11. Level Flight Performance and Level flight Envelope

We are interested in determining the maximum and minimum speeds that an aircraft can fly in level flight. If we do this for all altitudes, the locus of theses points as they change with altitude describes the flight envelope. It is important to characterize aircraft into two categories: 1) those whose power plant outputs are measured in terms of thrust (turbojets, and turbofans), and 2) those whose power plant outputs are measured in terms of power (piston propeller combinations and turboprops)

Power Plant Output in Terms of Thrust - General - Arbitrary Drag Polar

It is natural to look at these types of vehicles first since the equations that have to be satisfied for level flight are:

$$T = D$$
$$L = W \tag{1}$$

From the previous work on thrust and drag models, we know the functional form of both the thrust and drag. The drag (or thrust required) depends on the density (altitude), airspeed, and the vehicle weight, and therefore has the general form:

Thrust required:
$$D = D(h, V, W) \qquad \text{or} \qquad D = D(h, M, W) \tag{2}$$

The thrust available (the output of the engine) depends on the density (altitude), airspeed, and the throttle setting, and therefore has the general form:
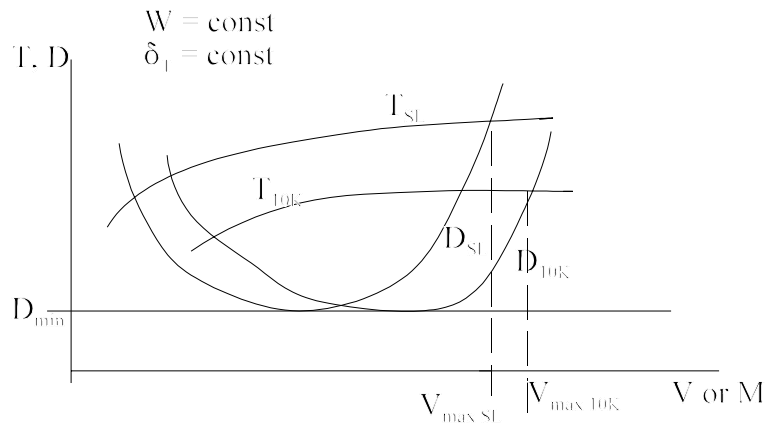
Thrust available:
$$T = T(h, V, \delta_T) \qquad \text{or} \qquad T = T(h, M, \delta_T) \tag{3}$$

Consequently, for a given altitude, weight, and throttle setting, the thrust available, and the thrust required (drag) become a function of only airspeed or Mach number. Then for level flight, we must satisfy Eq. (1). To solve this equation for a given throttle setting, altitude, and weight, we can plot the thrust available, and thrust required (drag), vs airspeed or Mach number and observe where the graphs cross. At that point, T = D and Eq. (1) will be satisfied, and we can read off the airspeed and the thrust. Then, at a given weight and throttle setting, we can plot the thrust available and thrust required curves vs airspeed for each altitude of interest. At a specified altitude, the thrust available curve and the thrust required curve will, in general, intersect at two points. The one at the highest airspeed is the maximum level flight airspeed for that aircraft at that particular weight and throttle setting at that altitude. The intersection at the lower airspeed is *a candidate* for the minimum airspeed at that altitude and is called the thrust-limited minimum airspeed. The actual minimum airspeed at a given altitude is whichever is the greater of the

thrust-limited minimum airspeed, or V_{stall},

The maximum speeds indicated are for the specified weight and throttle setting. Generally we are interested in the maximum possible airspeed and hence these plots are made with the throttle setting for maximum continuous thrust, or military thrust.

For military aircraft there are generally four levels of throttle settings and thrust available, 1) afterburner, 2) military thrust, 3) cruise, 4) idle. For commercial aircraft there are 1) emergency, 2) takeoff, 3) maximum continuous, 4) cruise, and 5) idle. The first two levels can not be used continuously for more than five minutes. From our study of thrust models, we know that the thrust decreases proportional to the density. Hence as we increase altitude and the density decreases, the thrust available at a given throttle setting decreases. Hence there will eventually be some altitude at which the thrust available is less than the minimum drag, or more precisely, there will be some altitude at which the thrust available curve will just touch the thrust required curve at one point (tangent point). The altitude at which that occurs is called the ceiling (however it will be defined in another way later).

Definition: Ceiling - The altitude above which T < D for all airspeeds.

Altitude-Speed Cross plots

The intersection of the thrust (thrust available) curve with the drag (thrust required) curve is the solution to Eq. (1), T = D. Consequently these points represent the maximum speed that the aircraft can fly at each altitude for the given weight and throttle setting. We can then make a ot of altitude vs speed and plot the points of maximum speed. Such a plot will represent the maximum speed flight envelope.

A similar plot can be made with minimum speeds. However here we must remember to take the greater of the thrust-limited minimum speed or of V_{stall}, and plot those points

In the figure on the right, the hatched area is outside of the level-flight envelope and as such the aircraft cannot fly in that region in straight and level flight. Note that at higher altitudes the lower speed is thrust limited, while at the lower altitudes, the lower speed is limited

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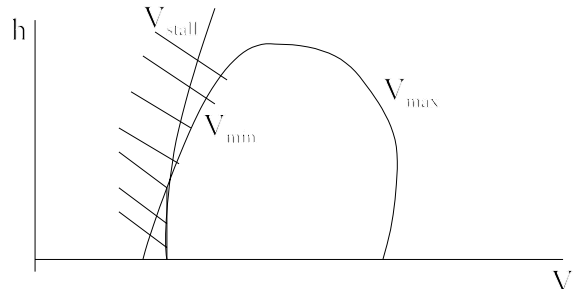
by the stall speed.

The procedures described previously work for any drag polar or thrust model. As long as you have the data, you can make the plots. Generally propulsion data is presented in terms of graphs and tables so that the thrust at each speed and altitude point must be calculated using these charts and graphs. Similar comments could be made for a detailed model of drag. It is not unusual to have a large table of drag vs Mach number for high performance aircraft. Regardless of how the data is presented, once the thrust available and thrust required (drag) curves are plotted we can determine the level-flight flight envelope as described above.

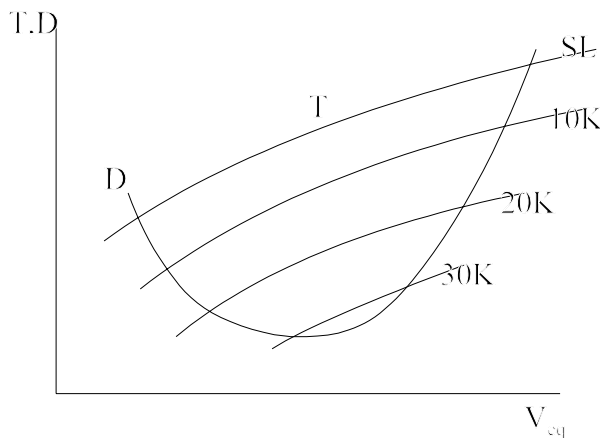### *Special Case - Parabolic Drag Polar - low performance aircraft*

The implications of a low performance aircraft with a parabolic drag polar is that the two drag parameters are constant. That is, $C_{D_{0L}}$, and K are constants. Under these circumstances, we have:

$$D = C_{D_{0L}} \, 1/2 \, \rho \, S \, V^2 \; + \; \frac{K\,W^2}{1/2 \, \rho \, S \, V^2} \tag{4}$$

Recall that $\bar{q} = 1/2 \, \rho \, V^2 = 1/2 \, \rho_{SL} \, V_{eq}^2$ so that Eq. (4) can be written as:

$$D = C_{D_{0L}} \, 1/2 \, \rho_{SL} \, S \, V_{eq}^2 \; + \; \frac{K\,W^2}{1/2 \, \rho_{SL} \, S \, V_{eq}^2} \tag{5}$$

Then we can plot drag vs equivalent airspeed, and the one curve will serve for all. So the procedure would be to plot the sea-level curve. Then on the same graph plot the thrust available vs equivalent airspeed. There is no savings here since you need to include the altitude effects in the thrust calculations. The result will look like

The cross plot of altitude vs airspeed will look like the figure to the right. Here the locus of the maximum equivalent airspeeds and altitude are plotted on the curve V_{eq}. Then to get the true airspeed you have to convert using
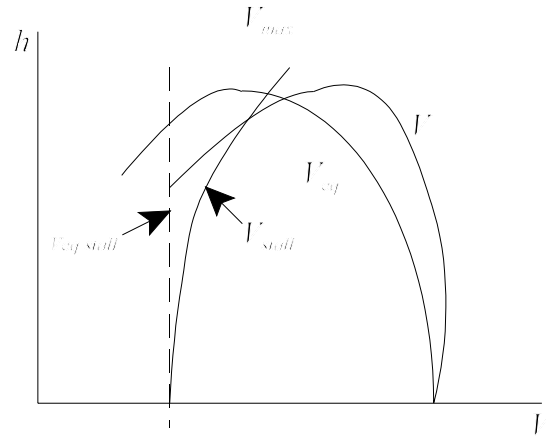
$$V = \sqrt{\frac{\rho_{SL}}{\rho}}\, V_{eq} \qquad (6)$$

On the plot, the equivalent stall airspeed is a constant with altitude. It can be converted using the above equation if necessary.

Real Special Case - Low Performance Aircraft, Parabolic Drag Polar, Constant Thrust

For the special case of a parabolic drag polar with constant parameters, and with thrust independent of airspeed, the maximum and thrust-limited minimum airspeeds can be solved for analytically. There are two ways to approach this problem. Both require the solution to a quadratic equation. The most direct way is to use the drag equation.

$$T = D = C_{D_{0L}} \bar{q} S + \frac{K W^2}{\bar{q} S} \qquad (7)$$

We can multiple through by $\bar{q} S$ and rearrange to obtain a quadratic equation in $\bar{q} S$:

$$C_{D_{0L}} (\bar{q} S)^2 - T(\bar{q} s) + K W^2 = 0 \qquad (8)$$

Then we get

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$$\left(\bar{q}\,S\right) = \frac{T}{2\,C_{D_{0L}}} \pm \frac{1}{2}\sqrt{\left(\frac{T}{C_{D_{0L}}}\right)^2 - \frac{4\,K\,W^2}{C_{D_{0L}}}}$$

$$V^2 = \frac{T}{C_{D_{0L}}\,\rho\,S} \pm \sqrt{\left(\frac{T}{C_{D_{0L}}\,\rho\,S}\right)^2 - \frac{4\,K\,W^2}{C_{D_{0L}}\,\rho^2\,S^2}} \tag{9}$$

If we evaluate the above equation at $\rho = \rho_{SL}$ we will get the equivalent airspeed. The correct thrust for any desired altitude must be entered to get the equivalent airspeed at that altitude. The true airspeed at altitude is then obtained from Eq. (6).

Alternate approach that deals with more friendly numbers is to deal with coefficients. From the original T = D and L = W equations, Eq. (1) we can substitute in coefficients in the form:

$$\frac{T}{W} = \frac{D}{L} = \frac{C_D}{C_L} = \frac{C_{D_{0L}} + K\,C_L^2}{C_L} \tag{10}$$

We can rearrange Eq. (10) to get a quadratic in the lift coefficient

$$K\,C_L^2 - \frac{T}{W}\,C_L + C_{D_{0L}} = 0 \tag{11}$$

The solution of Eq. (11) for C_L is

$$C_L = \frac{\dfrac{T}{W} \pm \sqrt{\left(\dfrac{T}{W}\right)^2 - 4\,C_{D_{0L}}\,K}}{2\,K} \tag{12}$$

Then the associated airspeeds are given by

$$V = \sqrt{\frac{W}{1/2\,\rho\,S\,C_L}} \tag{13}$$

where the maximum airspeed occurs with the small C_L, and the thrust limited minimum airspeed occurs at the largest C_L.

You may notice some familiar combination of terms in Eq. (12). These can be gathered and rearranged to give an nice, but probably completely useless result. See if you can derive it

$$\frac{C_L}{C_{L_{md}}} = e^{\pm \cosh^{-1}\left[\frac{T}{W}\left(\frac{L}{D}\right)_{max}\right]} = \frac{T}{W}\left(\frac{L}{D}\right)_{max} \pm \sqrt{\left[\frac{T}{W}\left(\frac{L}{D}\right)_{max}\right]^2 - 1}$$

Power Plant Output in Terms of Power - General - Arbitrary Drag Polar

The output of a piston engine driving a propeller is given in terms of power. The output of a turbine driving a propeller (turboprop) is given in terms of equivalent power (the power delivered to the propeller plus the small additional power associated with the exhaust thrust). In any case, the two are treated the same. Aircraft that use propellers are typically low performance aircraft so that most performance charts are given in terms of airspeed rather than Mach number.

Power is the rate of doing work, and work is the force times the distance. If we multiply the force times the rate that we cover that distance, we get power. Consequently the power necessary to overcome drag, or power required, is given by:

$$P_{req} = DV \tag{14}$$

with the proper units given in ft-lb/sec or Watts. Usually piston engine power is expressed in horsepower. The conversion is

$$\begin{aligned} 1 \text{ HP} &= 550 \text{ ft-lbs/sec} \\ &= 746 \text{ Watts} \end{aligned} \tag{15}$$

The power available is determined in a similar manner and is given by:

$$P_{av} = TV = P_s \eta_{prop} \tag{16}$$

where P_s = shaft power, and η_{prop} = propeller efficiency. Usually the engine and propeller data is given terms of power so Eq. (16) is primarily used to determine the thrust if we need it, as discussed in the section on thrust modeling.

Of interest to us in this section is the fact that if we look at our thrust and drag modeling, we can observe that the power available and power required have a functional form similar to the thrust and drag. We can write in general,

$$P_{req} = P_{req}(h, V, W)$$

$$P_{av} = P_{av}(h, V, \delta_T)$$

(17)

Therefore if we pick an altitude, weight, and throttle setting, the power available and power required depend only on airspeed, V. Hence we can plot each of these vs the airspeed.

The figure at the right represents two sets of curves of power available and power required for two different altitudes. Since T = D is satisfied if TV = DV, the level flight solution for a specified weight and throttle setting is where the power available curves and power required curves intersect. Again, the highest speed solution is the maximum level flight speed at altitude and throttle setting for given weight, but the minimum speed solution is only a candidate for the minimum airspeed. The true minimum airspeed is the greater of the power-limited minimum airspeed as read off this graph, or the stall speed at that altitude, whichever is greater.
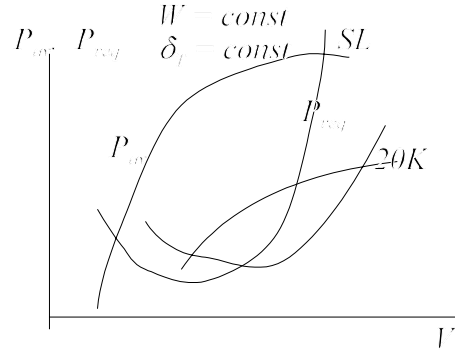
Power Available Curves for Turbojets

Later, we will see that power available and power required curves are important for predicting climb performance. As a result, it is necessary to generate power available curves for turbojet engines. You can recall that we suggested that a reasonable model for turbojet engines is that the thrust is constant with airspeed. If this is true, then the power output of the jet at zero airspeed is zero! Then for a turbojet (and turbofan)
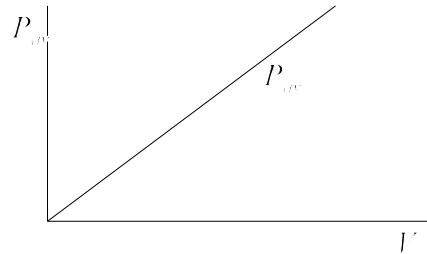
$$T \approx const$$

$$P_{av} = TV$$

and the power available curve for a turbojet looks something like the curve in the figure. Even if thrust varies slightly with airspeed, the figure will tend to look like this one.

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General Power Available and Power Required Curves

For the general case, the power available may be supplied in tabular and chart format. It is necessary to determine the power available from these charts and tables. Likewise for power required. Usually drag or thrust required charts and tables are given, and you must multiply them by the appropriate airspeed to get power required. Whatever the procedure, we require a plot like the one on the right for each altitude.
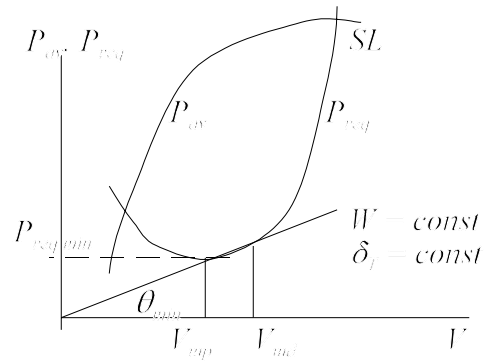
Just as the drag curves displayed a minimum, so do the power curves. The point on the curve tangent to a horizontal line is the minimum power value, and the airspeed associated with it is the minimum power airspeed at that altitude, V_{mp}. If we draw a line from the origin (0 power, 0 airspeed), so that it intersects the power required curve, it will, in general, intersect it in two locations. At those points we can calculate the associated drag by taking the tangent of the angle.

$$\tan\theta \ = \ \frac{P_{req}}{V} \ = \ \frac{D\,V}{V} \ = \ D \tag{18}$$

If we now draw the line so that it touches the power required curve in only one location, that is, it is tangent to the curve as shown in the figure, then the angle will be the smallest, and hence the tangent point is the minimum drag point,

$$\tan\theta_{min} \ = \ D_{min} \tag{19}$$

The corresponding airspeed is the ***minimum drag airspeed***. From the geometry we can observe a basic fact of life: The minimum drag airspeed is always greater than the minimum power airspeed.

$$V_{md} > V_{mp} \tag{20}$$

Power Available - Piston Aircraft

The power available is generally given by $p_{av} \ = \ P_s\,\eta_{prop}$. For most piston powered aircraft, the dependence of the shaft power on airspeed is small, so one can assume, for rough approximations, that the shaft power is constant with airspeed. The propeller efficiency is generally a relatively strong function of airspeed except if the propeller can adjust its pitch. In this case, except a very low speeds, the propeller efficiency is approximately constant. So, for first approximations, our power available model for a constant speed prop, is that the power available is nearly constant.

For a fixed-pitch propeller, the situation is quite different. The shaft power is still nearly constant, but the propeller efficiency is strongly dependent on air speed. To determine the power available curves, one must know or be able to calculate the propeller efficiency as a function of airspeed. Usually these curves have low efficiency at low speeds, increase their efficiency at some airspeed, and then the efficiency drops as the speed is increased more. If the peak efficiency is at a relatively low speed we say we have a performance or climb propeller. If the peak efficiency occurs at a higher speed, we say we have a cruise propeller. One does not want to take off from a short field with a cruise prop!

Special Case Low performance Aircraft, Parabolic Drag Polar

For the parabolic drag polar with constant parameters, we can write the power required by simply multiplying the drag by the airspeed. If we do that we get:

$$
\begin{aligned}
P_{req} &= D\,V = C_{D_{0L}}\,1/2\,\rho\,S\,V^3 \;+\; \frac{K\,W^2}{1/2\,\rho\,S\,V} \\[2mm]
&= A\,V^3 \;+\; \frac{B}{V}
\end{aligned}
\tag{21}
$$

We see that the power required goes up with V^3! As noted previously, a line drawn from the origin tangent to the power required curve touches it at the conditions for minimum drag.

$$
\tan\theta_{min} \;=\; \frac{P_{req}}{V} \;=\; \frac{D\,V}{V} \;=\; D_{min}
$$

We also know that the minimum drag is the same for all altitudes, so that the minimum theta line from the origin must be tangent to all minimum power curves at any altitude.

Special Case Minimum Power Conditions

For the special case of a low performance aircraft with a parabolic drag polar we can get an analytic expression for the conditions at minimum power. We can do this in several ways. The most straight forward way is to find the minimum power airspeed from Eq. (21)

$$
\frac{d\,P_{req}}{d\,V} \;=\; 3\,A\,V^2 \;-\; \frac{B}{V^2} \;=\; 0
$$

We can rearrange to get

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$$V^4 = \frac{B}{3A} = 4 \left(\frac{W}{\rho S} \right)^2 \frac{K}{3 C_{D_{0L}}}$$

$$V_{mp} = \sqrt{\frac{W}{1/2 \rho S}} \sqrt[4]{\frac{K}{3 C_{D_{0L}}}}$$

(22)

We can recognize from Eq. (22) that $C_{L_{mp}} = \sqrt{\dfrac{3 C_{D_{0L}}}{K}}$. However, we will obtain this information in another way. Consider the following approach. We want to write the power required in terms of lift and drag coefficients for a given weight and altitude. In order to do this, we need to recall the following relations:

$$D = W \frac{C_D}{C_L} \qquad \text{and} \qquad V = \sqrt{\frac{W}{1/2 \rho S C_L}}$$

Then we can write,

$$P_{req} = D V = W \frac{C_D}{C_L} \left(\frac{W}{1/2 \rho S C_L} \right)^{1/2} = \left(\frac{W^3}{1/2 \rho S} \right)^{1/2} \frac{C_D}{C_L^{3/2}}$$

Consequently, to minimize power required, we need to minimize $\dfrac{C_D}{C_L^{3/2}}$. Since C_D is a function of the lift coefficient, we can pose the problem as find the C_L that will minimize the ratio $\dfrac{C_D}{C_L^{3/2}}$ or:

$$\min_{C_L} \left(\frac{C_D}{C_L^{3/2}} \right) = \min_{C_L} \left(\frac{C_{D_{0L}} + K C_L^2}{C_L^{3/2}} \right)$$

(23)

Then

$$\frac{d \left(\dfrac{C_D}{C_L^{3/2}} \right)}{d C_L} = \frac{C_L^{3/2} (2 K C_L) - (C_{D_{0L}} + K C_L^2) 3/2 C_L^{1/2}}{C_L^3} = 0$$

10

or

$$3\,C_{D_{0L}} = K\,C_L^2$$

The conditions for minimum power are then given by:

$$C_{L_{mp}} = \sqrt{\frac{3\,C_{D_{0L}}}{K}}$$

$$C_{D_{mp}} = 4\,C_{D_{0L}} \tag{24}$$

The minimum power airspeed was given previously. The same result is obtained by using the expression for minimum power lift coefficient in Eq. (24) to calculate the airspeed.

Additional calculations can be made to determine the min-power L/D.

$$\left.\frac{L}{D}\right)_{mp} = \frac{C_{L_{mp}}}{C_{D_{mp}}} = \frac{\sqrt{\frac{3\,C_{D_{0L}}}{K}}}{4\,C_{D_{0L}}} = \sqrt{\frac{3}{16\,K\,C_{D_{0L}}}} \tag{25}$$

Finally, we can compare minimum drag and minimum power conditions:

$$V_{mp} = \sqrt[4]{\frac{1}{3}}\,V_{md} = 0.7598\,V_{md} \tag{26}$$

$$C_{L_{mp}} = \sqrt{3}\,C_{L_{md}} \tag{27}$$

$$C_{D_{mp}} = 2\,C_{D_{md}} \tag{28}$$

$$\left.\frac{L}{D}\right)_{mp} = \frac{\sqrt{3}}{2}\left.\frac{L}{D}\right)_{md} = 0.8660\left.\frac{L}{D}\right)_{md} \tag{29}$$

Summary (Parabolic drag polar with constant parameters)

	min drag	min power required
C_L	$\sqrt{\dfrac{C_{D_{0L}}}{K}}$	$\sqrt{\dfrac{3\,C_{D_{0L}}}{K}}$
C_D	$2\,C_{D_{0L}}$	$4\,C_{D_{0L}}$
L/D	$\dfrac{1}{2\sqrt{C_{D_{0L}}K}}$	$\dfrac{1}{4}\sqrt{\dfrac{3}{C_{D_{0L}}K}}$
V	$\sqrt{\dfrac{W}{1/2\,\rho\,S}}\sqrt[4]{\dfrac{K}{C_{D_{0L}}}}$	$\sqrt{\dfrac{W}{1/2\,\rho\,S}}\sqrt[4]{\dfrac{K}{3\,C_{D_{0L}}}}$

Consolidating Power Required Curves

For the case of a low performance aircraft with a parabolic drag polar, we can consolidate the power required curves for all altitudes into one. That means you only have to plot the sea-level curve and it will hold for all altitudes, similar to the drag vs equivalent airspeed curve. If we try the same thing it won't quite come out as we wish.

$$D\,V = \left[C_{D_{0L}}\,1/2\,\rho\,S\,V^2 \;+\; \frac{K\,W^2}{1/2\,\rho\,V^2\,S} \right] V$$

$$= \left[C_{D_{0L}}\,1/2\,\rho_{SL}\,S\,V_{eq}^2 \;+\; \frac{K\,W^2}{1/2\,\rho_{SL}\,V_{eq}^2\,S} \right] V$$

$$= \left[C_{D_{0L}}\,1/2\,\rho_{SL}\,S\,V_{eq}^2 \;+\; \frac{K\,W^2}{1/2\,\rho_{SL}\,V_{eq}^2\,S} \right] \frac{V_{eq}}{\sqrt{\sigma}}$$

If we multiply through by $\sqrt{\sigma}$ we obtain our consolidated curve:

$$P_{req}\sqrt{\sigma} \;=\; A_{SL}\,V_{eq}^3 \;+\; \frac{B_{SL}}{V_{eq}} \tag{30}$$

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where $\qquad A_{SL} = C_{D_{0L}} 1/2\, \rho_{SL} S$ $\qquad$ and $\qquad B_{SL} = \dfrac{K W^2}{1/2\, \rho_{SL} S}$

Consequently, if you plot the sea-level of power required times the square root of σ, vs equivalent airspeed, curve (σ = 1), then that curve is valid for all altitudes. You go in to the equivalent airspeed of interest, and read off the power required times the square root of σ. To get the power required you simply divide by $\sqrt{\sigma}$. It gets a little tricky when you plot the power available curves for each altitude because you multiply the available power at altitude by the $\sqrt{\sigma}$ before you plot it.