $0,6931472$, which gives for the product 179: we can therefore wager with advantage one against one that it will not happen in the interval of one hundred seventy-nine years.

Relatively to London,

$$p = 737629$$

and

$$q = 698958,$$

which gives

$$z_{a-1} = 18,3000$$

and

$$\Delta z_{a-1} = 0,0694;$$

so that, if we suppose the probability that the births of boys will not exceed those of girls equal to $18,3P$, this probability will surpass only about one fifteenth the truth. We find next

$$\log\left[1+\frac{a(p-q)}{q(p+q+2a)}\right]^{q+a}\left[1-\frac{a(p-q)}{q(p+q+2a)}\right]^{p+a}=0,0432414,$$
$$\log\frac{(p+q)^{\frac{3}{2}}\sqrt{(p+a)(q+a)}}{\sqrt{pq}(p+q+2a)^{\frac{3}{2}}}=\overline{1},9970020.$$

We have moreover, by carrying the precision to ten decimals,

$$\log p = 5,8678379827,$$
$$\log q = 5,8444510800,$$
$$\log(p+q) = 6,1573319321,$$

whence we deduce

$$\log\frac{p^a q^a}{\left(\frac{p+q}{2}\right)^{2a}}=\overline{4},8518990;$$

we will have therefore

$$\log P=\overline{6},6435674,$$

hence

$$18,3P=0,000080541=\frac{1}{12416}.$$

The probability that the births of boys in London, will not exceed those of girls, in one determined year, is therefore a little less than $\frac{1}{12416}$, so that we can wager with advantage 1 against 1 that this will not happen in the interval of eight thousand six hundred five years; this phenomenon is, as we see, much less probable in London than in Paris, which comes from this that, in the first of these cities, the ratio of the births of boys to those of girls is more considerable.

XXI.

The preceding theory supposes that we know the number of times that each simple event has happened; but, although this assumption extends to a great number of interesting problems, however it is still only a particular case of this part of the analysis of chances, which consists in reascending from events to the causes. We are going to exhibit, in the following articles, a general method to determine the possibilities of simple events, whatever be the composite event of which we have observed existence.

We will consider first two players A and B, playing on the conditions as in article III, that is that, A having m tokens at the beginning of each game, B has $n-m$ of them; that at each trial the one who loses gives a token to his adversary, and that the game must end only when one of them will have won all the tokens of the other. We suppose next that they have played in this manner a very great number of games, of which p had been won by A and q by B, and that we wish to determine their respective skills, or, what amounts to the same, their probabilities of winning a single trial. It is clear that the number of trials won or lost by each player is unknown, since each game can be composed of a greater or lesser number of trials: we do not know therefore here the number of times that each simple event has happened; but it is easy to extend to this case and to all the similar others the theory of the preceding articles, by observing that, if p and q are very great numbers, the probabilities of the two players A and B in order to win a game will be very nearly in the ratio of these numbers: now, these probabilities being known, we will have easily their respective skills or their probabilities to win a single trial; because, by naming X the probability of player A in order to win a game, and x his skill, we have, by article III,

$$X = \frac{1 - \left(\frac{1-x}{x}\right)^m}{1 - \left(\frac{1-x}{x}\right)^n}.$$

The only useful root in this equation is that which is positive and less than unity; now it is easy to see *a priori* that it can have only one of them which satisfies these conditions, since the skill x can increase or diminish without either the probability X increasing or decreasing; the value of x that we will deduce from this equation will enjoy therefore the same degree of probability as X; now, if we suppose p and q very large, it will be extremely probable, by article XVIII, that X differs very little from $\frac{p}{p+q}$; therefore, if we name a the positive and less than unity root of the equation

$$(a) \qquad 0 = qx^n + p(1-x)^n - (p+q)x^{n-m}(1-x)^m,$$

it will be very probable that the skill x is very close to a, so that, if p and q were infinite, it would be infinitely probable that the difference of x and of a is less than any given quantity. This value of x has moreover the advantage of making known us the ratio of the trials won to the trials lost by player A; because, if we name r the number of the firsts and s the one of the seconds, the skill x must be very little different from $\frac{r}{r+s}$, so that we have, very nearly,

$$\frac{r}{r+s} = a,$$

whence we deduce

$$\frac{r}{s} = \frac{a}{1-a}.$$

We suppose again that A and B have played with the preceding condition, and q games in which A had m' tokens and B $n' - m'$ at the beginning of each game. We suppose next that, out of these $p + q$ games, A has won r of them; this put, in order to determine the skills of these players, we will name x that of A, and v the unknown number of games which he has won out of the first p: equation (a) will give in this case

$$0 = (p-v)x^n + v(1-x)^n - px^{n-m}(1-x)^m.$$

The number of games that this player has won out of the last q is $r - v$; we have therefore again, by virtue of equation (a),

$$0 = (q-r-v)x^{n'} + (r-v)(1-x)^{n'} - qx^{n'-m'}(1-x)^{m'}.$$

By eliminating v from these two equations, we will have an equation in x, of which the positive and less than unity root is that which it is necessary to choose; now we will prove, as above, that there can be only one of them of this nature. If we name a this root, $\frac{a}{1-a}$ will be very nearly the ratio of the number of trials won to the number of trials lost by player A. We will have next

$$\frac{v}{p} = a^{n-m}\frac{a^m - (1-a)^m}{a^n - (1-a)^n},$$

and this will be the ratio of the number of first games won by player A to the total number p of these games.

XXII.

Here now is a direct and general method to determine the possibilities of simple events, whatever be the observed event.

If we designate by x and by $1-x$ the possibilities of two simple events, and if we seek, by the ordinary rules of the analysis of chances, the probability of the composite event of which there is concern, we will have for its expression a function of x, multiplied by any constant coefficient; if we name y this function and a the value of x, positive and less than unity which renders it a maximum, not only will this value be the most probable, but it will be again very near to the true possibility x: for example, if the observed event is the birth of p boys and q girls out of $p+q$ infants, by naming x the possibility of the birth of a boy and, consequently $1-x$ that of the birth of a girl, we will have

$$\frac{1.2.3\ldots(p+q)}{1.2.3\ldots p.1.2.3\ldots q}x^p(1-x)^q$$

for the probability of this event; in this case $y = x^p(1-x)^q$, and its maximum takes place when $x = \frac{p}{p+q}$; this value of x is therefore, very nearly, the true possibility of the birth of a boy, when p and q are very great numbers.

We suppose further that we draw three balls from an urn which contains an infinite number of white and black balls in an unknown proportion, and let A and B play with this condition that A will win the game if out of these three balls there are more whites than blacks, and that he will lose it if there are more blacks than whites. We suppose next that, out of $p+q$ games, A has won p of them and lost q; this put, if we name x the probability to extract a white ball, we will have $x^2(3-2x)$ for the expression of the probability that A will win a game, and $(1-x)^2(1+2x)$ for the probability that he will lose it; the probability of the observed event will be therefore

$$\frac{1.2.3\ldots(p+q)}{1.2.3\ldots p.1.2.3\ldots q}x^{2p}(3-2x)^p(1-x)^{2q}(1+2x)^q;$$

in this case,

$$y = x^{2p}(3-2x)^p(1-x)^{2q}(1+2x)^q,$$

and its maximum gives

$$0 = p(1-x)^2(1+2x) - qx^2(3-2x);$$

whence it follows that, if we name a the positive and less than unity root of this equation, the ratio of the white balls to the black balls in the urn will be very nearly equal to $\frac{a}{1-a}$.

The maximum of y indicates in an approximate manner the true value of x only as far as the values of y neighboring this maximum are incomparably greater than the others; because it is clear that the integral $\int y\,dx$, taken in a very small interval on both sides of the maximum, is then not very different from this same integral taken from $x = 0$ to $x = 1$: now the ratio of the first of these integrals to the second expresses the probability that the value of x is contained in this interval. The values of y neighboring the maximum will surpass considerably the others, when y will have the factors raised to great powers of the order $\frac{1}{\alpha}$, α being a very small coefficient and proportionally less as the observed event is more composed; if we take, in this case, the ratio of dy to $y\,dx$, we will be lead to an equation of this form

$$\frac{dy}{y\,dx} = \frac{1}{ax},$$

z being a function of x, which no longer contains powers of order $\frac{1}{\alpha}$. Thus, every time that we arrive to a similar equation, the values of x will decrease with a great rapidity in extending from the maximum, and the value of x corresponding to this maximum will be very near to the truth.

We see thence that the composite events are not all proper to make known the possibilities of the simple events: for example, A and B play to the same conditions as in article III, if A wins the game, by naming x his skill, we will have $\frac{1-\left(\frac{1-x}{x}\right)^m}{1-\left(\frac{1-x}{x}\right)^n}$ for the probability of this event. Now, if we suppose m and n very great numbers, the observed event will be composed of a great number of trials; but, as the values of y corresponding to x greater than $\frac{1}{2}$ are very little different from unity, this event cannot show in an approximate manner the value of x: all that we can conclude from it, is that it is extremely probable that A is stronger than B, because the values of y corresponding to x smaller than $\frac{1}{2}$ are incomparably less than the others.

XXIII.

The knowledge of the approximate values of possibilities of simple events which result from a composite event will be very imperfect if we were not in a state to appreciate how often it is probable that, by taking for these values those which correspond to the maximum of y, we will not be deceived, either to *more*, or to *less*, of a given quantity; for this, it is necessary, as we have seen in article XVIII, to determine the ratio of the integral $\int y\,dx$, taken on a small interval on both sides of the maximum, to this same integral taken from $x = 0$ to $x = 1$, and it is this which we have made, in the article cited, for the case where $y = x^p(1-x)^q$, p and q being very great numbers. We have now generalized these researches and extended them to all the values of y which lead to an equation of this form

$$y\,dx = \alpha z\,dy,$$

z being a function of x which contains no powers of order $\frac{1}{\alpha}$.

We take equation (λ) of article XVIII,

$$(\lambda) \qquad \int y\,dx = C + \alpha yz\left\{1 - \alpha\frac{dz}{dx} + \alpha^2\frac{d(zdz)}{dx^2} - \alpha^3\frac{d[zd(zdz)]}{dx^3} + \cdots\right\},$$

if we name
a the value of x corresponding to the maximum of y;
Y and Z the values of x and of z corresponding to $x = a - \theta$;
Y' and $-Z'$ the values of these same quantities corresponding to $x = a + \theta$;
if we observe moreover that, the two simple events being supposed to have taken place, we have $y = 0$ when $x = 0$ and when $x = 1$, the integral $\int y\,dx$ taken from $x = 0$ to $x = a - \theta$ will be

$$\alpha Y Z \left[1 + \alpha \frac{dZ}{d\theta} + \alpha^2 \frac{d(ZdZ)}{d\theta^2} + \cdots\right];$$

this same integral, taken from $x = a + \theta$ to $x = 1$, will be

$$\alpha Y' Z' \left[1 - \alpha \frac{dZ'}{d\theta} + \alpha^2 \frac{d(Z'dZ')}{d\theta^2} - \cdots\right].$$

By naming therefore k the integral $\int y\,dx$, taken from $x = 0$ to $x = 1$, we will have this same integral, taken from $x = a - \theta$ to $x = a + \theta$, by subtracting from k the two preceding integrals; by dividing next this remainder by k, we will have the probability that x will be contained within that interval. This probability will be, consequently, equal to

$$\begin{aligned} 1 &- \frac{\alpha Y Z}{k}\left\{1 + \alpha \frac{dZ}{d\theta} + \alpha^2 \frac{d(ZdZ)}{d\theta^2} + \alpha^3 \frac{d[Zd(ZdZ)]}{d\theta^3} + \cdots\right\} \\ &- \frac{\alpha Y' Z'}{k}\left\{1 - \alpha \frac{dZ'}{d\theta} + \alpha^2 \frac{d(Z'dZ')}{d\theta^2} - \alpha^3 \frac{d[Z'd(Z'dZ')]}{d\theta^3} + \cdots\right\} \end{aligned}$$

the question is reduced thus to determine k. We have attained it in article XVIII where, $y = x^p(1-x)^q$, by means of the beautiful theorem of Mr. Stirling on the value of the product $1.2.3\ldots u$, when u is a very great number; but this process is indirect, and it is natural to think that there exists a method to determine k directly, whatever be y, and of which this theorem is a corollary: that which I am going to exhibit has appeared to me to make complete this object in the most general manner.

Since the value of y leads, by the assumption, to an equation of this form $y\ dx = \alpha z\ dy$, we have

$$\log y = \frac{1}{\alpha}\int \frac{dx}{z},$$

so that $\log y$ is very great and of order $\frac{1}{\alpha}$; moreover, a being the value of x corresponding to the maximum of y, if we make $x = a + \theta$, and if we name A the greatest value of y or its value when $\theta = 0$, we will have, by reducing to series,

$$\alpha \log y = \alpha \log A - \theta^2(f + f'\theta + f''\theta^2 + \cdots),$$

the term multiplied by θ vanishing, because the equation $x = a$ or $\theta = 0$ renders y a maximum. We will have thus

$$y = Ae^{-\frac{\theta^2}{\alpha}(f + f'\theta + f''\theta^2 + \cdots)}$$

and

$$\int y\,dx = A\int e^{-\frac{\theta^2}{\alpha}(f+f'\theta+\cdots)},$$

e being the number of which the hyperbolic logarithm is unity. Let

$$\theta^2(f+f'\theta+f''\theta^2+\cdots)=\alpha t^2$$

or, what amounts to the same,

$$\log A-\log y=t^2,$$

we will have by the method of the reversion of series

$$\theta=\alpha^{\frac{1}{2}}t(h+h^{(1)}\alpha^{\frac{1}{2}}t+h^{(2)}\alpha t^2+h^{(3)}\alpha^{\frac{3}{2}}t^3+\cdots),$$

hence

$$d\theta=\alpha^{\frac{1}{2}}dt(h+2h^{(1)}\alpha^{\frac{1}{2}}t+3h^{(2)}\alpha t^2+\cdots),$$

which gives

$$\int y\,dx=\alpha^{\frac{1}{2}}A\int e^{-t^2}dt(h+2h^{(1)}\alpha^{\frac{1}{2}}t+3h^{(2)}\alpha t^2+\cdots)=k.$$

The integral $\int y\,dx$ must be taken from $x=0$ to $x=1$; now, x being null, we have $y=0$ and $\log y=-\infty$: therefore $t^2=\infty$. When $x=a$, we have $\theta=0$, hence $t=0$; moreover, when θ changes sign, t changes in it likewise, so that the values of t, corresponding to those of x, from $x=0$ to $x=a$, have a different sign from those which correspond to the values of x, from $x=a$ to $x=1$; now, x being 1, we have $y=0$, that which gives $t^2=\infty$; the values of t extend consequently from $t=-\infty$ to $t=\infty$. In this case,we have

$$\int t^{2n-1}e^{-t^2}dt=0,$$

because, $t^{2n-1}e^{-t^2}$ is changed into $-t^{2n-1}e^{-t^2}$ when t is negative, the sum of these two quantities is null; we have, by a similar reason,

$$\int t^{2n}e^{-t^2}dt=2\int t^{2n}e^{-t^2}dt$$

(the second integral being taken from $t=0$ to $t=\infty$); now this assumption gives

$$\int t^{2n}e^{-t^2}dt=\frac{2n-1}{2}\int t^{2n-2}e^{-t^2}dt;$$

similarly

$$\int t^{2n-2}e^{-t^2}dt=\frac{2n-3}{2}\int t^{2n-4}e^{-t^2}dt,$$

and thus in sequence; therefore

$$\int t^{2n}e^{-t^2}dt=\frac{1.3.5.7\cdots(2n-1)}{2^n}\int e^{-t^2}dt.$$

We will have therefore

$$k = 2\alpha^{\frac{1}{2}} A(h + 1.3\alpha\frac{h^{(2)}}{2} + 1.3.5\alpha^2\frac{h^{(4)}}{2^2} + 1.3.5.7\alpha^3\frac{h^{(6)}}{2^3} + \cdots) \int e^{-t^2}dt.$$

There is concern therefore only to have the integral $\int e^{-t^2}dt$ from $t = 0$ to $t = \infty$. For this, we consider the double integral

$$\iint e^{-s(1+u^2)}dsdu,$$

and we take it from $s = 0$ to $s = \infty$ and from $u = 0$ to $u = \infty$; by integrating first with respect to s, we will have

$$\iint e^{-s(1+u^2)}dsdu = \int \frac{du}{1+u^2}.$$

Now we have, as we know,

$$\int \frac{du}{1+u^2} = \frac{\pi}{2},$$

π being the ratio of the semi-circumference to the radius; therefore

$$\iint e^{-s(1+u^2)}dsdu = \frac{\pi}{2}.$$

If we take this double integral first with respect to u, by making $u\sqrt{s} = t$, it will become $\int e^{-s}\frac{ds}{\sqrt{s}} \int e^{-t^2}dt$; let $\int e^{-t^2}dt = B$ (the integral being taken from $t = 0$ to $t = \infty$), we will have

$$\iint e^{-s(1+u^2)}dsdu = \mathrm{B} \int e^{-s}\frac{ds}{\sqrt{s}}.$$

Now, by making $s = s'^2$, we have

$$\int e^{-s}\frac{ds}{\sqrt{s}} = 2\int e^{-s'^2}ds' = 2B$$

(the integral being taken from $s' = 0$ to $s' = \infty$); therefore

$$\iint e^{-s(1+u^2)}dsdu = 2\mathrm{B}^2 = \frac{\pi}{2},$$

whence we deduce $B = \frac{1}{2}\sqrt{\pi}$, hence

$$(s) \qquad k = A\sqrt{2\pi}(h + 1.3\frac{\alpha h^{(2)}}{2} + 1.3.5\frac{\alpha^2 h^{(4)}}{2^2} + 1.3.5.7\frac{\alpha^3 h^{(6)}}{2^3} + \cdots).$$

If we put the equation

$$\log A - \log y = t^2$$

under this form

$$\theta = t\sqrt{\frac{\theta^2}{\log A - \log y}},$$

we will have, in order to determine the coefficients $h,\ h^{(2)},\ h^{(4)},\ \ldots$ of the series

$$\theta = \alpha^{\frac{1}{2}} t(h + \ h^{(1)} \alpha^{\frac{1}{2}} t + \ h^{(2)}\, \alpha t^2 + \cdots),$$

the general expression

$$(z) \qquad \alpha^{n+\frac{1}{2}} h^{(2n)} = \frac{d^{2n}\left[\theta^{2n+1}(\log A - \log y)^{-n-\frac{1}{2}}\right]}{1.2.3\ldots(2n+1)d\theta^{2n}},$$

provided that we suppose $d\theta$ constant and $\theta = 0$ after the differentiations. [*See* on this the *Mémoires de l'Académie* for the year 1777, p. 115.[1]]

When $n = 0$, we have

$$\alpha^{\frac{1}{2}} h = \theta(\log A - \log y)^{-\frac{1}{2}};$$

now

$$\log y = \log A + \theta \frac{dy}{y\, d\theta} + \frac{\theta^2}{1.2}\left(\frac{d^2 y}{y\, d\theta^2} - \frac{dy^2}{y^2\, d\theta^2}\right) + \cdots,$$

$y,\ dy,\ d^2 y,\ \ldots$, in the second member of this equation, being that which these quantities become when we suppose $\theta = 0$; this assumption gives

$$y = A \quad \text{and} \quad \frac{dy}{d\theta} = 0;$$

we will have therefore

$$\alpha^{\frac{1}{2}} h = \sqrt{\frac{2A}{-\frac{d^2 y}{d\theta^2}}}$$

Hence we will have very nearly, when α is very small,

$$k = \frac{A^{\frac{3}{2}}\sqrt{2\pi}}{\sqrt{\frac{-d^2 y}{dx^2}}}$$

or, what amounts to the same,

$$(\mu) \qquad \left(\int y\, dx\right)^2 = \frac{2\pi y^2}{-\frac{d^2 y}{dx^2}},$$

the integral $\int y\, dx$ being taken from $x = 0$ to $x = 1$, and the quantities y and $\frac{d^2 y}{dx^2}$ of the second member of this equation being those which they become when we assume $x = a$.

[1] "Mémoire sur l'usage du Calcul aux différences partielle dans la théorie des suites," *Oeuvres de Laplace*, T. IX, p. 329.

XXIV.

By substituting $a+\theta$ in place of x in $\log y$ and by reducing in the series, the condition of the maximum of y makes the first power of θ disappear in this series; but this condition can, as we know, make the first, the second and the third power of θ disappear or the first, the second, the third, the fourth and the fifth power, and thus in sequence, provided that the number of the powers which disappear are odd. We see that which the integral $\int y\,dx$ becomes then, taken from $x=0$ to $x=1$.

We suppose that the first, the second and the third powers of θ disappear, we will have for $\alpha \log y$ a series of this form

$$\alpha \log y = \alpha \log A - \theta^2(f + f'\theta + f''\theta^2 + \cdots);$$

therefore, if we make

$$\theta^4(f + f'\theta + f''\theta^2 + \cdots) = \alpha t^4,$$

we will have

$$\log A - \log y = t^4$$

and

$$\theta = \alpha^{\frac{1}{4}} t(h + h^{(1)}\alpha^{\frac{1}{4}} t + h^{(2)}\alpha^{\frac{1}{2}} t^2 + h^{(3)}\alpha^{\frac{3}{4}} t^3 + h^{(4)}\alpha t^4 + \cdots),$$

whence we deduce

$$\int y\,dx = \alpha^{\frac{1}{4}} A \int e^{-t^4} dt (h + 2\,h^{(1)}\alpha^{\frac{1}{4}} t + 3\,h^{(2)}\alpha^{\frac{1}{2}} t^2 + \cdots).$$

We will prove, as in the preceding article, that the integral relative to t must be taken from $t=-\infty$ to $t=\infty$; now we have in this case

$$\int t^{2n-1} e^{-t^4} dt = 0$$

and

$$\int t^{2n} e^{-t^4} dt = 2\int t^{2n} e^{-t^4} dt,$$

the integral of the second member being taken from $t=0$ to $t=\infty$. If we suppose next $n=2i$, we will have

$$\int t^{2n} e^{-t^4} dt = \frac{1.5.9\ldots(4i-3)}{4^i}\int e^{-t^4} dt,$$

and, if $n = 2i+1$, we will have

$$\int t^{2n} e^{-t^4} dt = \frac{3.7.11\ldots(4i-1)}{4^i}\int t^2 e^{-t^4} dt;$$

by supposing therefore

$$\int e^{-t^4} dt = C,$$

$$\int t^2 e^{-t^4} dt = C'$$

we will have

$$\int y\,dx = 2\alpha^{\frac{1}{4}} AC\left(h + \frac{1.5}{4}\alpha h^{(4)} + \frac{1.5.9}{4^2}\alpha^2 h^{(8)} + \frac{1.5.9.13}{4^3}\alpha^3 h^{(12)} + \cdots\right)$$
$$+ 2\alpha^{\frac{1}{4}} AC'\left(3h^{(2)} + \frac{3.7}{4}\alpha h^{(6)} + \frac{3.7.11}{4^2}\alpha^2 h^{(10)} + \frac{3.7.11.15}{4^3}\alpha^3 h^{(14)} + \cdots\right)$$

and it is easy to conclude from it, by analogy, the values of $\int y\,dx$ in the case where the condition of the maximum of y causes a greater number of powers of θ to vanish.

All is reduced therefore to determine the values of C and C': we will observer first that, C being known, C' will be likewise, because, if we take the double integral $\iint e^{-s(1+u^4)} ds\,du$, from $s = 0$ to $s = \infty$ and from $u = 0$ to $u = \infty$, we will have, by integrating first with respect to s,

$$\iint e^{-s(1+u^4)} ds\,du = \int \frac{du}{1+u^4} = \frac{\pi}{2\sqrt{2}}.$$

If we make next $u\sqrt[4]{s} = t$, we will have

$$\iint e^{-s(1+u^4)} ds\,du = \int e^{-s}\frac{ds}{\sqrt[4]{s}} \int e^{-t^4} dt = C\int e^{-s}\frac{ds}{\sqrt[4]{s}}.$$

Let $s = s'^{\,4}$, and we will have

$$\int e^{-s}\frac{ds}{\sqrt[4]{s}} = 4\int s'^2 e^{-s'^4} ds' = 4C';$$

therefore

$$\iint e^{-s(1+u^4)} ds\,du = 4CC' = \frac{\pi}{2\sqrt{2}},$$

this which gives

$$C' = \frac{\pi}{8C\sqrt{2}}.$$

As for the value of C, it has not yet been possible, in spite of many attempts, to restore it to the arcs of a circle or to logarithms; but I have found that it depended on the rectification of the elastic rectangle curve or, what amounts to the same, on the integral $\int \frac{dx}{\sqrt{1-x^4}}$, taken from $x = 0$ to $x = 1$; if we designate by π' the value of this integral, we find[2]

$$\pi' = 1,31102877714605987$$

This put, we consider the double integral $\iint e^{-s^2(1+u^4)} ds\,du$, taken from $s = 0$ to $s = \infty$; by making $u\sqrt{s} = t$, it will become

$$\int e^{-s^2}\frac{ds}{\sqrt{s}} \int e^{-t^4} dt \quad \text{or} \quad C\int e^{-s^2}\frac{ds}{\sqrt{s}},$$

[2] *See* the treatise of Mr. Stirling, *De sommatione et interpolatione serierum*, p. 58.

Let $s = s'^2$, and we will have

$$\int e^{-s^2}\frac{ds}{\sqrt{s}} = 2\int e^{-s'^4}ds = 2C,$$

hence

$$\iint e^{-s^2(1+u^4)}ds\,du = 2C^2.$$

We suppose now $s\sqrt{1+u^4} = s''$, and we will have

$$\iint e^{-s^2(1+u^4)}ds\,du = \int\frac{du}{\sqrt{1+u^4}}\int e^{-s''^2} = \frac{1}{2}\sqrt{\pi}\int\frac{du}{\sqrt{1+u^4}};$$

by naming therefore E the integral $\int\frac{du}{\sqrt{1+u^4}}$, taken from $u = 0$ to $u = \infty$, we will have

$$2C^2 = \frac{1}{2}E\sqrt{\pi},$$

which gives

$$C = \frac{1}{2}\sqrt{E\sqrt{\pi}}.$$

If we make $\frac{1}{1+u^4} = s^4$, we will have

$$\int\frac{du}{\sqrt{1+u^4}} = \int\frac{ds}{(1-s^4)^{\frac{3}{4}}},$$

the integral relative to s being taken from $s = 1$ to $s = 0$, so that

$$\int\frac{du}{\sqrt{1+u^4}} = E = \int\frac{ds}{(1-s^4)^{\frac{3}{4}}},$$

the integral relative to s being taken from $s = 0$ to $s = 1$.

We consider presently the double integral $\iint\frac{dx\,dz}{(1-z^2-z^4)^{\frac{3}{4}}}$, taken[3] from $x = 0$ to $x = 1$ and from $z = 0$ to $z = 1$; by making $\frac{x}{(1-z^2)^{\frac{1}{4}}} = x'$, it will be changed into this here

$$\int\frac{dz}{\sqrt{1-z^2}}\int\frac{dx'}{(1-x'^4)^{\frac{3}{4}}}$$

these integrals being taken from $x' = 0$ and $z = 0$ to $x' = 1$ and $z = 1$, that which gives

$$\int\frac{dz}{\sqrt{1-z^2}} = \frac{\pi}{2} \quad \text{and} \quad \int\frac{dx'}{(1-x'^2)^{\frac{3}{4}}} = E;$$

[3]It is necessary without doubt to understand that this integral must be extended to all the positive values of x and of z verifying the inequality

$$1 - z^2 - z^4 > 0,$$

so that, for a given value of z, x varies from 0 to $(1-z^2)^{\frac{1}{4}}$; x increasing still to 1, the differential element would become imaginary. (*Note of the editor.*)

we will have therefore

$$\iint \frac{dx\,dz}{(1-z^2-z^4)^{\frac{3}{4}}} = \frac{\pi E}{2}.$$

If we make next $\frac{z}{\sqrt{1-x^4}} = z'$, we will have

$$\iint \frac{dx\,dz}{(1-z^2-z^4)^{\frac{3}{4}}} = \int \frac{dx}{\sqrt[4]{1-x^4}} \int \frac{dz'}{(1-z'^2)^{\frac{3}{4}}};$$

now we have

$$\int \frac{dx}{\sqrt[4]{1-x^4}} = \frac{\pi}{2\sqrt{2}}.$$

Moreover, if we suppose $1-z'^2 = t^4$, we will have

$$\int \frac{dz'}{(1-z'^2)^{\frac{3}{4}}} = -2\int \frac{dt}{\sqrt{1-t^4}},$$

the integral relative to t being taken from $t=1$ to $t=0$; this integral is evidently equal to $-\pi'$: therefore

$$\int \frac{dz'}{(1-z'^2)^{\frac{3}{4}}} = 2\pi',$$

which gives

$$\iint \frac{dx\,dz}{(1-z^2-z^4)^{\frac{3}{4}}} = \frac{\pi\pi'}{\sqrt{2}} = \frac{\pi E}{2},$$

hence

$$E = \pi'\sqrt{2},$$

whence we deduce

$$C = \frac{1}{2}\sqrt{\pi'\sqrt{2\pi}} \quad \text{and} \quad C' = \frac{\pi^{\frac{3}{4}}}{4\sqrt{\pi'\sqrt{2}}}.$$

XXV.

In order to apply the preceding theory to some examples, let

$$y = x^p(1-x)^q;$$

by making $p = \frac{1}{\alpha}$ and $q = \frac{\mu}{\alpha}$, we will have, in the case of the maximum of y,

$$x = \frac{1}{1+\mu},$$

hence (art. XXIII)

$$A = \frac{\mu^q}{(1+\mu)^{p+q}}$$

and

$$\alpha \log A = \mu \log\left(\frac{\mu}{1+\mu}\right) + \log\left(\frac{1}{1+\mu}\right);$$

we have besides

$$\alpha \log y = \log x + \mu \log(1 - x),$$

and, if we make

$$x = \frac{1}{1 + \mu} + \theta,$$

we will have

$$\alpha \log A = \mu \log \left(\frac{\mu}{1 + \mu} - \theta \right) + \log \left(\frac{1}{1 + \mu} + \theta \right);$$

therefore

$$\begin{aligned} \log A - \log y &= \frac{1}{\alpha} \log[1 + (1 + \mu)\theta] - \frac{\mu}{\alpha} \log \left(1 - \frac{1 + \mu}{\mu} \theta \right) \\ &= \frac{(1 + \mu)^2(1 + \mu)}{2\alpha\mu} \theta^2 + \frac{(1 + \mu)^3(1 - \mu^2)}{3\alpha\mu^2} \theta^3 + \frac{(1 + \mu)^4(1 + \mu^3)}{4\alpha\mu^3} \theta^4 + \cdots \end{aligned}$$

whence we will deduce, by virtue of the formula (z) of article XXIII,

$$\begin{aligned} \alpha^{\frac{1}{2}} h &= \frac{\sqrt{2\mu\alpha}}{(1 + \mu)^{\frac{3}{2}}}, \\ \alpha^{\frac{3}{2}} h^{(2)} &= \frac{\alpha\sqrt{2\mu\alpha}}{(1 + \mu)^{\frac{3}{2}}} \frac{[(1 + \mu)^2 - 13\mu]}{18\mu(1 + \mu)}, \\ &\cdots \end{aligned}$$

Formula (s) of article XXIII will give therefore

$$k = \frac{\sqrt{2\alpha\pi}\mu^{q + \frac{1}{2}}}{(1 + \mu)^{p + q + \frac{3}{2}}} \left\{ 1 + \frac{\alpha[(1 + \mu)^2 - 13\mu]}{12\mu(1 + \mu)} + \cdots \right\},$$

which is consistent with that which we have found in article XVIII.

If $\mu = 1$, or, what comes to the same, if $p = q$, we will determine more simply in the following manner the coefficients of the series in t, which expresses the value of θ; for this, we will observe that, in this case,

$$\log A - \log y = -\frac{1}{\alpha} \log(1 - 4\theta^2) = t^2,$$

that which gives

$$1 - 4\theta^2 = e^{-\alpha t^2}$$

and

$$2\theta = \left(1 - e^{-\alpha t^2} \right)^{\frac{1}{2}}.$$

Let

$$\left(1 - e^{-\alpha t^2} \right)^{\frac{1}{2}} = 2\alpha^{\frac{1}{2}} t (l + l'\alpha t^2 + l''\alpha^2 t^4 + l'''\alpha^3 t^6 + \cdots);$$

by taking the logarithmic differentials of the two members of this equation and cross multiplying them, we will have

$$\alpha t^2 e^{-\alpha t^2}(l + l'\alpha t^2 + l''\alpha^2 t^4 + \cdots) = (l + 3l'\alpha t^2 + 5l''\alpha^2 t^4 + \cdots)(1 - e^{-\alpha t^2});$$

now we will have

$$e^{-\alpha t^2} = 1 - \alpha t^2 + \frac{\alpha^2 t^4}{1.2} + \frac{\alpha^3 t^6}{1.2.3} + \cdots$$

If we substitute this value into the preceding equation, we will have among the coefficients l, l', l'', l''', ... the following equations

$$\begin{aligned} 2l' + \frac{t}{1.2} &= 0, \\ 4l'' - \frac{l'}{1.2} - \frac{2l}{1.2.3} &= 0, \\ &\cdots \end{aligned}$$

and generally

$$\begin{aligned} 0 = 2il^{(i)} &- (2i-3)\frac{l^{(i-1)}}{1.2} + (2i-6)\frac{l^{(i-2)}}{1.2.3} \\ &- (2i-9)\frac{l^{(i-3)}}{1.2.3.4} + (2i-12)\frac{l^{(i-4)}}{1.2.3.4.5} - \cdots \end{aligned}$$

by continuing this series until we arrive at the coefficient l. We will determine therefore easily $l', l'', l''', \ldots$ when this coefficient will be known; now, if we neglect the powers of t superior to unity, we have

$$\left(1 - e^{-\alpha t^2}\right)^{\frac{1}{2}} = \alpha^{\frac{1}{2}} t;$$

therefore $l = \frac{1}{2}$; formula (s) will give next, by observing that in this case $A = \frac{1}{2^p}$,

$$k = \frac{\sqrt{\alpha\pi}}{2^p}\left(1 + 1.3\alpha\frac{l'}{2} + 1.3.5\frac{\alpha^2 l''}{2^2} + \cdots\right).$$

We have generally

$$k = \int x^p(1-x)^q dx = \frac{1.2.3\ldots p.1.2.3\ldots q}{1.2.3\ldots(p+q+1)};$$

the assumption of $p = q$ gives

$$\frac{1.2.3\ldots 2p}{(1.2.3\ldots p)^2} = \frac{1}{(2p+1)k};$$

now the first member of this equation is the middle term of the binomial $(1+1)^{2p}$; we will have the value of this term by a very convergent series, when p is a very great number. If we compare the manner in which we have arrived there with those which

Messers. Stirling and Euler have employed, the first in his work *De transformatione et interpolatione serierum*, and the second in his *Institutions de Calcul différentiel*, we will find, if I do not deceive myself, that, independently of its generality, it has the advantage of being more direct, in that the processes of the two illustrious authors suppose that we know in advance the expression, in factors, of the ratio of the semi-circumference to the radius, an expression that Wallis has given; the one of Mr. Euler is, moreover, based on the value in series of the product $1.2.3\ldots p$, when p is a great number; this value is yet very easy to determine by our method. For this, let

$$y = x^p e^{-x},$$

we will have, by integrating from $x = 0$ to $x = \infty$,

$$\int x^p e^{-x} dx = p \int x^{p-1} e^{-x} dx,$$

whence it is easy to conclude

$$\int x^p e^{-x} dx = 1.2.3\ldots p.$$

The maximum of y takes place when $x = p$, that which gives $p^p e^{-p}$ for this maximum; let therefore $p = \frac{1}{\alpha}$ and $x = \frac{1}{\alpha} + \theta$, we will have

$$\log y - \log p^p e^{-p} = \frac{1}{\alpha} \log(1 + \alpha\theta) - \theta;$$

therefore

$$\int y\, dx = p^p e^{-p} \int e^{\frac{1}{\alpha} \log(1+\alpha\theta) - \theta} d\theta.$$

If we make

$$\log(1 + \alpha\theta) - \alpha\theta = -\alpha t^2;$$

we will have

$$\frac{\alpha\theta^2}{2} - \frac{\alpha^2\theta^3}{3} - \frac{\alpha^3\theta^4}{4} - \cdots = t^2;$$

let

$$\theta = \frac{1}{\sqrt{\alpha}} \left(h + h'\alpha^{\frac{1}{2}} t + h''\alpha t^2 + h'''\alpha^{\frac{3}{2}} t^3 + \cdots \right),$$

we will find

$$h = \sqrt{2}, \quad h' = \frac{2}{3}, \quad h'' = \frac{\sqrt{2}}{18}, \quad \ldots,$$

and we will have

$$d\theta = \frac{dt}{\sqrt{\alpha}} \left(h + 2h'\alpha^{\frac{1}{2}} t + 3h''\alpha t^2 + \cdots \right),$$

therefore

$$\int y\, dx = p^{p+\frac{1}{2}} e^{-p} \int dt \left(h + 2h'\alpha^{\frac{1}{2}} t + 3h''\alpha t^2 + \cdots \right) e^{-x}.$$

The integral in x must be taken from $x = 0$ to $x = \infty$; now, x being null, we have $\theta = -\frac{1}{\alpha}$, and consequently, $t^2 = \infty$; x being equal to ∞, we have $\theta = \infty$, hence $t^2 = \infty$; we must therefore take the integral relative to dt from $t = -\infty$ to $t = \infty$, whence we deduce, by article XXIII,

$$\int y\,dx = p^{p+\frac{1}{2}} e^{-p}\sqrt{\pi}\left(h + 1.3\frac{\alpha h''}{2} + 1.3.5\frac{\alpha^2 h^{iv}}{2^2} + \cdots\right),$$

hence

$$1.2.3\ldots p = p^{p+\frac{1}{2}} e^{-p}\sqrt{2\pi}\left(1 + \frac{1}{12}\alpha + \cdots\right).$$

We could apply this method to many other examples, and thence to extend and to perfect the theory of series; but this digression would separate us too far from our object.

XXVI.

The preceding method gives a quite simple solution to an interesting problem, which it is perhaps very difficult to resolve by other methods: we have seen (art. XIX) that the ratio of the births of boys to those of girls is sensibly greater in London than in Paris; this difference seems to indicate in London a greater facility for the birth of boys: the question is to determine how much this is probable.

For this, let
u be the probability of the birth of a boy in Paris;
p be the number of births of boys observed in this city;
q be the one of girls;
$u - x$ be the possibility of the birth of a boy in London;
p' be the number of births of boys that we have observed;
q' be the one of girls.
We will have, for the probability of this double event,

$$Hu^p(1-u)^q(u-x)^{p'}(1-u+x)^{q'},$$

H being a constant coefficient; therefore, if we name P the probability that the birth of a boy is less possible in London than in Paris, we will have

$$P = \frac{\iint u^p(1-u)^q(u-x)^{p'}(1-u+x)^{q'}\,dx\,du}{\iint u^p(1-u)^q(u-x)^{p'}(1-u+x)^{q'}\,dx\,du},$$

the integral of the numerator being taken from $u = 0$ to $u = x$ and from $x = 0$ to $x = 1$. That of the denominator must be taken over all the possible values of x and u; now, if we make $u - x = s$, this denominator will become

$$\iint u^p(1-u)^q s^{p'}(1-s)^{q'}\,du\,ds,$$

the double integral being taken from $u = 0$ to $u = 1$ and from $s = 0$ to $s = 1$: we will have thus

$$P = \frac{\iint u^p(1-u)^q(u-x)^{p'}(1-u+x)^{q'}\,dx\,du}{\iint u^p(1-u)^q s^{p'}(1-s)^{q'}\,ds\,du}.$$

We determine first the integral of the numerator.

By naming y the quantity

$$u^p(1-u)^q(u-x)^{p'}(1-u+x)^{q'},$$

we will have very nearly, by formula (μ) of article XXIII,

$$\int y\,dx = \frac{\sqrt{2\pi}y^{\frac{3}{2}}}{\sqrt{-\frac{\partial^2 y}{\partial u^2}}},$$

by substituting for u, in the second member of this equation, its value in x, which renders y a maximum; let X be this value, we have

$$\frac{\partial y}{\partial u} = y\left(\frac{p}{u} - \frac{q}{1-u} + \frac{p'}{u-x} - \frac{q'}{1-u+x}\right)$$

and

$$\begin{aligned}-\frac{\partial^2 y}{\partial u^2} =& y\left[\frac{p}{u^2} - \frac{q}{(1-u)^2} + \frac{p'}{(u-x)^2} - \frac{q'}{(1-u+x)^2}\right] \\ &- \frac{\partial y}{\partial u}\left(\frac{p}{u} - \frac{q}{1-u} + \frac{p'}{u-x} - \frac{q'}{1-u+x}\right).\end{aligned}$$

If we substitute X in place of u, we have, by the condition of the maximum $\frac{\partial y}{\partial u} = 0$, hence

$$-\frac{\partial^2 y}{\partial u^2} = y\left[\frac{p}{X^2} - \frac{q}{(1-X)^2} + \frac{p'}{(X-x)^2} - \frac{q'}{(1-X+x)^2}\right],$$

whence we deduce

$$\int y\,du = \frac{\sqrt{2\pi}y}{\sqrt{\frac{p}{X^2} + \frac{q}{(1-X)^2} + \frac{p'}{(X-x)^2} + \frac{q'}{(1-X+x)^2}}},$$

X being determined by the equation

$$(t) \qquad 0 = \frac{p}{X} - \frac{q}{1-X} + \frac{p'}{X-x} - \frac{q'}{1-X+x}$$

or

$$(t') \qquad \left\{\begin{aligned}0 =& X(1-X)[(p+p')(1-X) - (q+q')X] \\ &+ x\{(p'+q')X(1-X) + (1-2X)[qX - p(1-X)]\} \\ &+ x^2[qX - p(1-X)].\end{aligned}\right.$$

Let, for brevity,

$$R = \sqrt{\frac{p}{X^2} + \frac{q}{(1-X)^2} + \frac{p'}{(X-x)^2} + \frac{q'}{(1-X+x)^2}};$$

the question is reduced to determining the integral $\sqrt{2\pi}\int \frac{y\,dx}{R}$, from $x = 0$ to $x = 1$. In place of this integral, we can consider this one $\sqrt{2\pi}\int \frac{y\,dx}{RdX}dX$, x being regarded as function of X; but it is necessary to take this last integral from the value of X which takes place when $x = 0$ to that which takes place when $x = 1$; now, by making $x = 0$, equation (t') becomes

$$0 = (p + p')(1 - X) - (q + q')X,$$

hence

$$X = \frac{p + p'}{p + p' + q + q'}.$$

By making $x = 1$, this equation gives $X = 1$; we must therefore take the integral $\int \frac{y\,dx}{RdX}dX$, from $X = \frac{p+p'}{p+p'+q+q'}$ to $X = 1$.

We suppose $\frac{y\,dx}{RdX} = y'$: the condition of the maximum of y' gives the equation

$$0 = \frac{dy}{y} + \frac{d\left(\frac{1}{R}\frac{dx}{dX}\right)}{\frac{1}{R}\frac{dx}{dX}};$$

now, y being equal to $X^p(1 - X)^q(X - x)^{p'}(1 - X - x)^{q'}$, we have

$$\frac{dy}{y} = \left(\frac{p}{X} - \frac{q}{1 - X} + \frac{p'}{X - x} - \frac{q'}{1 - X + x}\right)dX + \left(\frac{q'}{1 - X + x} - \frac{p'}{X - x}\right)dx;$$

this equation is reduced, by virtue of equation (t), to this one

$$\frac{dy}{y} = \frac{q'dx}{1 - X + x} - \frac{p'dx}{X - x} = \left(\frac{p}{X} - \frac{q}{1 - X}\right)dx.$$

Now, p and q being very great numbers, it is clear that $\frac{dy}{y}$ is incomparably greater than $\frac{d\left(\frac{1}{R}\frac{dx}{dX}\right)}{\frac{1}{R}\frac{dx}{dX}}$, and that thus we can neglect the second of these two differentials with respect to the first; we will have therefore, very nearly, in the case of the maximum of y',

$$0 = \frac{p}{X} - \frac{q}{1 - X},$$

hence

$$X = \frac{p}{p + q}.$$

This value of X is less than $\frac{p+p'}{p+p'+q+q'}$, when $\frac{p'}{p'+q'}$ is, as we suppose it here, greater than $\frac{p}{p+q}$; the two limits in which it is necessary to take the integral $\int y'dx$ are, consequently, beyond the value of X which renders y' a maximum; thus we must, to determine this integral, make usage of the series (λ) of article XVIII.

We have, very nearly,

$$\frac{dy'}{y'} = \frac{dy}{y} = \left(\frac{p}{X} - \frac{q}{1 - X}\right)dx;$$

moreover, by differentiating the equation

$$0 = \frac{p}{X} - \frac{q}{1-X} + \frac{p'}{X-x} - \frac{q'}{1-X+x},$$

we find

$$\frac{dx}{dX} = \frac{R^2}{\frac{p'}{(X-x)^2} + \frac{q'}{(1-X+x)^2}};$$

therefore

$$\frac{dy'}{y'} = \frac{R^2}{\frac{p'}{(X-x)^2} + \frac{q'}{(1-X+x)^2}} \frac{p-(p+q)X}{X(1-X)} dX.$$

Let

$$p = \frac{1}{\alpha}, \quad q = \frac{\mu}{\alpha}, \quad p' = \frac{\nu}{\alpha}, \quad q' = \frac{\nu'}{\alpha};$$

the quantity which we have named z in article XVIII will be therefore

$$\frac{X(1-X)\left[\frac{\nu}{(X-x)^2} + \frac{\nu'}{(1-X+x)^2}\right]}{\left[\frac{1}{X^2} + \frac{\mu}{(1-X)^2} + \frac{\nu}{(X-x)^2} + \frac{\nu'}{(1-X+x)^2}\right][1-(1+\mu)X]},$$

X being the principal variable of which x is a function; if we observe next that, X being equal to unity, we have $y' = 0$, the series (γ) of the article cited will give

$$\int y'dx = -\alpha y' z \left\{1 - \alpha\frac{dz}{dX} + \alpha^2 \frac{d(zdz)}{dX^2} - \alpha^3 \frac{d[zd(zdz)]}{dX^3} + \cdots\right\},$$

by substituting, after the differentiation in the second member of this equation $\frac{p+p'}{p+p'+q+q'}$ for X and by making $x = 0$ in it.

If we suppose $X = \frac{p}{p+q} + \theta$, θ will be equal to $\frac{p+p'}{p+p'+q+q'} - \frac{p}{p+q}$, and we will have

$$z = -\frac{X(1-X)\left[\frac{\nu}{(X-x)^2} + \frac{\nu'}{(1-X+x)^2}\right]}{(1+\mu)\theta\left[\frac{1}{X^2} + \frac{\mu}{(1-X)^2} + \frac{\nu}{(X-x)^2} + \frac{\nu'}{(1-X+x)^2}\right]};$$

now, θ being very small, the successive differences of z grow principally by the differentiation of the factor θ which is found in the denominator, so that, if we suppose

$$F = \frac{X(1-X)\left[\frac{\nu}{(X-x)^2} + \frac{\nu'}{(1-X+x)^2}\right]}{(1+\mu)\left[\frac{1}{X^2} + \frac{\mu}{(1-X)^2} + \frac{\nu}{(X-x)^2} + \frac{\nu'}{(1-X+x)^2}\right]},$$

we will have, very nearly,

$$\frac{dz}{dX} = \frac{F}{\theta^2},$$
$$\frac{d(zdz)}{dX^2} = \frac{3F^2}{\theta^4},$$
$$\cdots$$

hence

$$\int y' dx = \frac{\alpha y' F}{\theta}\left(1 - \frac{\alpha F}{\theta^2} + \frac{3\alpha^2 F^2}{\theta^4} - \cdots\right),$$

y' and F being that which these quantities become when we suppose $x = 0$ and $X = \frac{p+p'}{p+p'+q+q'}$, that which gives

$$y'F = \frac{(p+p')^{p+p'+\frac{3}{2}}(q+q')^{q+q'+\frac{3}{2}}}{(1+\mu)(p+p'+q+q')^{p+p'+q+q'+\frac{7}{2}}}.$$

It is easy to see by the analysis of article XVIII that $\int y' dx$ is less than $\frac{\alpha y' F}{\theta}$, greater than $\frac{\alpha y' F}{\theta}\left(1 - \frac{\alpha F}{\theta^2}\right)$ and less than $\frac{\alpha y' F}{\theta}\left(1 - \frac{\alpha F}{\theta^2} + \frac{3\alpha^2 F^2}{\theta^4}\right)$, so that we have in this manner the limits in which the value of $\int y' dx$ is narrowed.

We seek now the value of the double integral

$$\iint u^p (1-u)^q s^{p'} (1-s)^{q'} \, ds \, du.$$

Formula (μ) of article XXIII gives, very nearly,

$$\int u^p (1-u)^q du = \sqrt{2\pi} \frac{u^{p+1}(1-u)^{q+1}}{\sqrt{[p(1-u)^2 + qu^2]^2}},$$

by substituting for u the value which renders $u^p(1-u)^q$ a maximum; now this value is $\frac{p}{p+q}$; we have therefore

$$\int u^p (1-u)^q du = \sqrt{2\pi} \frac{p^{p+\frac{1}{2}} q^{q+\frac{1}{2}}}{(p+q)^{p+q+\frac{3}{2}}}.$$

By changing p to p' and q to q', we will have

$$\int s^{p'} (1-s)^{q'} ds = \sqrt{2\pi} \frac{p'^{p'+\frac{1}{2}} q'^{q'+\frac{1}{2}}}{(p'+q')^{p'+q'+\frac{3}{2}}},$$

hence

$$\iint u^p (1-u)^q s^{p'} (1-s)^{q'} \, ds \, du = 2\pi \frac{p^{p+\frac{1}{2}} q^{q+\frac{1}{2}} p'^{p'+\frac{1}{2}} q'^{q'+\frac{1}{2}}}{(p+q)^{p+q+\frac{3}{2}} (p'+q')^{p'+q'+\frac{3}{2}}}.$$

If we suppose this quantity equal to k, we will have for the sought probability P

$$P = \frac{\alpha y' F \sqrt{2\pi}}{\theta h}\left(1 - \frac{\alpha F}{\theta^2} + \frac{3\alpha^2 F^2}{\theta^4} - \cdots\right);$$

there is no longer concern but to determine the numerical values of the different terms of this expression, by starting from the preceding data. These data are

$$p = 251527, \qquad p' = 737629,$$
$$q = 241945, \qquad q' = 698958;$$

whence it is easy to conclude

$$\log F = \overline{2}, 9767121,$$
$$\log \theta = \overline{3}, 4457598,$$
$$\log \alpha = \overline{6}, 5994154$$

and, consequently,

$$\frac{\alpha F}{\theta^2} = 0,048374,$$
$$\frac{3\alpha^2 F^2}{\theta^4} = 0,007020.$$

We have next, by carrying the precision to a dozen decimals,

$$\log p = 5,400584610947,$$
$$\log q = 5,383716651469,$$
$$\log(p+q) = 5,693262515480,$$

whence we deduce

$$\log\left(\frac{p}{p+q}\right)^p = \overline{73617}, 6047065,$$
$$\log\left(\frac{p}{p+q}\right)^q = \overline{74894}, 9259319.$$

We have similarly

$$\log p' = 5,867837982735,$$
$$\log q' = 5,844451080009,$$
$$\log(p'+q') = 5,157331932083,$$

whence we deduce

$$\log\left(\frac{p'}{p'+q'}\right)^{p'} = \overline{213540}, 8676364,$$
$$\log\left(\frac{p'}{p'+q'}\right)^{q'} = \overline{218691}, 4253961.$$

We find again

$$\log(p+p') = 5,995264741371,$$
$$\log(q+q') = 5,973544853243,$$
$$\log(p+p'+q+q') = 6,285570585161,$$

whence we deduce

$$\log\left(\frac{p+p'}{p+p'+q+q'}\right)^{p+p'} = \overline{287158}, 2327801,$$
$$\log\left(\frac{p+p'}{p+p'+q+q'}\right)^{q+q'} = \overline{293586}, 0527612.$$

We have finally

$$\log(1+\mu) = 0,2926796,$$
$$\log 2\pi = 0,7981799,$$

hence

$$\log \frac{\alpha y' F\sqrt{2\pi}}{\theta} = \overline{580751},4993272,$$
$$\log k = \overline{580745},0942543,$$

that which gives

$$\frac{\alpha y' F\sqrt{2\pi}}{\theta k} = 0,0000025414;$$

therefore

$$P = 0,0000025414(1 - 0,048374 + 0,007020 - \cdots).$$

If we take the first three terms of the series, we will have

$$P = \frac{1}{410458};$$

this value of P is a little too great; but, since taking in it only the first two terms of the series we would have a value too small, it is easy to conclude from it that the preceding can differ from the truth by the $\frac{1}{142}$ part of its value, so that it is a good approximation: there is therefore odds of more than four hundred thousand against one that the births of boys are more facile in London than in Paris. Thus we can regard as a very probable thing that there exists, in the first of these two cities, a cause more than in the second, which facilitates the births of boys, and which depends either on the climate or on the nourishment of the mothers.

XXVII.

It is easy to extend the theory of the preceding articles to the case of three or of a greater number of simple events.

Let us name, in fact, x the possibility of the first simple event, x' that of the second and, consequently, $1 - x - x'$ that of the third; in seeking, by the ordinary methods, the probability of the observed event, we will have for its value a function of x, x' and $1 - x - x'$, multiplied by any constant. Let y be this function, for which the observed event can indicate in an approximate manner the possibilities of the simple events, it is necessary, as we have observed in art. XXII, that $\frac{\frac{\partial y}{\partial x}}{y}$ and $\frac{\frac{\partial y}{\partial x'}}{y}$ are some very great functions of x of order $\frac{1}{\alpha}$, α being a coefficient proportionally lesser as the observed event is more composite; this put, if we integrate $\int y\,dx'$, from $x' = 0$ to $x' = 1 - x$, we will have for result a function of x, that the method of art. XXIII will give by a highly convergent series. Let u be the value of x' to x which renders y a maximum, x being supposed constant, and if we represent by Y this maximum, we will have, by the article cited, for $\int y\,dx'$, an expression of this form

$$\int y\,dx' = Y\sqrt{\alpha\pi}\left(h + 1.3\frac{\alpha h''}{2} + 1.3.5\frac{\alpha^2 h^{iv}}{2^2} + \cdots\right),$$

Y, h, h'', h^{iv}, ... being some function of x. The value of x which renders the second member of this equation a maximum will be very near to the true possibility of the first event; let a be this value, we will have for the expression of the probability P that x will be contained in the limits $a-\theta$ and $a+\theta$

$$P = \frac{\int Y\,dx\left(h + 1.3\frac{\alpha h''}{2} + 1.3.5\frac{\alpha^2 h^{iv}}{2^2} + \cdots\right)}{\int Y\,dx\left(h + 1.3\frac{\alpha h''}{2} + 1.3.5\frac{\alpha^2 h^{iv}}{2^2} + \cdots\right)},$$

the integral of the numerator being taken from $x = a-\theta$ to $x = a+\theta$, and that of the denominator being taken from $x = 0$ to $x = 1$; now we will determine easily these integrals by the method of art. XXIII.

The value a is determined by equating to zero the difference of $Y\left(h + 1.3\frac{\alpha h''}{2} + \cdots\right)$, which gives

$$0 = \frac{dY}{Y} + \frac{dh + 1.3\frac{\alpha dh''}{2} + \cdots}{h + 1.3\frac{\alpha h''}{2} + \cdots};$$

$\frac{dY}{Y}$ is, by assumption, a very great quantity of the order $\frac{1}{\alpha}$; by neglecting therefore, vis-à-vis of it, the quantity

$$\frac{dh + 1.3\frac{\alpha dh''}{2} + \cdots}{h + 1.3\frac{\alpha h''}{2} + \cdots}$$

we will have, in order to determine a, the equation

$$0 = \frac{\partial Y}{\partial x}$$

Now we have

$$\frac{\partial Y}{\partial x} = \frac{\partial y}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial x'}{\partial x},$$

by substituting into the second member of this equation, in place of x', its value u in x; but this value renders null the quantity $\frac{\partial y}{\partial x'}$; we will have therefore the two equations

$$\frac{\partial y}{\partial x} = 0 \quad \text{and} \quad \frac{\partial y}{\partial x'} = 0.$$

It follows thence that a is, in the quantities near to order α, the value of x which renders y a maximum, by making vary at the same time x and x'; we can therefore take, without sensible error, the value of x corresponding to this maximum, for the possibility of the first simple event, and it is clear that we can make some analogous remarks on the possibilities of two other simple events.

We suppose, for example, that there is in an urn an infinity of white balls, reds and blacks, in an unknown proportion, and that out of the number $p+q+r$ drawings we can bring forth p white balls, q red balls and r black balls; by naming x the facility to bring forth a white ball, x' that of bringing forth a red ball and, consequently, $1-x-x'$ that of bringing forth a black ball, we will have, for the probability of an observed event,

$$\frac{1.2.3\ldots(p+q+r)}{1.2.3\ldots p.1.2.3\ldots q.1.2.3\ldots r}x^p x'^q(1-x-x')^r.$$

In this particular case,

$$y = x^p x'^q (1 - x - x')^r,$$
$$\int y\,dx' = \frac{1.2.3\ldots q.1.2.3\ldots r}{1.2.3\ldots(q+r+1)} x^p (1-x)^{q+r+1};$$

the value of x which renders $\int y\,dx$ a maximum is $\frac{p}{p+q+r+1}$; this fraction is consequently the most probable value of x. When p, q and r are great numbers, it is reduced to very nearly to this one $\frac{p}{p+q+r}$; which corresponds to the maximum of y.

XXVIII.

Until now we have assumed the law of possibility of the simple events constant from zero to unity, and this assumption is, as we have observed in article XVII, the sole one which we must adopt, when we have none other given relatively to these possibilities; but, if their law were exactly known, we could again apply the preceding researches. For this, we will consider only two simple events, and we call x the possibility of the first and $1 - x$ the possibility of the second; we will calculate the probability of the observed event, by starting from these possibilities, and we will have for its expression a function of x, that we will designate by y; if we represent next by u the facility of the possibility of x of the first event, u being function of x, and by s the facility of the possibility $1-x$ of the second event, we will have, by article XV, $\frac{usy\,dx}{\int usy\,dx}$ for the probability that the observed event is due to the possibilities x and $1 - x$, the integral of the denominator being taken from $x = 0$ to $x = 1$; therefore, if we name P the probability that the value of x is contained within the given limits, we will have

$$P = \frac{\int usy\,dx}{\int usy\,dx},$$

provided that the integral of the numerator is taken only within the extent of these limits. We see thus that this case returns to those which we have considered in the preceding articles, and that the value of P will be determined easily by the method of these articles.

The value of x which renders usy a maximum will be very close to the truth, if the observed event is very composite and if we have $ydx = \alpha z dy$, α being a very small coefficient; now we have, by equating to zero the differential of $usdy$,

$$0 = \frac{d(us)}{us} + \frac{dy}{y},$$

hence

$$0 = \frac{\alpha d(us)}{us} + \frac{1}{z}.$$

We will have therefore, by neglecting the quantities of order α, $0 = \frac{1}{z}$, whence it follows that the value of x which renders y a maximum is very close to the truth, whatever be moreover the law of facilities of the possibilities of the two simple events.

XXIX.

After having determined the possibilities of the simple events which result from a composite event proper to making them known, there remains for us to consider the influence of this event on the probability of any future event, and the manner in which we must calculate this probability. If we name x and $1-x$ the possibilities of two simple events, s the facility of x, and s' that of $1-x$, we will calculate the probabilities, as much of the observed event as of the future event, by starting from these possibilities, and we will have for result two functions of x, of which we will represent the first by y and the second by u; this put, if we name P the sought probability of the future event, we will have, by articles XIV and XV,

$$P = \frac{\int ss'uy\,dx}{\int ss'y\,dx},$$

the integrals of the numerator and of the denominator being taken from $x = 0$ to $x = 1$. When the observed event will be very composite, the method of article XXIII will give these integrals by a very rapid approximation, that which indicates the extent of this method and its utility in these matters.

If we have nothing given on the law of possibility of the two simple events, which is the most ordinary case, we must suppose (art. XVII) s and s' equal to unity, which gives

$$P = \frac{\int uy\,dx}{\int y\,dx};$$

now we have, very nearly, by article XXIII,

$$\left(\int y\,dx\right)^2 = \frac{2\pi y^3}{-\frac{d^2y}{dx^2}}$$
$$\left(\int uy\,dx\right)^2 = \frac{2\pi u'^3 y'^3}{-\frac{d^2(u'y')}{dx^2}}$$
$$\cdots$$

y and $\frac{d^2y}{dx^2}$ being that which the quantities become when we substitute in them for x the value which renders y a maximum, and u', y' and $\frac{d^2y'}{dx^2}$ being that which u, y and $\frac{d^2y}{dx^2}$ become when we substitute in them for x the value which renders uy a maximum; we will have therefore

$$(\alpha) \qquad P^2 = \frac{u'^3y'^3}{y^3}\frac{\frac{d^2y}{dx^2}}{\frac{d^2(u'y')}{dx^2}}.$$

We suppose that the future event of which we calculate the probability is very little composite, by equating to zero the differential of uy, we will have

$$0 = \frac{dy}{y\,dx} + \frac{du}{u\,dx};$$

but we have, by assumption,

$$\frac{dy}{y\,dx} = \frac{1}{\alpha z} = \frac{1}{\alpha} z',$$

by making $\frac{1}{z} = z'$; the preceding equation will become thus

$$0 = \alpha \frac{du}{u\,dx} + z'.$$

Let a be the value of x which renders y a maximum, and consequently z' null; the value of x which renders uy a maximum can therefore be represented by $a + \alpha h$, h being any coefficient, and we will have

$$y' = y + \alpha h \frac{dy}{dx} + \frac{\alpha^2 h^2}{1.2} \frac{d^2 y}{dx^2} + \frac{\alpha^3 h^3}{1.2.3} \frac{d^3 y}{dx^3} + \cdots,$$

y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, …being that which these quantities become when we make $x = a$ in them; we have next

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\alpha} y z', \\
\frac{d^2 y}{dx^2} &= y \left(\frac{1}{\alpha^2} z'^2 + \frac{1}{\alpha} \frac{dz'}{dx} \right), \\
\frac{d^3 y}{dx^3} &= y \left(\frac{1}{\alpha^3} z'^3 + \frac{3}{\alpha^2} z' \frac{dz'}{dx} + \frac{1}{\alpha} \frac{d^2 z'}{dx^2} \right), \\
&\cdots
\end{aligned}$$

The assumption of $x = a$ gives $z' = 0$, and, consequently,

$$\begin{aligned}
\alpha h \frac{dy}{dx} &= 0, \\
\frac{\alpha^2 h^2}{1.2} \frac{d^2 y}{dx^2} &= \frac{\alpha h^2}{1.2} y \frac{dz'}{dx}, \\
\frac{\alpha^3 h^3}{1.2.3} \frac{d^2 y}{dx^2} &= \frac{\alpha^2 h^3}{1.2.3} \frac{d^2 z'}{dx^2}, \\
&\cdots;
\end{aligned}$$

we will have therefore, by neglecting the terms multiplied by α,

$$y = y'.$$

We have moreover

$$\frac{d^2(u'y')}{dx^2} = u' \frac{d^2 y'}{dx^2} + \frac{2du'}{dx} \frac{dy'}{dx} + y' \frac{d^2 u'}{dx^2};$$

now it is clear, by that which precedes, that $\frac{d^2y'}{dx^2}$ is much greater than $\frac{dy'}{dx}$ and than y', so that we can suppose, very nearly

$$\frac{d^2(u'y')}{dx^2} = u' \frac{d^2 y'}{dx^2},$$

and we will prove, as we just did for y', that $\frac{d^2y'}{dx^2}$ can be supposed equal to $\frac{d^2y}{dx^2}$, hence

$$\frac{d^2(u'y')}{dx^2} = u'\frac{d^2y}{dx^2}.$$

Formula (α) will become therefore

$$P^2 = u'^2;$$

but, if we name v that which u becomes when we make $x = a$ in it, we will have, by neglecting the quantities of order α, $u' = v$; therefore

$$P = \nu;$$

whence it follows that we can calculate the probability P of the future event by employing for x the value which renders y a maximum.

This theorem would cease to be exact if the future event in question was itself quite composite, because then $\frac{du}{udx}$ would be very great, and the value of x, which gives the equation

$$0 = \alpha\frac{du}{u\,dx} + z',$$

could no longer be represented by $a + \alpha h$; we could no longer moreover suppose $\frac{d^2(u'y')}{dx^2}$ equal to $u'\frac{d^2y}{dx^2}$. If we represent, in general, by $u + \alpha^n h$ the root of the equation

$$0 = \alpha\frac{du}{u\,dx} + z,$$

n being an exponent less than unity, we will have

$$y' = y + \alpha^n h\frac{dy}{dx} + \frac{\alpha^{2n}h^2}{1.2}\frac{d^2y}{dx^2} + \cdots,$$

and we will find

$$\alpha^n h\frac{dy}{dx} = 0,$$
$$\frac{\alpha^{2n}h^2}{1.2}\frac{d^2y}{dx^2} = \frac{\alpha^{2n-1}h^2}{1.2}y\frac{dz'}{dx},$$
$$\cdots,$$

hence

$$y' = y + \frac{\alpha^{2n-1}h^2}{1.2}y\frac{dz'}{dx} + \cdots$$

This value of y' is reduced to y, in the case of α infinitely small, only when $2n - 1$ is positive, that which supposes $n > \frac{1}{2}$, and it is easy to see likewise that it is only under this supposition that $\frac{d^2(u'y')}{dx^2}$ is reduced to $\frac{n'd^2y}{dx^2}$; the preceding theorem can therefore hold only in the case where $2n$ is greater than unity.

Let $\frac{du}{u\,dx} = \frac{\lambda}{\alpha^{1-n'}}$, λ being a function of x, the equation

$$0 = \frac{\alpha\,du}{u\,dx} + z'$$

will become

$$0 = \alpha^{n'-1}\lambda + z',$$

that which gives for x an expression of this form

$$x = a + \alpha^{n'-1}h;$$

now the truth of the preceding theorem requires that we have $n' - 1 > \frac{1}{2}$, and , consequently, $1 - n' < -\frac{1}{2}$; therefore, so that this theorem subsists, it is necessary that the future event be little enough composite relatively to the observed event, in order that $\left(\frac{du}{u\,dx}\right)^2$ be a function of x, very small relatively to $\frac{dy}{y\,dx}$.

If the future event is exactly the same as the observed event, so that $u = y$, the value a of x, which render y a maximum, will render similarly uy a maximum, so that we will have $y' = y$ and $u' = v$. We will have next

$$\frac{d^2(u'y')}{dx^2} = 2y\frac{d^2y}{dx^2} + \frac{2dy^2}{dx^2};$$

but the substitution of a for x gives $\frac{dy}{dx} = 0$, hence

$$\frac{d^2(u'y')}{dx^2} = 2y\frac{d^2y}{dx^2}.$$

Formula (α) will become therefore

$$P^2 = \frac{\nu^2}{2},$$

ν being that which u or y becomes when we make $x = a$ in it; thence results this sufficiently remarkable theorem:

The probability of a future event, similar to the one that we have observed, is to that same probability, determined by employing for the possibilities of the simple events those which result from the observed events, as 1 *is to* $\sqrt{2}$.

If we have observed, for example, that out of $p + q$ infants, there are born p boys and q girls, and if we seek the probability P, that out of $p + q$ infants who should be born there will be p boys and q girls, we will have

$$P = \frac{1.2.3\ldots(p+q)}{1.2.3\ldots p.1.2.3\ldots q}\frac{p^p q^q}{\sqrt{2}(p+q)^{p+q}};$$

this is what results similarly from formula (ϖ) of article XVII.

In general, if we seek the probability P that the observed event will be followed by a number n of similar events, we will have $u = y^n$, and we will find

$$P = \frac{\nu^n}{\sqrt{n+1}},$$

ν being that which y becomes, when we substitute for x in it the value a which renders y a maximum, and this equation takes place equally, n being fractional. We will be exposed therefore then to some considerable errors, by employing, in the calculation of the probability of future events, the possibilities of the simple events which result from the observed event: in reality, it is clear that the small error which we can commit, by making usage of these possibilities, is accumulated by reason of the number of simple events which enter into the future event, and must occasion a sensible error when they are in very great number. Besides, whatever be this event, we can determine the probability of it by means of formula (α), which is always valid, quite nearly, when the observed event is very composite.

XXX.

One of the most useful problems of this part of analysis of chances, which consists in reascending from the events to the causes which have produced them, is the one of the determination of the mean which it is necessary to choose among the results of several observations. I have given, in Book VI of the *Mémoires des Savants étrangers*[4], the principles on which it seems to me that the solution of this problem must be founded; three illustrious geometers, Messrs. de la Grange, Daniel Bernoulli and Euler, have since exercised themselves on this object: the first in Book V of the *Mémoires de la Societé royale de Turin*, and the two others in the first part of the *Mémoires de Pétersbourg* for the year 1777; but their principles were different from those to which I have availed myself, this consideration engaged me to resume here this matter and to present my results in a way to leave no doubt on their exactitude.

We suppose, in order to fix the ideas, that the question is of a phenomenon which has been noticed by many observers at some different instants; each observation has been able to be deviated more or less from the truth and to fix thus the instant of the phenomenon earlier or later than it has happened. We will suppose, that which is very natural, that the facilities of the same errors, either to plus, or to minus, are equal among themselves, and we will designate by $\phi(x)$ the facility so much for the positive error x as for the negative error $-x$, relatively to the first observer; by $\phi'(x)$, $\phi''(x)$, ... these same facilities for the second, third, ... observers. By naming next first the observation which fixes earliest the phenomenon, the second, third observations, etc. the different observations in the order of their distances to that one, we will name p, p', p'', ... these distances; by supposing therefore x the error of the first observation, the errors of the following observations will be $p-x$, $p'-x$, $p''-x$, ... and the probability that all these observations will have among them the respective distances p, p', p'', ... will be

$$\phi(x)\phi'(p-x)\phi''(p'-x),\ldots;$$

now the probabilities of the different values of x are among them, by article XV, as the probabilities that, these values taking place, the observations will be deviated among them with the observed quantities p, p', p'', ... Therefore, if we construct a curve of which the equation is

$$y = \phi(x)\phi'(p-x)\phi''(p'-x),\ldots,$$

the ordinates of this curve will be proportional to the probabilities of the corresponding abscissas x, and for this reason we will name it the *curve of the probabilities*.

[4]'Mémoire sur la probabilité des causes par les evenemens," *Oeuvres de Laplace*, T. VIII, p. 27.

Now, we can intend an infinity of different things by the *mean* or the *mean result* of any number of observations, according as we subject this result to such or such condition. For example, we can require that the mean be such that the sum of the errors to fear to the *plus* be equal to the sum of the errors to fear to the *minus*; we can require that the sum of the errors to fear to the plus, multiplied by their respective probabilities, be equal to the sum of the errors to fear to the minus, multiplied by their respective probabilities. We can again subject this mean to be the point where it is the most probable that it must fall the true instant of the phenomenon, as Mr. Daniel Bernoulli has made in the Memoir cited: in general we can impose an infinity of other similar conditions which will give each a different mean; but they are not all arbitrary. There is one of them which keeps to the nature of the problem and which must serve to fix the mean that it is necessary to choose among many observations: this condition is that, by fixing at this point the instant of the phenomenon, the error which results from it is a minimum; now since, in the ordinary theory of chances, we evaluate the advantage by making a sum of the products of each advantage to hope, multiplied by the probability to obtain it, likewise here the error must be estimated by the sum of the products of each error to fear, multiplied by its probability; the mean which it is necessary to choose must therefore be such that the sum of its products is less than that for all other instants.

We suppose now that, in the curve of probabilities of which the equation is

$$y = \phi(x)\phi'(p-x)\ldots,$$

the value of x can be extended from $-f$ to $+f$, so that the interval in which x can vary is c; if we make $x = z - f$, it is clear that z can vary from $z = 0$ to $z = c$, and that the probabilities of different values of z will be proportional to y or to $\phi(z-f)\phi'(p-z+f)\cdots$, so that we can represent them by ky, k being a constant coefficient. Let h be the value of z that we must take for the true instant of the phenomenon, we will have $k\int(h-z)y\,dz$ for the sum of the errors to fear from $z = 0$ to $z = h$, multiplied by their respective probabilities, the preceding integral being taken in all the extent of these limits; we will have next $k\int'(z-h)y\,dz$ for the sum of the errors to fear from $z = h$ to $z = c$, multiplied by their probabilities, the sign $\int'$ serving to indicate that the integral must be taken for all the extent of these last limits. We will have

$$k\int(h-z)y\,dz + k\int'(z-h)y\,dz$$

for the entire sum of the errors to fear, multiplied by their probabilities, and h must be such that this sum is a minimum. Now if we make h vary with the infinitely small quantity δh, it is clear that the variation of $\int(h-z)y\,dz$ will be $\delta h\int y\,dz$ and that that of $\int'(z-h)y\,dz$ will be $-\delta h\int' y\,dz$; the variation of the preceding quantity will be therefore

$$k\delta h\left(\int y\,dz - \int' y\,dz\right).$$

By equating this quantity to zero by the property of the minimum, we will have

$$\int y\,dz = \int' y\,dz.$$

The ordinate corresponding to the abscissa h, which determines the mean which it is necessary to choose, must therefore divide into two equal parts the area of the curve of probabilities, contained between $z = 0$ and $z = c$, this which gives a very simple way to determine this mean, and we see that it has again the property of being such, that it is equally probable that the true instant of the phenomenon fall above or below, so that we can name it the *mean of probability*.

XXXI.

Whenever the functions $\phi(x)$, $\phi'(x)$, $\phi''(x)$, ... which express the law of facility of errors of the observations are known, the determination of the mean which it is necessary to choose among several observations will be reduced, by the preceding article, to dividing a given area into two equal parts, that which is a problem of pure Analysis. But, these functions being most often unknown, it is to the Calculus of probabilities to furnish the means to make up for this ignorance; now we have seen, in article XIII, that if, in this case, $\pm a$, $\pm a'$, $\pm a''$, ... are the limits of the errors of the first, of the second, ... observation, we must assume

$$\phi(x) = \frac{1}{2a} \log \frac{a}{x}, \quad \phi'(x) = \frac{1}{2a'} \log \frac{a'}{x}, \quad \ldots$$

Thus there remains no more, in the research of the mean result of several observations, than the inevitable difficulties of Analysis; but it is necessary to agree that they render the preceding method of a very difficult usage: likewise my object, by exposing it, has been rather to make known all that which the analysis of chances can give to illuminate on this matter, than to present to the observers a practical method and of a convenient usage; we can, however, employ in some very delicate occasions, such as those of the passage of Venus on the disk of the Sun, in which it is necessary to obtain the greatest precision. The most simple way for this object is to square by parts the curve of the probabilities and to determine thus the ordinate which divides the area into two equal parts.

XXXII.

The ordinary rule of the arithmetic means is deduced from this method, by supposing $a = a' = a'' = \cdots = \infty$, as it is easy to be convinced of it; but we have demonstrated a much more general theorem by showing that this rule holds all the time: 1° that the law of facility of errors is the same for all the observations; 2° that the same errors, either to *plus*, or to *minus*, are equally possible; 3° that they can be infinites and that the functions which express their facilities decreases from a finite quantity only when x is infinite, but that then it always decreases to the point of becoming null.

For this, let $\phi(\alpha x)$ be the law of facility of the errors of observations, α being an infinitely small quantity; let moreover q be the value of $\phi(\alpha x)$, when $\alpha x = 0$ and, consequently, when x is a finite quantity. It is evident that the ordinate of the curve of probabilities, from $-x = 0$ to $-x = \infty$, will be

$$y = \phi(\alpha x)\phi(\alpha p + \alpha x)\phi(\alpha p' + \alpha x) \cdots$$

By supposing the number of observations equal to n and by neglecting the quantities of order α^2, we will have

$$y = \phi(\alpha x)^n + \alpha(p + p' + p'' + \cdots + p^{(n-1)})\phi(\alpha x)^{n-1}\frac{d\phi(\alpha x)}{d(\alpha x)};$$

now, if we take the integral $\int \alpha\phi(\alpha x)^{n-1}\frac{d\phi(\alpha x)}{d(\alpha x)}dx$, from $x = 0$ to $x = \infty$, and if we recall that $\phi(\alpha x) = q$ when $x = 0$ and $\phi(\alpha x) = 0$ when $x = \infty$, we will have

$$\int \alpha\phi(\alpha x)^{n-1}\frac{d\phi(\alpha x)}{d(\alpha x)}dx = -\frac{1}{n}q^n;$$

let therefore A be the integral $\int \phi(\alpha x)^n dx$, taken from $x = 0$ to $x = \infty$, and we will have

$$A - \frac{1}{n}(p + p' + p'' + \cdots + p^{(n-1)})q^n$$

for the integral $\int y\,dx$ corresponding to the negative values of x.

This same integral, taken from $x = 0$ to $x = p^{(n-1)}$, is $p^{(n-1)}q^n$, because we can, in this interval, suppose

$$\phi(\alpha x) = \phi(\alpha p - \alpha x) = \cdots = q,$$

consequently the ordinate $y = q^n$.

Since $x = p^{(n-1)}$ to $x = \infty$, we have

$$y = \phi(\alpha x)\phi(\alpha x - \alpha p)\phi(\alpha x - \alpha p')\cdots$$

or

$$y = \phi(\alpha x)^n - \alpha(p + p' + p'' + \cdots + p^{(n-1)})\phi(\alpha x)^{n-1}\frac{d\phi(\alpha x)}{d(\alpha x)};$$

now the integral $\int \alpha\phi(\alpha x)^{n-1}\frac{d\phi(\alpha x)}{d(\alpha x)}dx$, taken from $x = p^{(n-1)}$ to $x = \infty$, is $-\frac{1}{n}q^n$. Moreover, the integral $\int \phi(\alpha x)^n dx$, taken in the same interval, is evidently equal to $A - p^{(n-1)}q^n$; we will have therefore

$$A - p^{(n-1)}q^n + \frac{1}{n}(p + p' + p'' + \cdots + p^{(n-1)})$$

for the value of $\int y\,dx$, taken in this interval. Hence, the entire area of the curve of probabilities is equal to $2A$. Now, by naming h the abscissa of which the ordinate divides this area into two equal parts, the part of the area which is to the left of this ordinate will be clearly equal to

$$A - \frac{1}{n}(p + p' + p'' + \cdots + p^{(n-1)})q^n + hq^n;$$

by equating it to A, we will have

$$h = \frac{1}{n}(p + p' + p'' + \cdots + p^{(n-1)}),$$

which gives for h the same value as the rule of the arithmetic means. The assumptions which have led us to this result being beyond all likelihood, we see how much it is necessary, in the delicate occasions, to make usage of the method that we have proposed.

XXXIII.

It is easy to apply the preceding theory to the correction of instruments; for this, we suppose that, in verifying an instrument and by repeating a great number of times the same verification, we have found n different errors p, p', p'', ... Let i, i', i'', ... be the number of times each of them has been repeated; by representing by x, x', x'', ... their respective facilities, we will have $kx^i x'^{i'} x''^{i''} \ldots, \ldots$ for the probability of the observed event, k being a constant coefficient; the probability of this system of facilities will be therefore

$$\frac{x^i x'^{i'} x''^{i''} \ldots dx dx' dx'' \ldots}{\int^n x^i x'^{i'} x''^{i''} \ldots dx dx' dx'' \ldots},$$

the integrals of the denominator being taken for all the possible values of x, x', x'', ... In order to conclude the probability of x, we will integrate the function $x^i x'^{i'} x''^{i''} \ldots dx dx' dx'' \ldots$ first with respect to x', from $x' = 0$ to $x' = 1 - x - x'' - \cdots$, next with respect to x'', from $x'' = 0$ to $x'' = 1 - x - x''' - \cdots$, and thus in sequence, which gives for the last integral

$$\frac{1.2.3\ldots i'.1.2.3\ldots i''.1.2.3\ldots i''' \ldots}{1.2.3.4\ldots(i' + i'' + i''' + \cdots)} x^i (1-x)^{i'+i''+i'''+\cdots+n-1} dx.$$

We will have therefore, for the probability that the facility x will be contained in the given limits,

$$\frac{\int x^i (1-x)^{i'+i''+i'''+\cdots+n-1} dx}{\int x^i (1-x)^{i'+i''+i'''+\cdots+n-1} dx},$$

the integral of the numerator being taken in the extent of these limits and that of the denominator being taken from $x = 0$ to $x = 1$; now this probability will be determined by the formula of article XVIII by changing p into i and q into $i' + i'' + i''' + \cdots + n - 1$.

We examine now the correction that it is necessary to make to a new observation made with this instrument: we suppose that there is a quarter circle and that, in taking a great number of times one same apparent height a, we have found between this height and the real height n differences which extend from $a - \alpha$ to $a + \alpha'$. We suppose moreover that, by partitioning the interval $\alpha + \alpha'$ into $n - 1$ very small parts, we have found that the error $-\alpha$ has been repeated i times; that the error $-\alpha + \frac{\alpha+\alpha'}{n-1}$ has been repeated i' times; that the error $-\alpha + \frac{2(\alpha+\alpha')}{n-1}$ has been repeated i'' times and thus in sequence; let finally x, x', x'', ... be the facilities of these errors. We will have, by article XIV,

$$\frac{\int^n x^{i+1} x'^{i'} x''^{i''} \ldots dx dx' dx'' \ldots}{\int^n x^i x'^{i'} x''^{i''} \ldots dx dx' dx'' \ldots}$$

for the probability that the error of a new height a, observed with this quarter circle, will be $-\alpha$, the integrals of the numerator and of the denominator being taken for all the possible values of x, x', x'', ..., which recur to integrate the one and the other, first with respect to x, from $x = 0$ to $x = 1 - x' - x'' - \cdots$, next with respect to x', from $x' = 0$ to $x' = 1 - x'' - \cdots$, and thus of the rest. We will find in this manner that the preceding fraction is reduced to $\frac{i+1}{i+i'+i''+\cdots n+1}$; this quantity expresses therefore the probability that the error of observation will be $-\alpha$; by changing in it i successively into i', i'', ..., and reciprocally, we will have the probabilities that the error of observation will be $-\alpha + \frac{\alpha+\alpha'}{n-1}$ or $-\alpha + \frac{2(\alpha+\alpha')}{n-1}$,.... We will imagine therefore raised on the extremities and on each of the divisions of the interval $\alpha + \alpha'$ some ordinates equal or proportional to these probabilities and of which the extremities, because of the smallness of the divisions, will form sensibly a curved line; this put, the abscissa of which the ordinate will partition the area of this curve into two equal parts will be, by article XXIX, that of which it is necessary to make usage, so that, if we name h this abscissa computed from the origin of the interval $\alpha+\alpha'$ which corresponds to the error $-\alpha$, the correction which should be made to the observed height a will be $h - \alpha$ and, consequently, we should suppose the real height equal to $a + h - \alpha$.

Thence results this quite simple rule to correct the instrument: *Add continually the quantities $i + i' + 2$, $i' + i'' + 2$, $i'' + i''' + 2$, ... until when you have arrived at a sum equal or immediately smaller by any quantity μ than the half of the sum*

$$i + 2i' + 2i'' + \cdots + 2i^{(n-2)} + i^{(n-1)} + 2n - 2.$$

Let r be the number of quantities $i + i' + 2$, $i' + i'' + 2$, ... that you have thus added; l the number of parts $\frac{\alpha+\alpha'}{n-1}$ contained in α; the correction that it is necessary to make to the height a or, what amounts to the same, the quantity that it is necessary to add to it will be, very nearly,

$$\left(r - l + \frac{\mu}{i^{(r)} + i^{(r+1)} + 2}\right) \frac{\alpha + \alpha'}{n-1}.$$

If, instead of fixing the true height at the point of the abscissa of which the ordinate divides the area of the curve into two equal parts, we fixed it at the point of which the ordinate passes through the center of gravity of this area, we would have the same correction that the method of the arithmetic mean gives: this method returns therefore, in this case, to taking for *mean* the point where the sum of the errors to the *less*, multiplied by their probabilities, is equal to the sum of the errors to the *plus*, multiplied by their probabilities.

When once we know the law of the facility of the errors of the instrument, we can conclude from it that of the errors of any result deduced from observations made with this instrument, such as the middle concluded from two corresponding heights. In fact, if we name z, z', z'', ... the errors of the observations that we will suppose here very small, the correction that we should make to the result will be $Az + A'z' + A''z'' + \cdots$, A, A', A'', ... being some constant coefficients depending on the nature of the result that we deduce from the observations. If we suppose this correction equal to x, we will have

$$Az + A'z' + A''z'' + \cdots = x.$$

There is no longer question than to determine, by the method of article VII, the probability of this equation by means of the law of facility of errors z, z', z'', ...; we will have thus for this probability a function of x, which we will designate by $\phi(x)$, so that the equation of the curve of the probabilities of values of x will be $y = \phi(x)$. Now, if we take the integral $\int y\,dx$ for all the extent of the limits in which x can vary, the abscissa h which will divide into two equal parts the area which represents this integral will be the correction that should be made to the proposed result.