The set ω_1, the first uncountable ordinal

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Construction of ω_1

Let X be an uncountable set and let $\leq$ be a well ordering on X. Well ordering means that each non-empty subset $A \subseteq X$ has a least element $a \in A$, that is $b \in A$ implies $a \leq b$. (For the moment we will accept the existence of such an ordering.) For each $a \in X$, let

$$\mathbf{seg}(a) = \{x \in X | x < a\}$$

be the initial segment, the set of all points of X strictly that are strictly less than a in the well ordering (see §7.7 of the text).

Let $Y = \{y \in X | \, \mathbf{seg}(y) \text{ is uncountable}\}$ and let assume Y is non-empty so that there is a least element $y_0 \in Y$. Then $\omega_1 = \mathbf{seg}(y_0)$ is a well ordered uncountable set, so that

$$\forall \alpha \in \omega_1, \mathbf{seg}\, \alpha \text{ is countable}$$

(If Y is empty, the we can take the whole space X as ω_1.)

(The space ω_1 should not be confused with ω. Indeed $\omega = \{0, 1, 2, \dots\} = \mathbb{N} = \aleph_0$ is countable, while $\omega_1 = \aleph_1$ is uncountable. Another way of noticing the difference is the say ω is the (infinite countable) collection of finite ordinals and ω_1 is the (uncountable) collection of countable ordinals)

The fundemental property of ω_1

For every sequence $\{\alpha_n\} \subseteq \omega_1$ there is a $\beta \in \omega_1$ so that

$$\cup_n \mathbf{seg}(\alpha_n) = \mathbf{seg}(\beta)$$

That is no sequence can reach the end of ω_1. The proof is simple, the union is countable because it is a countable union of countable sets. The must be points in ω_1 outside the union, so there must be a least element β. There are some routine things to check, but eventually the result is clear.

Exercise: complete the proof of $\cup_n \mathbf{seg}(\alpha_n) = \mathbf{seg}(\beta)$.

The interlacing lemma

For sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_n \leq \beta_n \leq \alpha_{n+1}$ for all n, let α_∞ and β_∞ be defined by

$$\cup_n \mathbf{seg}(\alpha_n) = \mathbf{seg}(\alpha_\infty) \text{ and } \cup_n \mathbf{seg}(\beta_n) = \mathbf{seg}(\beta_\infty)$$

then $\alpha_\infty = \beta_\infty$ and both sequences converge to α_∞ in the order interval topology.

The basic open sets for the order interval topology are sets of one of the forms:

1. $(-\infty, \alpha) = [0, \alpha) = \{\gamma \in \omega_1 | \gamma < \alpha\}$
2. $(\alpha, \infty) = \{\gamma \in \omega_1 | \alpha < \gamma\}$
3. $(\alpha, \beta) = \{\gamma \in \omega_1 | \alpha < \gamma < \beta\}$

The void set $\emptyset$ is open, and a set $U \subseteq \omega_1$ is open if each $u \in U$ there is a basic open set B with $u \in B \subseteq U$. A set is closed if its complement is open.

To see that $\lim \alpha_n = \alpha_\infty$, let $B = (\gamma, \delta)$ a basic open set with $\alpha_\infty \in B$. Thus $\gamma < \alpha_\infty < \delta$ and γ is not an upper bound to $\{\alpha_n\}$, so eventually for large n, $\gamma < \alpha_n \leq \alpha_\infty$ and $\alpha_n \in B$.

(Although it is not needed for the proof, each $\alpha \in \omega_1$ is either a sucessor ordinal if $\mathbf{seg}(\alpha)$ has maximum β; in which case $\alpha = \beta^+ = \beta + 1$; or its a limit ordinal if $\mathbf{seg}(\alpha)$ has no maximum. If $\alpha_\infty = \beta^+$ then the sequence $\{\alpha_n\}$ is eventually constant and equal to α_∞.)

Disjoint closed sets

If A and B are disjoint closed subsets of ω_1, then at least one of them is countable. If both A and B are uncountable, then for each $\alpha_n \in A$ there is a $\beta_n \in B$ with $\alpha_n < \beta_n$ and a $\alpha_{n+1} \in A$ with $\beta_n < \alpha_{n+1}$. The interlacing lemma says $\alpha_\infty = \beta_\infty$. Since A is closed $\alpha_\infty = \lim \alpha_n \in A$ and since B is closed $\beta_\infty = \lim \beta_n \in B$. Thus $A \cap B \neq \emptyset$, contardicting the fact A and B are disjoint.

The continuous real-valued functions on ω_1

The result is that every real-values continouous function $f : \omega_1 \to \mathbb{R}$ is eventually constant. That is there is some $\beta \in \omega_1$ and $c \in \mathbb{R}$ so that

$$\beta \leq \alpha \Rightarrow f(\alpha) = c$$

Composing f with $1/2 + \arctan(x)/\pi$, gives a continuous function from ω_1 to $(0, 1) \subseteq \mathbb{R}$, so we can assume the range of f is contained in $[0, 1]$. (That is the composition is eventually constant exactly when f is eventually constant.)

For each m, And $0 < i < 2^m - 1$, let $A_i^m = [i/2^m, (i+1)/2^m]$, each of these sets are closed and if $i \neq j, j - 1 \mathrm{or} j + 1$ then $A_i^m \cap A_j^m = \emptyset$. Continuity implies $B_i^m = f^{-1}[A_i^m]$ are closed set whose union is all of ω_1. Thus at least one of the B_i^m must be uncountable, and by the disjoint closed set property at most two can be uncountable (Either B_{i-1}^m or B_{i+1}^m could also be uncountable but not both). Let C_m be the union of the uncountable B_i^m and let α_n be such that $\beta \geq \alpha_n$ implies $f(\beta) \in C_m$, note the diameter of C_m is at most 2^{m-1}. Obviously $C_m \supseteq C_{m+1}$ and we can take $\alpha_{n+1} > \alpha_n$. Since the C_m are closed with diameters decreasing to zero, there is a unique $c \in \cap C_m$ and for $\alpha \geq \beta = \lim \alpha_n$, $f(\alpha) = c$.

More to come: ϵ_0

The countable ordinal ϵ_0 is the limit of the ordinals

$$\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots$$

where $\omega^{\lim \alpha_n} = \lim \omega^{\alpha_n}$ and $\omega^{\alpha+1} = \omega^\alpha \omega$ which is a countable union of ω^α's one after the other. This ordinal has the property that

$$\omega^{\epsilon_0} = \epsilon_0$$