Al-KĀSHĪ'S DETERMINATION OF π TO 16 DECIMALS IN AN OLD MANUSCRIPT

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In memory of A. Qorbānī

1. *Introduction*

One of the highlights of the medieval Islamic mathematical tradition is the determination of π to 16 decimals by Jamshīd ibn Masʿūd al-Kāshī or al-Kāshānī (died A.H. 832/1429 CE).[1] He spent the first half of his life in Kāshān in Iran, where he observed a lunar eclipse in A.H. 808/1406 CE. In A.H. 824/1421 CE he moved to Samarkand to work as a mathematician and astronomer at the court of Ulugh Beg.[2] The symbol π is of course modern; in the terminology of al-Kāshī and his predecessors, π corresponds to the ratio between the circumference and diameter of a circle. Before al-Kāshī π had been determined with an accuracy equivalent to seven decimals in China, and less in other cultures.

Al-Kāshī published his determination of π in an Arabic treatise entitled *al-risāla al-muḥīṭiyya*, *Treatise on the Circumference.* Manuscript no. 5389 in the Holy Shrine Library in Meshed[3] is one of the eight manuscripts of the treatise which are known to exist to date. This manuscript was studied for the first time in the book *Kāshānī-nāmeh* by the Iranian historian of mathematics A. Qorbānī, with facsimiles of six pages [19, 124-130]. On the last page of the manuscript, the following is stated: "This has been written by its author, the most insignificant servant of God Most High, Jamshīd ibn Masʿūd ibn Maḥmūd ibn Muḥammad, the Physician, al-Kāshānī, called Ghiyāth, may God treat him well, in the middle of the great month Shaʿbān of the year 827 of the Hijra" (corresponding to the end of July 1424 CE).Thus Professor Qorbani believed the manuscript to be an autograph by al-Kāshī.

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[1] On the life and works of al-Kāshī see, e.g., [29], [2], and [16, I:480-486].

[2] The date of his move has been determined by Qorbānī in [19, 7-9].

[3] See [17, III:52, no. 162; VIII:42].

Shortly before his death,[4] Professor Qorbānī kindly made available a photocopy of the Meshed manuscript, in order that it could be published in facsimile.

The Meshed manuscript is of good quality; the figures are drawn carefully, and there are few scribal errors in the numerical tables and in the letters designating points in the geometrical figures. The colophon does not prove that the manuscript is an autograph, because colophons were often copied by scribes; that the manuscript cannot be an autograph by al-Kāshī is shown by errors which will be discussed below. The following is stated on the second page of the Meshed manuscript, which is the cover page of the *Treatise on the Circumference*:[5] "The Treatise on the Circumference; it is the original text in the handwriting of the author, the most glorious and most excellent master, the Ptolemy of his time, our master Ghiyāth al-Dīn Jamshīd al-Kāshī; it was edited by the poor Muḥammad Bahā' al-Dīn al-ʿĀmilī." This mathematician was well-known in Iran in the tenth/sixteenth century, and his irrigation system can still be seen in Isfahan.[6] Al-ʿĀmilī cannot have been the scribe because the owner's marks on the next page of the manuscript include the statement that "it (i.e., the manuscript) was transmitted to me (*intaqala ilayya*) ... Bahā' al-Dīn al-ʿĀmilī." In any case, the Meshed manuscript must be old.

The good quality of the Meshed manuscript is a sign of the competence of the anonymous scribe, and the manuscript is probably as close as we will ever get to the original text of al-Kāshī's work. A facsimile of this important document can be found in the appendix to this paper, and al-Kāshī's computation is introduced in Section 2 below.

Because al-Kāshī's text is not yet available in an English translation, incorrect or confused statements often appear in the Western literature on the history of π. Al-Kāshī is not mentioned at all in Petr Beckmann's popular *A History of Pi* [5], in which Islamic mathematics is placed in the chapter "Night", between the chapters on "Dusk" (late antiquity) and "Awakening" (the European Renaissance). The recent survey of π-determinations [4] is more accurate, but states that al-Kāshī computed π to 14 decimals, and that 15 decimals were found by Romanus (Adriaan van Roomen) in 1593. Actually, al-Kāshī's world record was broken in 1596 by the Dutch mathematician Ludolf van Ceulen in his work *Vanden Circkel*. The interesting similarities between this work and the computations of al-Kāshī will be discussed in Section 3 of this paper. Van Ceulen was unaware of the work of al-Kāshī, which remained unknown in Europe until the twentieth century.

[4] See the obituary by M. Bagheri in *Historia Mathematica* 29 (2002), no. 3, pp. 244-246.

[5] See for a reproduction of this page [19, 125].

[6] On al-ʿĀmilī (A.H. 953-1031/1547-1622 CE) see [19, 160].

In 1925, al-Kāshī's π-determination was mentioned for the first time in the Western literature by David Eugene Smith [27, II:238, 240]. Smith had been informed by the Turkish scholar Salih Mourad, who had apparently studied the manuscript of al-Kāshī's *Treatise on the Circumference* in Istanbul, Askeri Müze 756. The historian of mathematics and Arabist Paul Luckey prepared a German translation with commentary of al-Kāshī's *Treatise on the Circumference*, published posthumously in 1953 in [15]. The editors of [15] also printed the Arabic text which Luckey had prepared for himself on the basis of the Istanbul manuscript only. Luckey did not plan to publish a critical edition, because he did not have access to the manuscripts in Iran.

By means of the Meshed manuscript, which is reproduced in the present paper, the Arabic edition of al-Kāshī's text in [15] can be slightly improved. Changes to the Arabic text will be listed in Section 4, and the corresponding changes to the German translation can be found in Section 5. On the basis of the imperfect Istanbul manuscript, Luckey made conjectural restorations to the text, and as we will see in Section 4, many of his conjectures are confirmed by the Meshed manuscript.

More than half of al-Kāshī's *Treatise on the Circumference* consists of a series of 28 large tables for successive square-root extractions. These tables have never been published in full. In his translation, Luckey included only the first, second, fifteenth and 28th of these tables and he omitted the rest. Only the first two tables are printed in the Arabic edition in [15, 81-82], and in the Russian translation with commentary which was published in 1956 by B.A. Rosenfeld [24, 265-308, 367-375]. In [24, 383-424], Rosenfeld included a facsimile of the Istanbul manuscript, but the photos are so vague that many numbers in the tables are illegible. In the facsimile publication in the present paper, the tables, which form the core of al-Kāshī's computation, are much clearer. This new material will facilitate future research of al-Kāshī's computational methods.

2. *Summary of al-Kāshī's treatise*

The decimal system for fractions was not well-known in the time of al-Kāshī. Al-Kāshī made his computations in the sexagesimal system, which had been developed in Babylon, and which was widely used in later Greek and medieval Islamic astronomy. Al-Kāshī probably believed that it is impossible to find an exact numerical expression for what is now called π, i.e., the ratio beween the circumference and diameter of a circle.[7]

The purpose of al-Kāshī's *Treatise on the Circumference* is to compute π with such an accuracy that the resulting uncertainity in the circumference of the largest circle in the physical universe is less than the breadth of one hair. Al-Kāshī and his contemporaries accepted the cosmological ideas of Ptolemy (ca. 150 AD, see [21]), who believed that the earth is surrounded by the concentric spheres of the moon, Mercury, Venus, the sun, Mars, Jupiter, Saturn, the fixed stars, and by an outermost sphere. To a modern reader, the geocentric models of Ptolemy and the medieval Islamic astronomers may appear primitive. However, these models are mathematically equivalent to the later Copernican models [22], and they enabled the astronomers to predict the celestial phenomena with such an accuracy that the errors could hardly be noticed by the naked eye. Ptolemy and his Islamic successors determined the distance from the earth to the moon and the sun on the basis of measurements of lunar parallax and the apparent sizes of the moon, the sun, and the earth shadow during solar and lunar eclipses. These method is mathematically correct but sensitive to observational errors. The Greek astronomy made a small error in the measurement of the earth shadow. The resulting value for the distance between the earth and the sun is only 1/20-th of the actual value.

Ptolemy assumed that the maximal distance between the earth and the sun is equal to the minimal distance between the earth and Mars. The Ptolemaic model produces essentially correct ratios between the maximal and minimal distances from the earth to any planet (but not the values of the distances themselves). From the supposed minimal distance of Mars, and the ratio between its minimal and maximal distance, Ptolemy could now find the maximal distance of Mars to the earth. He then assumed this distance to be equal to the minimal distance of Jupiter to the earth, and so on. Finally, he assumed that the maximal distance of Saturn to the earth is equal to the minimal distance of the fixed stars to the earth. Ptolemy believed that all fixed stars were attached to a thin sphere which had a very slow (precessional) motion with respect to the outermost sphere containing the celestial equator. This ninth sphere rotated once every day around the center of the universe, which coincided with the center of the earth. In this way Ptolemy concluded that the radius of the universe was approximately 20,000 earth-radii.

On the basis of new astronomical observations, medieval Islamic astronomers changed some of the parameters in Ptolemy's models, but they computed the radius of the universe along the same lines with similar results. In his *Sullam al-Samā'* ("Stairway to Heavens"), al-Kāshī assumes that the radius of the universe is 26,328 earth-radii [3, 251].

Al-Kāshī's computation will now be summarized in modern algebraical notation, which was unavailable in his time. In his *Treatise on the Circumference*, al-Kāshī requires that in a circle with radius R equal to $600,000$ earth-radii, the inaccuracy in the circumference $2\pi R$ should be less than the breadth of a hair. Then the inaccuracy is much less than the breadth of a hair for all circles which can exist in his physical universe.

For the approximation of π, al-Kāshī uses a method of Archimedes, which is as follows in modern notation. Consider a circle with an inscribed and circumscribed hexagon. If the diameter of the circle is 1, the circumferences of the inscribed hexagon, the circle, and the circumscribed hexagon are 3, π, and $2\sqrt{3}$ respectively. The circumference of the circle is greater than the circumference of the inscribed hexagon and less than the circumference of the circumscribed hexagon. Hence we obtain $3 < \pi < 2\sqrt{3} = 3.46\ldots$. Using lower and upper bounds of the sides of an inscribed and circumscribed regular $n-$gon, Archimedes computed lower and upper bounds of the sides of the inscribed and circumscribed regular $2n$-gon. Thus he approximated the sides of the inscribed and circumscribed regular 12-, 24-, 48-, and 96-gons, and he finally obtained $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

The modern algebraic expressions of the sides of these polygons involve irrational numbers, which Archimedes did not use. His estimates of the ratios between the sides of inscribed and circumscribed polygons and the diameter of the circle boil down in modern terms to inequalities such as $265/153 < \sqrt{3} < 1351/780$. Archimedes did not use a

[7] The fact that π is an irrational number was proved in 1766 by the Swiss mathematician Lambert [7, 141-6].

decimal or sexagesimal system for fractions.[8]

Al-Kāshī mentions Archimedes' approximation but remarks that the result $3\frac{10}{71} < \pi < 3\frac{1}{7}$ is much too inaccurate for his purpose.

After the 96-gon ($96 = 3 \cdot 2^4$), al-Kāshī considers 24 more polygons, namely the 192-gon, the 384-gon, and so on, until the $3 \cdot 2^{28}$-gon. He shows that the circumferences of the inscribed and circumscribed $3 \cdot 2^{28}$-gon of a circle with radius 600,000 earth-radii differ by less than a breadth of a hair, so his approximation of the circle by one of these circumferences produces a sufficiently accurate value of π.

Al-Kāshī works in a circle with radius 60 units, as was usual in the trigonometry of his time. He simplifies Archimedes' method of computation in the following way in modern terms:

The sides of the inscribed and circumscribed n-gons in a circle with radius 60 are given by the formulas

$$120 \sin \frac{180^o}{n}, \ 120 \tan \frac{180^o}{n}.$$

Al-Kāshī first computed, for $n = 6, 12, ..., 3 \cdot 2^{28}$,

$$k_n = 120 \cos \frac{180^o}{n}.$$

The quantities k_n satisfy the simple relation

$$k_{2n} = \sqrt{60(120 + k_n)},$$

equivalent to the modern formula

$$2 \cos \frac{\alpha}{2} = \sqrt{2 + 2 \cos \alpha}.$$

Thus, al-Kāshī computes, in modern notation:

$$k_6 = 60\sqrt{3}, \ k_{12} = 60\sqrt{2 + \sqrt{3}}, \ k_{24} = 60\sqrt{2 + \sqrt{2 + \sqrt{3}}}, \ldots$$

up to $k_{3 \cdot 2^{28}}$.

Al-Kāshī shows that the computations are sufficiently accurate if the root extractions are carried out in 20 sexagesimals (two integer and 18 fractional). On fol. 11 of the facsimile of the Meshed manuscript, the reader will find al-Kāshī's computation of $\sqrt{3 \cdot 60^2} =$

$$= 1, 43; 55, 22, 58, 27, 57, 56, 0, 44, 25, 31, 42, 1, 56, 22, 42, 48, 58, 57 \ldots$$

[8] For Archimedes' method see [1, 93-94], reprinted in [7, 9-14].

(meaning: $= 1 \cdot 60 + 43 + \frac{55}{60} + \frac{22}{60^2} \ldots$). Figure 1 is the transcription of this computation by Luckey [15, 12].

By means of 27 further computations of this type, al-Kāshī finds $k_{3 \cdot 2^{28}}$. Then the circumference I of the inscribed regular $3 \cdot 2^{28}$-gon is

$$I = 3 \cdot 2^{28} \times \sqrt{120^2 - k_{3 \cdot 2^{28}}^2} = 3 \cdot 2^{28} \times \sqrt{7200 - k_{3 \cdot 2^{27}}}.$$

Al-Kāshī then determines the circumference C of the circumscribed polygon by a method involving similar triangles, equivalent to the formula $C/I = 120/k_n$ for $n = 3 \cdot 2^{28}$. In this way he finds upper and lower bounds for the circumference of the circle with radius 60, corresponding to the following upper and lower bounds for 2π:

$$2\pi < 6; 16, 59, 28, 1, 34, 51, 46, 14, 50, 15$$

$$2\pi > 6; 16, 59, 28, 1, 34, 51, 46, 14, 49, 45$$

(here 6;16,59 ... means $6 + \frac{16}{60} + \frac{59}{60^2} \ldots$).

Initially, al-Kāshī finds 46 as the last sexagesimal of the lower bound, but he then estimates the next sexagesimal place in all his computations, and corrects 46 to 45.

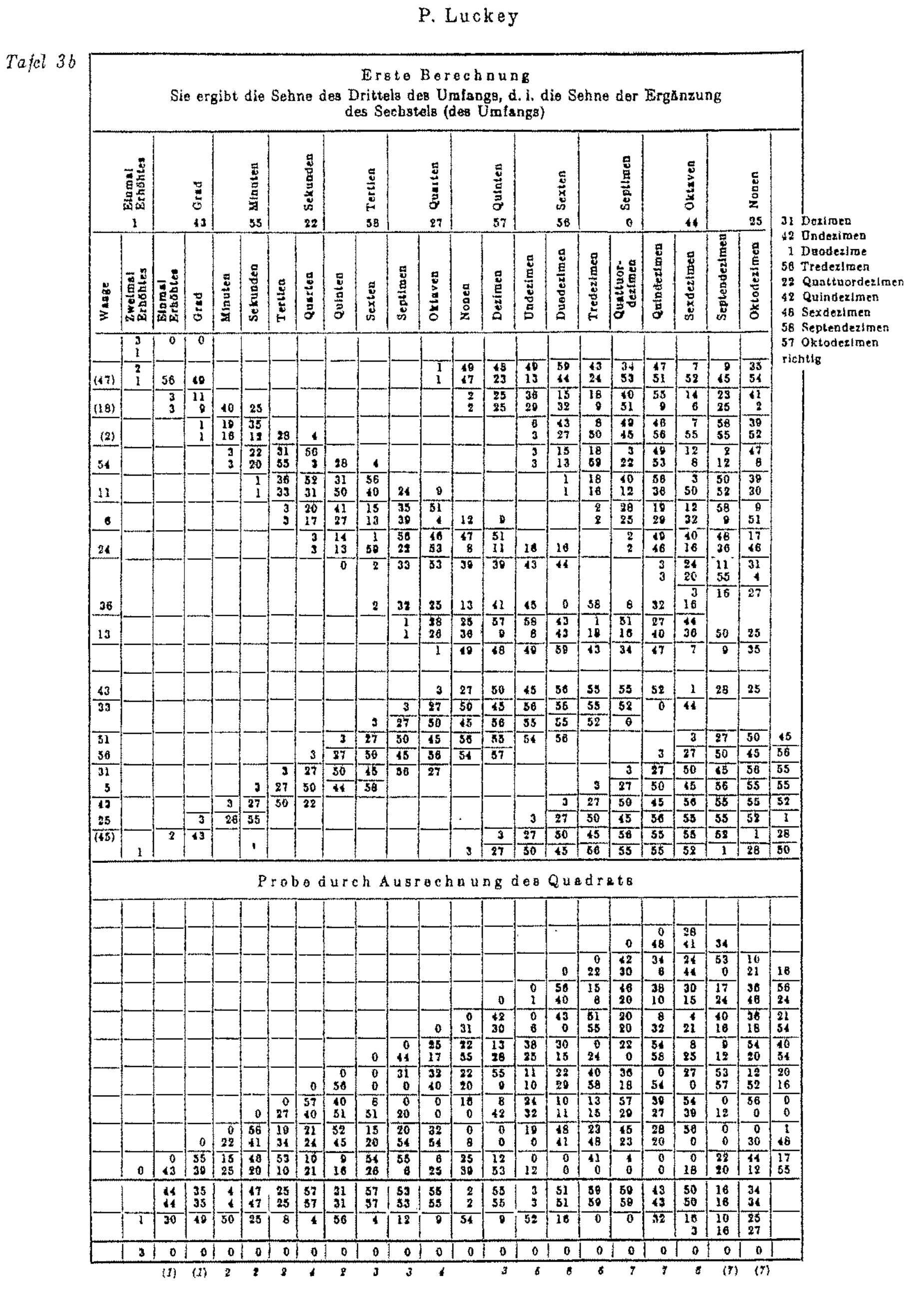
Al-Kāshī chooses for 2π the average 6;16,59,28,1,34,51,46,14,50, and he converts this number into the decimal system of fractions as 6.28318 53071 79586 5 in modern notation.[9]

Al-Kāshī then presents tables for integer multiples of 2π in sexagesimal and decimal numbers. The table for decimal numbers (on fol. 46 below) is especially interesting because it contains on the fifth line the multiple $5 \cdot 2\pi = 10\pi = 31.4159265358979325$; thus the reader can see the successive decimals of π in a direct copy of al-Kāshī's own handwriting. Because the decimals are written in Hindu-Arabic number symbols, they can be recognized without knowledge of the Arabic language.[10]

Al-Kāshī concludes his treatise by an analysis of the less accurate π-determinations by al-Būzjānī (328/940 - ca. 388/998) and al-Bīrūnī (362/972 - 440/1048).

[9] Al-Kāshī indicates only 16 decimals, but his upper and lower bounds of 2π are equivalent to $3.14159\ 26535\ 89793\ 230 < \pi < 3.14159\ 26535\ 89793\ 254$.
The average $\pi = 3.14159\ 26535\ 89793\ 242$ is correct to 17 decimals [15, 67].

[10] In the table, Hindu-Arabic numbers are used, but no decimal point. The decimals appear in columns for "tens," "times the diameter," and multiples of 10^{-n} times the diameter for $1 \leq n \leq 16$. The fifth line actually ends with 255, but the last 5 refers to the multiple $5 \cdot 2\pi$.



P. Luckey

Tafel 3b

Erste Berechnung

Sie ergibt die Sehne des Drittels des Umfangs, d. i. die Sehne der Ergänzung des Sechstels (des Umfangs)

Einmal Erhöhtes	Grad	Minuten	Sekunden	Tertien	Quarten	Quinten	Sexten	Septimen	Oktaven	Nonen
1	43	55	22	58	27	57	56	0	44	25

31 Dezimen
42 Undezimen
1 Duodezime
58 Tredezimen
22 Quattuordezimen
42 Quindezimen
48 Sexdezimen
58 Septendezimen
57 Oktodezimen
richtig

Waage	Zweimal Erhöhtes	Einmal Erhöhtes	Grad	Minuten	Sekunden	Tertien	Quarten	Quinten	Sexten	Septimen	Oktaven	Nonen	Dezimen	Undezimen	Duodezimen	Tredezimen	Quattuor-dezimen	Quindezimen	Sexdezimen	Septendezimen	Oktodezimen	
	3 1	0	0																			
(47)	2 1	56	49								1 1	49 47	48 23	49 13	59 44	43 24	34 53	47 51	7 52	9 45	35 54	
(18)		3 3	11 9	40	25							2 2	25 25	38 29	15 32	18 9	40 51	55 9	14 6	23 25	41 2	
(2)			1 1	19 18	35 12	28	4							8 3	43 27	8 50	49 45	48 56	7 55	58 55	39 52	
54				3 3	22 20	31 55	56 3	28	4					3 3	15 13	18 59	3 22	49 53	12 8	2 12	47 8	
11					1 1	36 33	52 31	31 50	56 40	24	9				1 1	18 16	40 12	56 36	3 50	50 52	39 30	
6						3 3	20 17	41 27	15 13	35 39	51 4	12	9			2 2	28 25	19 29	12 32	58 9	9 51	
24							3 3	14 13	1 59	56 22	46 53	47 8	51 11	16	16		2 2	49 46	40 16	48 36	17 46	
								0	2	33	53	39	39	43	44			3 3	24 20	11 55	31 4	
36									2	32	25	13	41	45	0	58	6	32	3 16	16	27	
13										1 1	38 26	25 36	57 9	58 8	43 43	1 19	51 16	27 40	44 36	50	25	
											1	49	48	49	59	43	34	47	7	9	35	
43											3	27	50	45	56	55	55	52	1	28	25	
33										3	27	50	45	56	55	55	52	0	44			
									3	27	50	45	56	55	55	52	6					
51								3	27	50	45	56	55	54	56				3	27	50	45
56							3	27	50	45	56	54	57					3	27	50	45	56
31						3	27	50	45	56	27						3	27	50	45	56	55
5					3	27	50	44	56							3	27	50	45	56	55	55
43				3	27	50	22								3	27	50	45	56	55	55	52
25			3	26	55									3	27	50	45	56	55	55	52	1
(45)		2	43										3	27	50	45	56	55	55	52	1	28
	1											3	27	50	45	56	55	55	52	1	28	50
Probe durch Ausrechnung des Quadrats																						
																	0	0 48	28 41	34		
															0	0 22	42 30	34 6	24 44	53 0	10 21	18
													0	0 1	56 40	15 8	46 20	38 10	30 15	17 24	36 46	56 24
											0	0 31	42 30	0 6	43 0	51 55	20 20	8 32	4 21	40 16	36 18	21 54
									0	0 44	25 17	22 55	13 28	38 25	30 15	0 24	22 0	54 58	8 25	9 12	54 20	40 54
							0	0 56	0 0	31 0	32 40	22 20	55 9	11 10	22 29	40 58	36 18	0 54	27 0	53 57	12 52	20 16
					0	0 27	57 40	40 51	6 51	0 20	0 0	18 0	8 42	24 32	10 11	13 15	57 29	39 27	54 39	0 12	56 0	0 0
			0	0 22	56 41	19 34	21 24	52 45	15 20	20 54	32 54	0 8	0 0	19 0	48 41	23 48	45 23	28 20	58 0	0 0	0 30	1 48
	0	0 43	35 39	15 25	48 20	53 10	10 21	9 16	54 26	55 6	6 25	25 39	12 53	0 12	0 0	41 0	4 0	0 0	0 18	22 20	44 12	17 55
		44 44	35 35	4 4	47 47	25 25	57 57	31 31	57 57	53 53	55 55	2 2	55 55	3 3	51 51	59 59	59 59	43 43	50 50	16 16	34 34	
	1	30	49	50	25	8	4	56	4	12	9	54	9	52	16	0	0	32	16 3	10 16	25 27	
	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
		(1)	(1)	2	2	2	4	2	3	3	4		3	6	6	6	7	7	8	(7)	(7)	

Figure 1

3. *Al-Kāshī, Adriaan van Roomen and Ludolph van Ceulen*

Archimedes' *Measurement of the Circle* was known to Regiomontanus in 1461; the Greek text was printed in 1544 in Basel and a Latin translation by Commandinus appeared in 1558. Soon afterward, several European mathematicians started to improve the Archimedean estimate $3\frac{10}{71} < \pi < 3\frac{1}{7}$ using inscribed and circumscribed polygons. In 1579 [28], François Viète (1540-1603)[11] computed the circumferences of the inscribed and circumscribed regular polygon of $3 \cdot 2^{17} =$ 393,216 sides in a circle with diameter 100,000, and he found the approximation $314,159\frac{265,36}{100,000}$ for the circumference of the circle. The first nine decimals of π are correct; only the last decimal 6 should be 5. Independently of Viète, the Belgian mathematician Adriaan van Roomen (1561-1615)[12] considered a circle with diameter divided into 2,0000,0000,0000,0000 parts in his notation. He computed the circumferences of the inscribed and circumscribed regular $15 \cdot 2^{24} =$ 251,658,240-gon, and showed that the circumference of the circle is less than 6,2831,8530,7179,5863 and more than 6,2831,8530,7179,5861 parts in his notation. This is equivalent in modern terms to the approximation π = 3.14159 26535 89793 1, of which the first 15 decimals are correct, and only the last 1 should be 2. Van Roomen actually prints these decimals in his approximation of the area of the circle, but he does not use a symbol equivalent to the modern decimal point.[13] Thus van Roomen did not break al-Kāshī's world record.

In 1596, however, van Roomen's friend Ludolph van Ceulen (1540-1610)[14] published a determination of π in 20 decimals in a Dutch work entitled "Vanden Circkel" (On the Circle) [11], which was published in Delft. The (striking) similarities between van Ceulen's work and al-Kāshī's treatise were briefly noticed by Luckey [15, 54]. Because the work by van Ceulen is not easily accessible in English, I will present some details here, in modern notation.

Van Ceulen computed for $n = 48, 96, \ldots, 3 \cdot 2^{31}$ the quantity $c_n = 2\cos(180^o/n)$ (my notation), using the formula $c_{2n} = \sqrt{2 + c_n}$. His results are displayed in Figure 2. As we have seen above, al-Kāshī computed $k_n = 60c_n$. Van Ceulen performed his computations in the

[11] On Viète see [10].

[12] On van Roomen see [9].

[13] See Liber Tertius, Tabulae Quartae Pars Vigesima Quinta in [23, 97], and also the summary on the last page of the introduction in [23].

[14] On van Ceulen see [13].

decimal system, which had been developed by the Dutch mathematician Simon Stevin (1548-1620), independently of al-Kāshī and other medieval Islamic mathematicians.

By transforming the number c_n of van Ceulen into the sexagesimal system, we obtain the number k_n in al-Kāshī's computation. The multiplication by 60 causes a shift of one position in the sexagesimal system. Thus, on line 28, van Ceulen's number $c_{3.2^{28}} = 1\ 99999\ 99999\ 99999\ 98478\ 13027\ 08290\ 02173\ 7702$ corresponds to the number 1 59; 59 59 59 59 59 59 59 50 47 52 12 30 48 37 49 54 40 at the end of al-Kāshī's 28th computation.[15]

Because van Ceulen carried the computation three steps further (to a $3 \cdot 2^{31}$-gon), and because he used 39 decimals in the final part, his computation produces π to 18 decimals.[16] He then continued by repeating the whole computation for a $15 \cdot 2^{32}$-gon, and determined π to 20 decimals. After *Vanden Circkel* was published in 1596, van Ceulen continued his computations, and he eventually found 35 decimals of π. The 35 decimals were published posthumously on his tomb in a church (the Pieterskerk) in Leiden. The tomb was destroyed in the nineteenth century, but a replica was placed in the church on the occasion of the World Mathematics Year 2000, see [18].

Al-Kāshī and van Ceulen found their approximations of π by taking the average of the circumferences I and C of an inscribed and circumscribed polygon of the same number of sides. Neither al-Kāshī nor van Ceulen realized that the weighted average $\frac{2}{3}I + \frac{1}{3}C$ leads to a much better approximation of π. This method was first published in the *Cyclometricus* by the Dutch mathematician Willibrord Snel in 1621 (see [6]).

Van Ceulen's approximation of π to 20 decimals in [11] apparently became known in Iran one century after his death. The Iranian author Muḥammad Bāqir Yazdī, grandson of the famous mathematician Muḥammad Bāqir Yazdī, says around A.H. 1100/1700 CE in a passage cited by Qorbānī [20, 5] that some mathematician from Europe had shown the following: if the diameter of a circle is 100,000,000,000, the circumference is 314 159 265 481. I have been unable to identify the author of this approximation, which could have been found by means

[15] The conversion of van Ceulen's number produce further sexagesimals 26, 21 ... after al-Kāshī's last sexagesimal 40. The last decimals 702 of van Ceulen's number are incorrect and should be 692.

[16] Van Ceulen obtained an estimate of π between ...238 and ...239, see the bottom of Figure 2. The inaccuracy is 10^{-18}, but the computation did not show whether the 18th decimal is 8 or 9.

1	7071067811865475244008443621048490.
2	1224744871391589049098642037352945$\frac{1}{10}$.
3	9828897227476208222891150538571256.
4	19957178464772070134761395825455552.
5	1998929174952731288859672892485719.
6	19997322758191235657254942981201998.
7	199993306783480220691520762111582781.
8	1999983266888701298295172411376694.
9	19999958167178003620833274486537009.
10	19999989541791766552221964749280281.
11	19999997385447770740971503103434821.
12	19999999346361932004174777442982778.
13	19999999836590482333476932760602833.
14	199999999959147620541646310504917746.
15	199999999989786905132803894957070750.
16	1999999999974467262830379935724244463.
17	19999999999936168157074931213267808 9.
18	199999999999840420392686691391892765.
19	199999999999960105098171633057893393.
20	19999999999999002627454290577759336 1.
21	199999999999997506568635726288968341.
22	19999999999999937664215893156252771 0.
23	19999999999999998441605397328900247792 9.
24	19999999999999996104013493322246824804.
25	199999999999999990260033733305614690336.
26	1999999999999999975650084333264035243543.
27	1999999999999999993912521083316008718242.
28	1999999999999999998478130270829002173 7702.
29	19999999999999999996195325677072505430806.
30	19999999999999999999048831419268126357451.
31	199999999999999999999762207854817031589348.
	1000000000000000000000000000000000000000
	1000000000000000000000000000000000000000

Sub 1999999999999999999990488314192681263574 51

Rest meerd als $\frac{93116831807311873642345}{100000000000000000000000000000000000000}$

Dit rest is het Quadraet der syde eener Figuer/in den Circkel ghescreven/van 6442450944 houcken.

$3\frac{141592653589793238}{1000000000000000000}$ / ende minder als $3\frac{141592653589793239}{1000000000000000000}$.

Figure 2. Van Ceulen's computation of the side of a 3.2^{31}-gon in [11, 13]. Lines 4-31 display the quantities $c_{3.2^n}$ for $n = 4 \ldots 31$. At the bottom one can read the resulting upper and lower bounds of π.

Figure 3. π on the front page of van Ceulen's work *Vanden Circkel* [11].

of an inscribed and circumscribed $2^{16} = 65536$-gon.[17]

Muḥammad Bāqir Yazdī then says that someone else found by a more accurate computation: if the diameter is 1 followed by 20 zeros, the circumference is between 314 159 265 358 979 323 847 and ... 846. The "someone else" is probably van Ceulen, because the approximation is expressed in the same way in [11], even on the front page (Figure 3). The late professor Qorbānī considers this transmission from Europe to the Islamic world as the event defining the end of the medieval Islamic period in mathematics [20, 4].

To explain the mathematical similarities between the methods of al-Kāshī on one hand, and van Ceulen and van Roomen on the other hand, it is not necessary to assume that al-Kāshī's work was transmitted to Western Europe. When van Roomen completed his work [23], he and van Ceulen did not even know the above-mentioned π-determination of François Viète, let alone al-Kāshī's work. Between 1580 and 1600, there was a general interest in Western Europe in the ancient Greek problem of the quadrature of the circle. Some European scholars arrogantly claimed that they had found exact methods, which implied exact values for π (such as $\pi = \sqrt{10}$). For van Ceulen and van Roomen, it was a pleasure to refute false quadratures [8, vol. 1, p. 173-175]. In this way, increasingly accurate determinations of π were discovered in the late 16th and early 17th centuries.

Al-Kāshī, however, was not in such a fortunate situation. As far as we know, none of his contemporaries was working on the same problem. In the Islamic tradition before al-Kāshī, very little attention had been paid to the determination of π. The values of π that had been found by al-Būzjānī and al-Bīrūnī were by-products of computations of the sine of one half or one-quarter of a degree, involving a regular polygon of at most 720 sides. In the determination of π, and in computational mathematics as a whole, al-Kāshī was a pioneer.

[17] The difference between half the average of the inscribed and circumscribed n-gon in a circle with diameter 1 and π, that is half the circumference of the circle, is approximately $\frac{\pi^3}{6n^2}$. Assuming that only polygons were used whose number of sides was $2^n.3 \cdot 2^n, 5 \cdot 2^n$ and $15 \cdot 2^n$, only the 65536-gon comes close; the resulting approximation is 314 159 265 479 ....

4. *Al-ʿĀmilī's manuscript and Luckey's edition*

In this section the following abbreviations will be used:

A: Manuscript Meshed, Holy Shrine Library, 5389, see the facsimile below.

I: Manuscript Istanbul, Askeri Müze 756, which Luckey used; a poorly legible facsimile is found in [24, 383-424].

L: The Arabic text which was prepared by Luckey on the basis of **I**, and which the editors Siggel and Gieseke included in the posthumous publication [15]; also Luckey's translation and commentary in [15].

A notation such as **A** 55:3 or **L** 55:3 refers to line 3 of page 55 of **A** or **L**; **L** 85n37 refers to footnote 37 of page 85 of **L**. The notation **I 1b:5** means line 5 of folio 1b in the Istanbul manuscript; some of the line numbers in the Istanbul manuscript are indicated in the margin of the Arabic edition in **L**.

A detailed investigation of the relations between **A** and **I** is premature because the other extant manuscripts of the *Treatise on the Circumference* should be involved as well. Here I only make a few remarks on the text in **A**.

A contains some corrections to the text and a number of marginalia in the same hand as the main text. These corrections and marginalia must be due to the scribe. Examples: he corrected the word *akthar* "more" in **A** 52:9 to *aqall* "less", as in **L** 91:16 and **I** 21a:8. There are four instances where he changed the word *hindī*, "Indian" from the masculine form to the feminine form *hindiyya*.

We will see below that the quality of **A** is much superior to that of **I**. But it is interesting that there are a few errors in the numerical tables in **A** which do not appear in **I** (note that the numbers in the tables are written in *abjad*-notation): In **L** 13, Table 4a, in the column of the "Undezimen," the seventeenth number is 14 (*bd*) in **A** 12, although it should have been 11 (*bā*) as in Luckey's transcription and in **I** 4a. In **L** 25, the second row of the table at the bottom, **A** has incorrectly 44 (*md*) instead of the correct number 45 (*mh*) in **L** and **I** 19b. These errors appear in the middle of computations, which continue in the correct way. A possible explanation is that the two manuscripts **A** and **I** are based on different autographs of the text by al-Kāshī. According to the date in **A**, al-Kāshī must have been in Samarkand when he wrote the (lost) autograph manuscript on which **A** is based. Because the *Treatise on the Circumference* does not contain a dedication, he may have written the very first autograph of the treatise before his arrival in Samarkand.

An interesting error is in **A** 55, second line from the bottom of the table, where the scribe put the wrong diacritical marks on a word which appears as *yalīh*. The context requires *thalātha*, "three" (as in **I** 22a:16 and **L** 94, row 17 of the right column of the table), and this is certainly what al-Kāshī must have written. The scribal error confirms that manuscript **A** is not an autograph.

Of course, al-Kāshī was not infallible. The following errors are common to **A** and **I** and therefore may be oversights by al-Kāshī: In **L** 87:19n51 the word *nisf*, corresponding to *Hälfte* in **L** 23:10, is necessary but missing in **A** 47:22 and in **I** 20b:25; in **L** 89:9n54, the passage *wa-nisf dhirāʿ* corresponding to **L** 24:7 [*und eine halbe*] is missing in **A** 49, fourth line from bottom, and also in **I** 19b:4; in **L** 92:24n64 and **L** 92:25n65, the word *tamām*, corresponding to [*Ergänzung*] in **L** 28:27, **L** 28:30, is missing in **A** 54:3 and **A** 54, margin, and also in **I** 21b:5 (two times). The following two inessential mathematical errors in **A** and **I** are also due to al-Kāshī:

1. **L** 78:1 = **I** 2b:14-15 = **A** 7:1-2. The text contains a number "366 and a fraction," which Luckey interpreted as 120π with π approximated by $3\frac{1}{7}$. Luckey then emended the text to "377 and a fraction," see **L** 7n1, **L** 55:8. Actually $120\pi = 376.99112\ldots$, so I believe that the "366 and a fraction" in the text is a slip of al-Kāshī's mind for "376 and a fraction." This interpretation is rejected by Luckey on the grounds that $120 \cdot 3\frac{1}{7} > 377$, but I think that al-Kāshī did not have the estimate $\pi \approx 3\frac{1}{7}$ in mind in the particular passage of the text.

2. In two instances, namely **L** 80:3 = **I** 3a:10-11 = **A** 9:4-6 and **L** 83:2 = **I** 18a:2 = **A** 40:3, al-Kāshī gives the decimal expression of $3 \cdot 2^{28}$ as 800, 335, 168. Actually $3 \cdot 2^{28} = 805,306,368$, as noted by Qorbānī [19, 148]. The error is not essential because al-Kāshī computed in the sexagesimal positional system. His sexagesimal expression $3 \cdot 2^{28} = 1, 2, 8, 16, 12, 48$ $(= 1 \cdot 60^5 + 2 \cdot 60^4 + 8 \cdot 60^3 \ldots)$ is of course correct.

Manuscript **I** contains numerous errors which are not found in **A**, and many corrections which Luckey made to the manuscript text in

I are confirmed by A.[18] These corrections are a witness of Luckey's excellent editorial skills.

I have collated manuscript A with the printed Arabic text in L. Here is a list of (minor) corrections to the printed text suggested by the readings in A:

1. L 75:8 = I 1b:5 change *hādhayn* to A 1:12 *hādhayn al-miqdārayn.*
2. L 75:10 = I 1b:7 change *wa-mashkūka* to A 1:15 *aw mashkūka.*
3. L 75:20 = I 1b:15 change *wa-kasr* to A 2:9 *wa-kasran.*
4. L 75:29 = I 1b:22 change *mukhīla* to A 3:4 *mukhtalla.*
5. L 76:4 = I 2a:2 change *wa l-wahhāb* to A 3:11 *al-wahhāb.*
6. L 76:11 = I 2a:9 change *mutashābihayn li-muthallath* $\overline{adb}$ to A 4:4-5 *mutashābihayn wa-mushābihayn li-muthallath* $\overline{adb}$.
7. L 76:15 = I 2a:14 after *mutasāwiyatān*, A 4:12 inserts *yṭ j q* (i.e., maqāla), which is a reference to prop. 19 of Book III of the *Elements*, corresponding to *Elements* III:20 in the Greek [12, 218-221].
8. L 76:16 = I 2a:15 after *wa-ḍilʿay* A 4:13 inserts *kw a q*, which is a reference to prop. 26 of Book I of the *Elements* [12, 62-67].
9. L 80:11 = I 3a:21 *yaḥṣulu* : A 9:18 has *la-ḥaṣala.*
10. L 80:16 the reading *sahw wa-yushrā* (cf. I 3a:25, L 80n20) is uncertain; A 10:5 has *sahw aw-yasrī* (?)
11. L 80:17 = I 3a:26 change *yaḥtāj* to A 10:12 *naḥtāj.*
12. L 83:5-6. Here I 18a:5-7 and A 40 have four small Hindu-Arabic numbers in the text: 1 before *hadā l-muḥīṭ*, 2 above *faḍl*, 3 above *niṣf*, and 4 above *faḍl* in the last line of the second table (Tafel 18a, 9ff.). Since the numbers occur in both manuscripts I and A, they were probably used by al-Kāshī, in order to indicate the first second, third and fourth terms in a proportion of the form $a : b = c : d$.

[18] These are the emendations to the Arabic text in L 75-95 indicated by notes 2, 15, 17, 22, 25, 27, 28, 29, 31, 33, 34, 35, 42, 43, 47, 50, 56, 58, 60, 61, 62, 67, as well as the following corrections to the tables: L 80 note 4 to p. 81; L 80 notes 1, 2, 4, 5 to p. 82; L 83 note 1; L 90 notes 1, 2, 3 ,4; L 93 notes 1, 2, 3; L 94 note 1.

13. L 83. In the second row of the header of the first table, A 40 writes (correctly) *al-maḍrūb fīhi* instead of the vertically printed *al-maḍrūb* in L. This confirms Luckey's conjecture on p. 37 that the *fīhi* had dropped out in the heading of his *Tafel Blatt 18a rechts.*
14. L 84:2n24 change *al-qisma* to A 41:3 *qismatuhu* (I 18a:10 has *qismat*).
15. L 85:6n37, and L 85:30n46 change Luckey's emendation *wa l-nāqiṣa* to the manuscript text *aw al-bāqiya* in A 44:2, 45:17 and I 20a:1, 20a:20.
16. L 85:11 = I 20a:4 after *bi-mithlihi* add *al-kasr*, found in A 44:9 below the line.
17. L 85:12 change *baqiya* to *yabqā* as in I 20a:5 and A 44:11.
18. L 85:18 = I 20a:10 after *wa li-l-sābiʿ ʿashara* add *zāʾid* as in A 45:1. The word *zāʾid* disappeared at the end of a line in I 20a:10.
19. L 85:23 = I 20a:14 change *al-zāʾidāt* to *zāʾidāt* as in A 45:7.
20. L 86, in line 4 of the text between the two tables, Luckey deleted *wa-li-suhūlat al-ʿamal bihi ayḍan* from I 20b:2, but this passage is genuine because it also occurs in the margin of A 46.
21. In the header of the second table, change *al-quṭr* to *wa l-quṭr* as in A 46 and I 20b. In the second table, change *khams marrāt* in I to *mukarrar khams marrāt* as in A 46, see the third row of the second column to the right.
22. L 87:4 = I 20b:9 change *akhadhnā* to *akhadhnāhā* as in A 46, line 2 from bottom.
23. See L 87:6-7, L 22. A Note that 47:2 has a somewhat different vocalization of the Arabic mnemonic verse which al-Kāshī composed for the decimal digits of $2\pi = 6.28318\ 53071\ 79586\ 5$, see also [19, 152]. In al-Kāshī's Persian verse for the decimal digits of 2π, Luckey (L 22) read the two words *yek rā* as 1 and he noted that the following digit 7 is missing in the verse. A 47:4 has *zā* instead of *rā*, and Qorbānī points out [19, 152] that the abjad number *zā* represents 7.

24. **L** 87:13 = **I** 20b:20 change *lam yaḥtaj* to **A** 47:13 *lam naḥtaj*.

25. Change **L** 89:5 = **I** 19a line 1 from bottom *waqt* to **A** 49:5 *diqqa*.

26. **L** 89:9 = **I** 19b:4 change *yaktafī* to **A** 49:22 *yaktafī bihi*.

27. **L** 89:12n55 change *al-marātib* to **A** 50:4 *marātibihi*, **I** 19b:6 has *marātib*.

28. **L** 90 = **I** 19b, in the header of the second table, add the words *al-kusūr* above 125 and *al-ṣiḥāḥ* above 0650844 as in **A** 51.

29. **L** 90 = **I** 19b, in the right column of the second table, change *ʿasharātuhu* to to *ʿasharāt al-ulūf* as in **A** 51.

30. **L** 91:7n57 Change *al-aʿdād* to *li-l-aʿdād* in **A** 51, margin.

31. **L** 91:16 = **I** 21a:7, between *māʾit* and *wa-sabʿīn* add *sabʿa* as in **A** 52, margin.

32. **L** 91:17 = **I** 21a:9 add *an* between *min hādhā ʿulima* and *idhā*, as in **A** 52, margin.

33. **L** 92:5 = **I** 21a:13 change $\overline{dz}$ to **A** 52:18 $\overline{zd}$.

34. **L** 92:20n63 has *ḍilʿ* where **I** 21b:1 has *watar ḍilʿ* Note that **A** 53:19 also has *watar ḍilʿ*, although the terminology "chord of the side" is odd.

35. **L** 92:23 = **I** 21b:4, change *bayyana* to **A** 54:2 *yubayyina*.

36. **L** 93 = **I** 21b, right column of table, in row 4 change *baqiya* to *yabqā* as in **A** 54.

37. **L** 93 = **I** 21b, right column of table, in row 17 change *watar* to *watar qʿz*, i.e. the chord of 177 in abjad-notation. See the left side of the table in **A** 54.

38. **L** 93, bottom row, change *tamayyuzan* to *tamyīzan* as in **A** 54, last line.

39. **L** 94, in the right column of the table: row 6, change *baqiya* to *yabqā* as in **I** 22a and **A** 55; in row 13 change *al-tafāḍul* as in **I** 22a to *al-tafāḍul baynahumā* as in **A** 55.

40. **L** 95:10 = **I** 22b delete *tamma l-kitāb bi-ʿaun Allāh al-wahhāb*. The passage is missing in **A**, so it was probably added to the text of **I** by the scribe.

5. *Changes to Luckey's translation*

Some but not all changes in the Arabic text entail a change in the German translation by Luckey. The changes in the translation are as follows, using the same numbering as in Section 4:

1. In **L** 3:12 change [*Brüchen*] to *Grössen*.

2. In **L** 3:14 change *und unsicher* to *oder unsicher*.

6. In **L** 4, line 3 from bottom, change *die dem Dreieck adb* to *die einander und dem Dreieck adb*

7. In **L** 5:6, between *gleich* and *; also*, insert: *nach dem neunzehnten (Lehrsatz) des dritten Buchs (der Elemente)*. This theorem corresponds to *Elements* III:20 in the Greek version [12, 218-221].

8. In **L** 5:8 after *einander gleich sind* insert: *nach dem sechsundzwanzigsten (Lehrsatz) des ersten Buchs (der Elemente)*.

12. In **L** 17, in the text after Table 18a and in the last line of Table 18a, Zeile 9, add the numbers 1, 2, 3, 4 as follows: *ist das Verhältnis (1) dieses Umfangs zum (2) Überschuß der Summe ... gleich dem Verhältnis der nachfolgend angegebenen (3) Hälfte ... zu folgendem (4) Überschuß....*

15. In **L** 20:9 (title of Chapter 7) and **L** 21:15 change *überschießenden und mangelhaften Brüchen* to *überschießenden oder übrig bleibenden Brüchen*.

16. In **L** 20:18 change *den gleichen Betrag* to *den gleichen Bruch*.

18. In **L** 20:30 change [*überschiessend*] to *überschiessend*.

20. In **L** 21:30 delete the two pointed brackets around *ebenfalls zur Erleichterung der Rechnung mit ihm*.

21. In Table 20b on **L** 22 delete the parentheses around *wiederholten*.

22. In **L** 21 line 4 from bottom, change *fingen wir bei einem Ausgangspunkt (oder: Nenner) an* to *haben wir sie mit einem Ausgangspunt (oder: Nenner) angenommen.*

24. In **L** 23:1 change *keine Verfeinerung nötig* to *wir keine Verfeinerung brauchen.*

25. In **L** 24, in the text in the bottom part of Table 19a, change the word at the end, *Rechenzeit*, to *Rechengenauigkeit.*

28. In **L** 26 add to the header of Table 19a *Ganze Zahlen* (above 0650844) and *Brüche* (above 125).

31. In **L** 26:13, change [*sieben*] to *sieben*.

33. In **L** 27, line 14 from bottom, change *dz* to *zd*.

34. In **L** 28:18, delete the pointed brackets around the word *Sehne*; the terminology is odd but genuine.

35. In **L** 28:25-26 change *bewies er* to *wird ... bewiesen*.

37. In **L** 29, fifth row from the bottom of the table change *die Sehne [von 177]* to *die Sehne von 177.*

40. In **L** 31 delete the last line: *Es endet die Schrift mit Hilfe Gottes, des Freigebigen.* This passage is probably an addition by the scribe of **I**.

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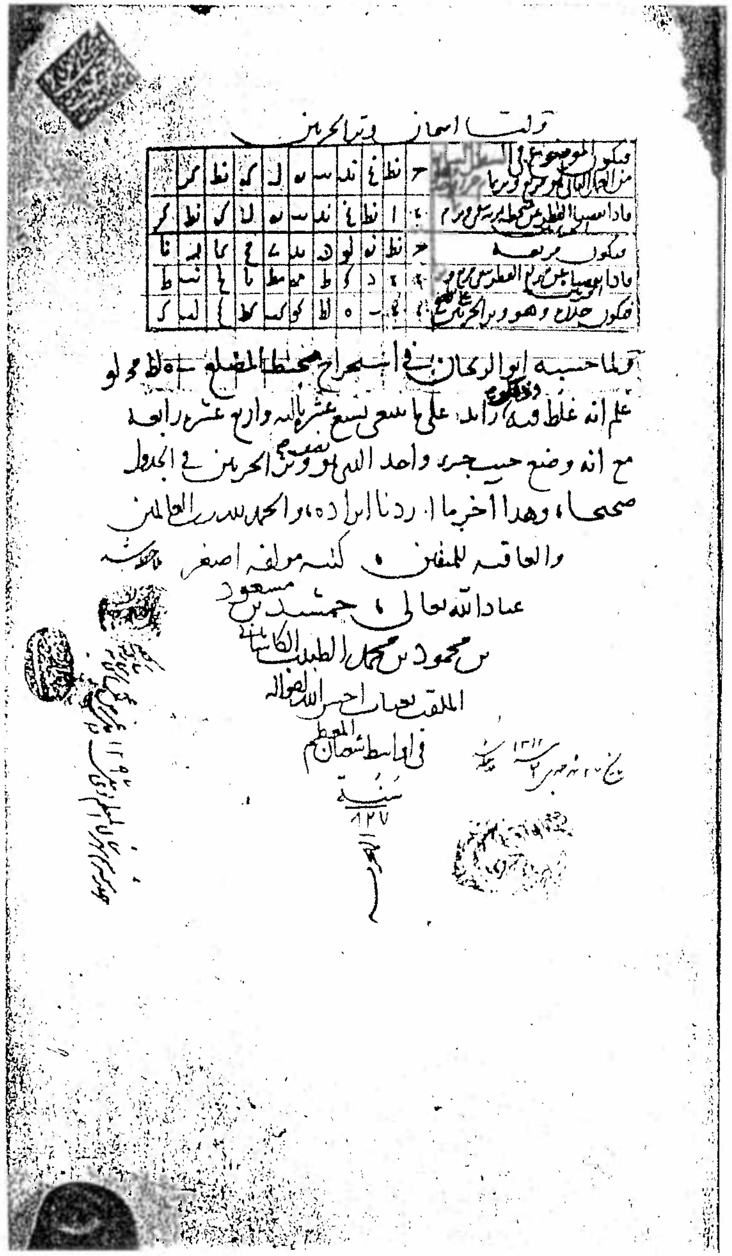
APPENDIX

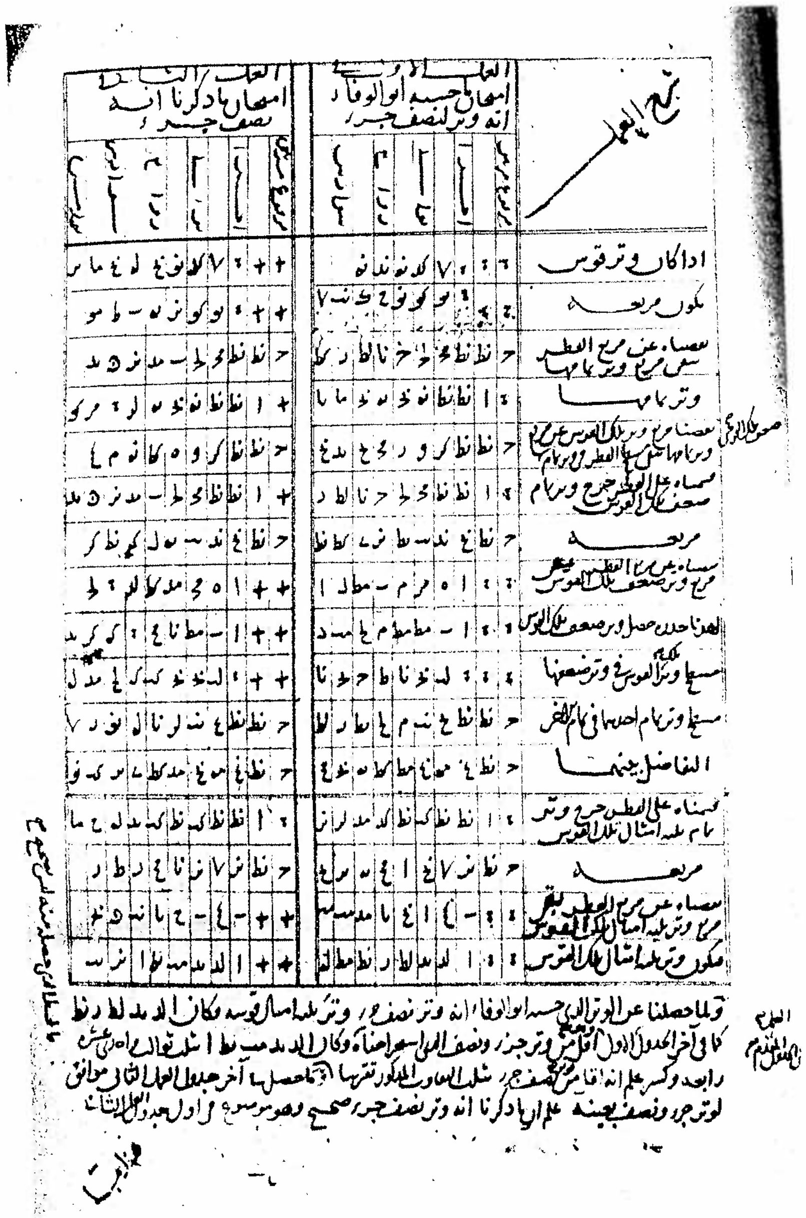
Al-Kāshī, Treatise on the Circumference

Facsimile of Manuscript Meshed,

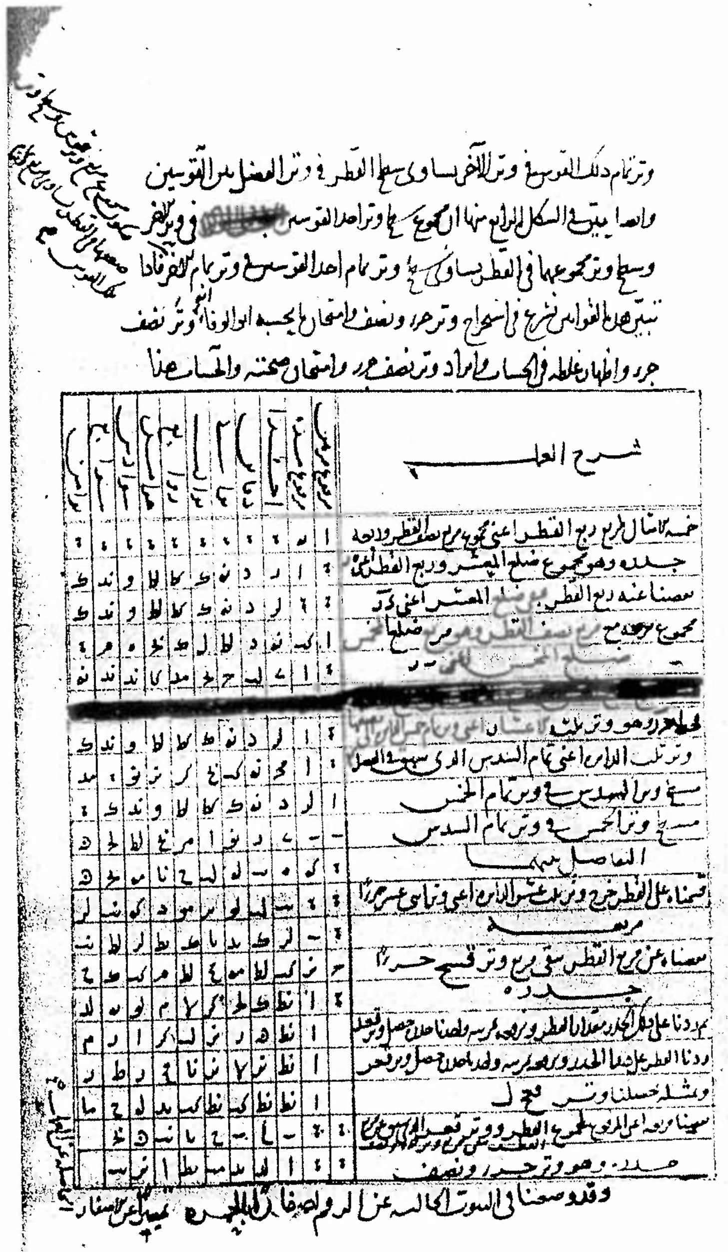
Holy Shrine Library, 5389, pp. 1-56.



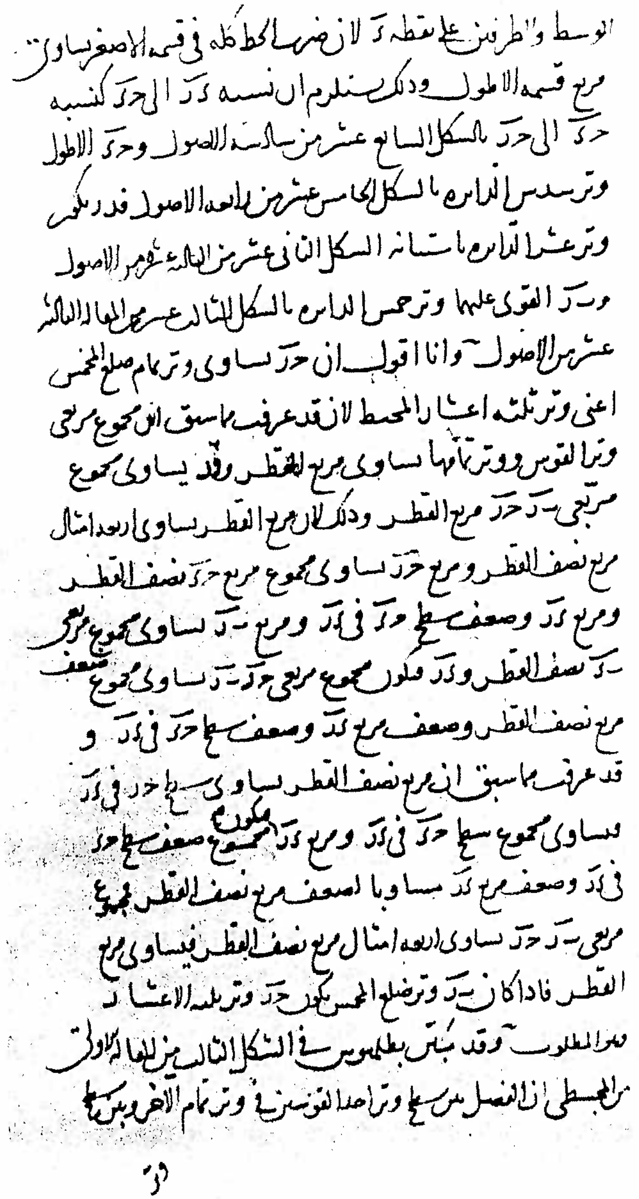
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 56



Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 55



Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 54

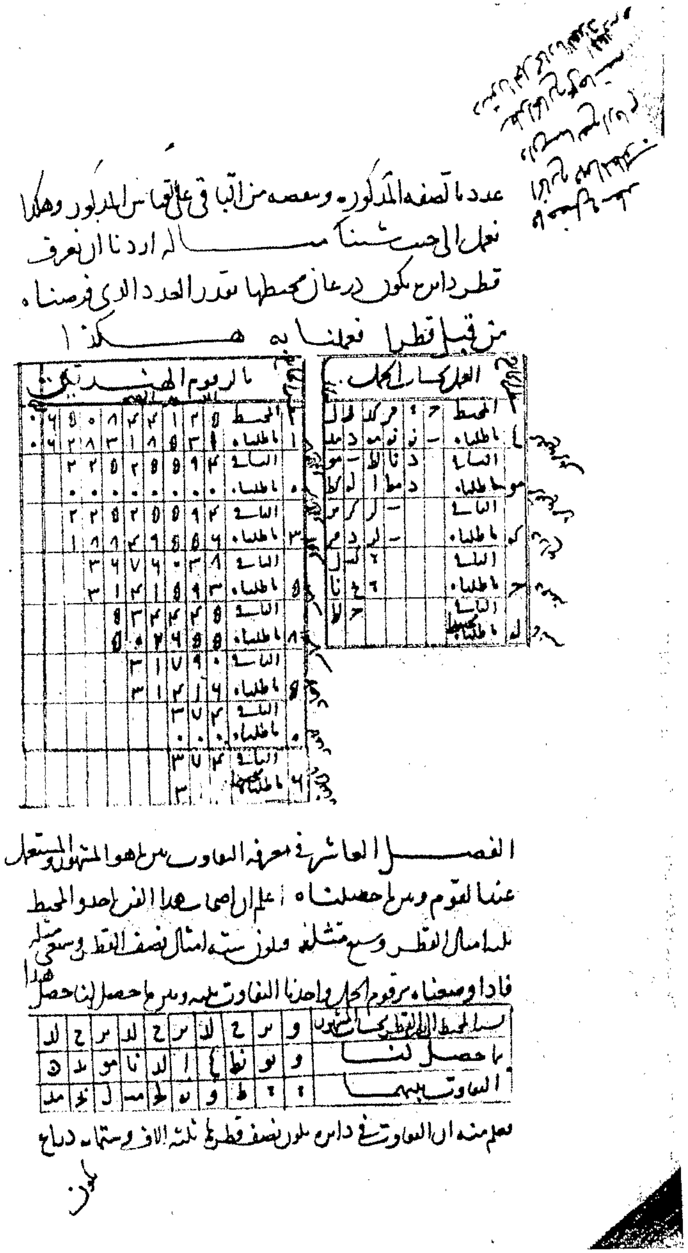


الوسط والطرفين على نقطة د لان ضرب الخط كله في قسمه الاصغر يساوي
مربع قسمه الاطول وذلك يستلزم ان نسبة دز الى حز كنسبة
حز الى حز بالشكل السابع عشر من سادسة الاصول وحز الاطول
وتر سدس الدائرة بالشكل الخامس عشر من رابعة الاصول فدز يكون
وتر عشر الدائرة بالشكل الثاني عشر من الثالثة عشر من الاصول
ودز القوي عليهما وتر خمس الدائرة بالشكل العاشر من المقالة الثالثة
عشر من الاصول وانا اقول ان حز يساوي وتر تمام ضلع المخمس
اعني وتر ثلثة اعشار المحيط لانه قد عرفت مما سبق ان مجموع مربعي
وتري القوس ووتر تمامها يساوي مربع القطر وقد يساوي مجموع
مربعي دز حز مربع القطر وذلك لان مربع القطر يساوي اربعة امثال
مربع نصف القطر ومربع حز يساوي مجموع مربع حز نصف القطر
ومربع دز وضعف سطح حز في دز ومربع دز يساوي مجموع مربعي
دز نصف القطر ودز فيكون مجموع مربعي حز دز يساوي مجموع ضعف
مربع نصف القطر وضعف مربع دز وضعف سطح حز في دز و
قد عرفت مما سبق ان مربع نصف القطر يساوي سطح حز في دز
يساوي مجموع سطح حز في دز ومربع دز المكون مجموع ضعف سطح حز
في دز وضعف مربع دز مساويا لضعف مربع نصف القطر فمجموع
مربعي دز حز يساوي اربعة امثال مربع نصف القطر فيساوي مربع
القطر فاذا كان دز وتر ضلع المخمس يكون حز وتر ثلثة الاعشار
وهو المطلوب وقد بين بطلميوس في الشكل الثالث من المقالة الاولى
من المجسطي ان الفضل من سطح وتر احد القوسين في وتر تمام الاخرى وبين سطح

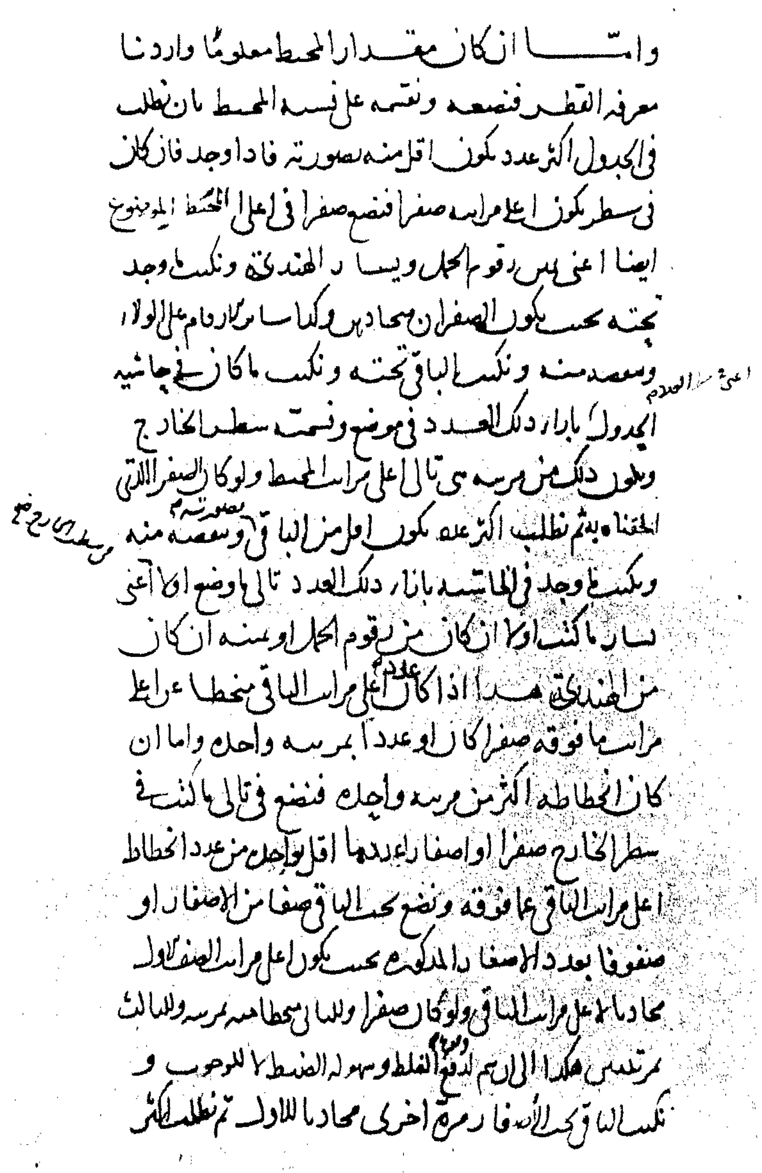
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 53

يكون تسعة اذرع وعشر ذراع تقريبا وقد اورد صاحب التحفة
الشاهية فيها ان نصف قطر محدب الثوابت سبعون الفا
وثلثة وسبعون مثلا لقطر الارض ونصف مثل واسم المحيط
منه على انه ثلثة امثاله وسبع مثله وهو اثنان وعشرون الفا
واربعمائة واثنان وستون مثلا لقطر الارض فاذا حسبناه بما
لنا هكذا
يكون التفاوت بينهما
مائة ... مثلا
لقطر الارض ... يكون ... درجة واحدة
من محدب الثوابت مثل نصف قطر الارض تقريبا واما علم
الخاتمة في اثبات غلط ... الى الوفاء والى ...
سداودنا الشكل الاول من المقالة الاولى من المجسطي ولكن
اسح نصف دائرة على قطر اح ومركز د و دب عمودا على القطر
ونصف حد على ه ونصل به ونجعل هز مساويا له
ونصل بز فيكون دز ضلع
المعشر وبز ضلع المخمس
لان حد تنصف على ه وزيد
فيه دز فسطح حز في دز مع مربع هد يساوي مربع هز
بالشكل السادس من ثانية الاصول يساوي مربع هب بل يساوي
مربعي هد دب ونلقي مربع هد المشترك يبقى سطح حز في دز
مساويا لمربع دب اعني مربع حد فخط حز مقسوم بنسبة ذات

Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 52

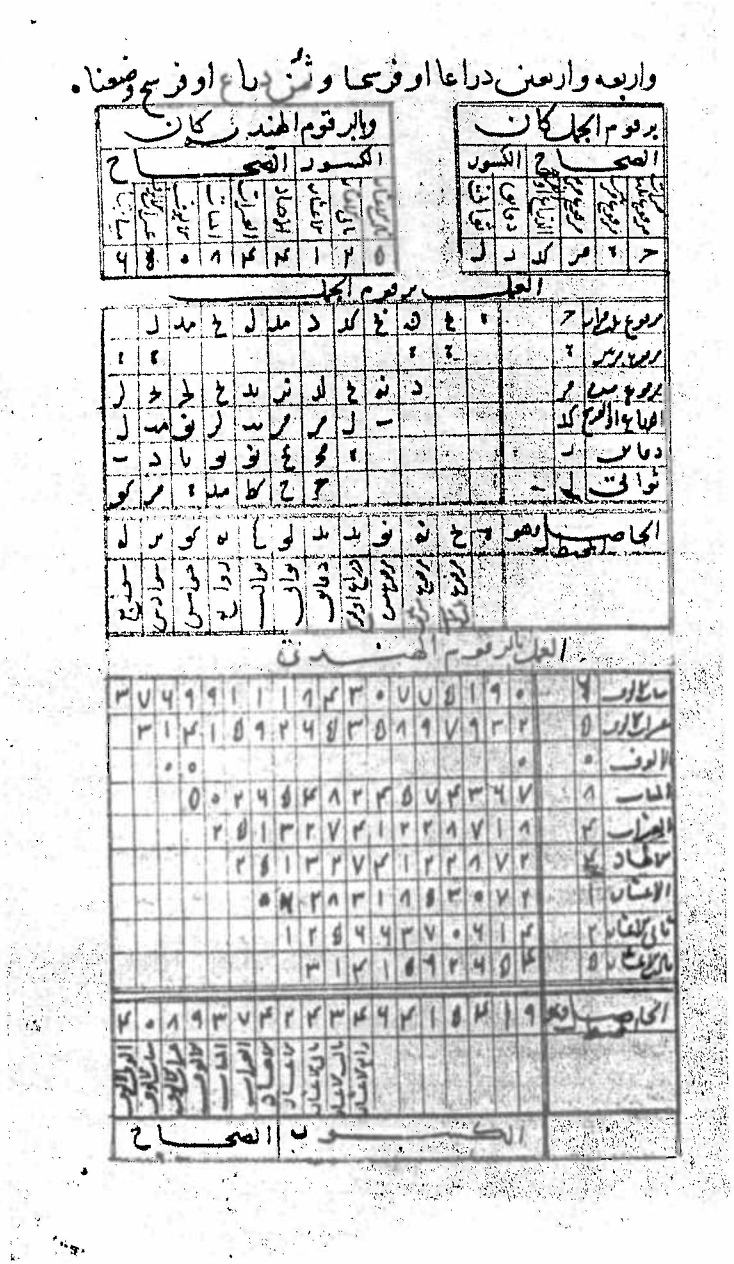


Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 51

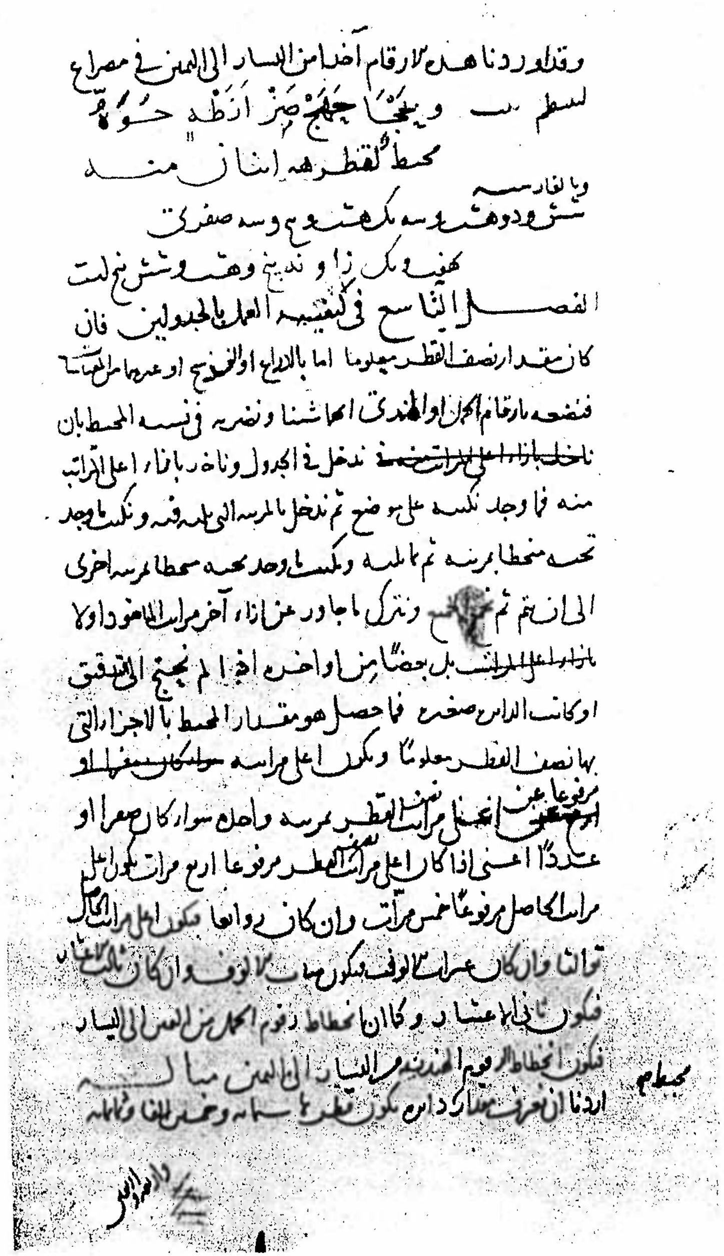


Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 50

Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 49

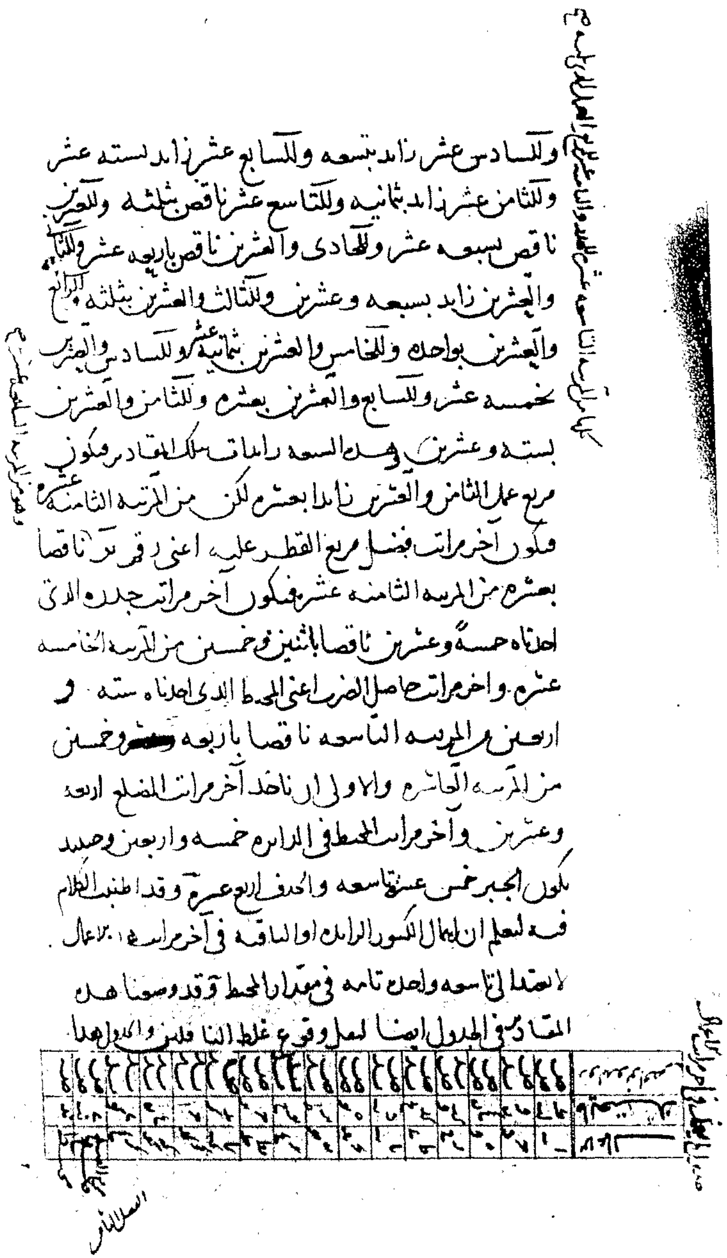


Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 48



Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 47

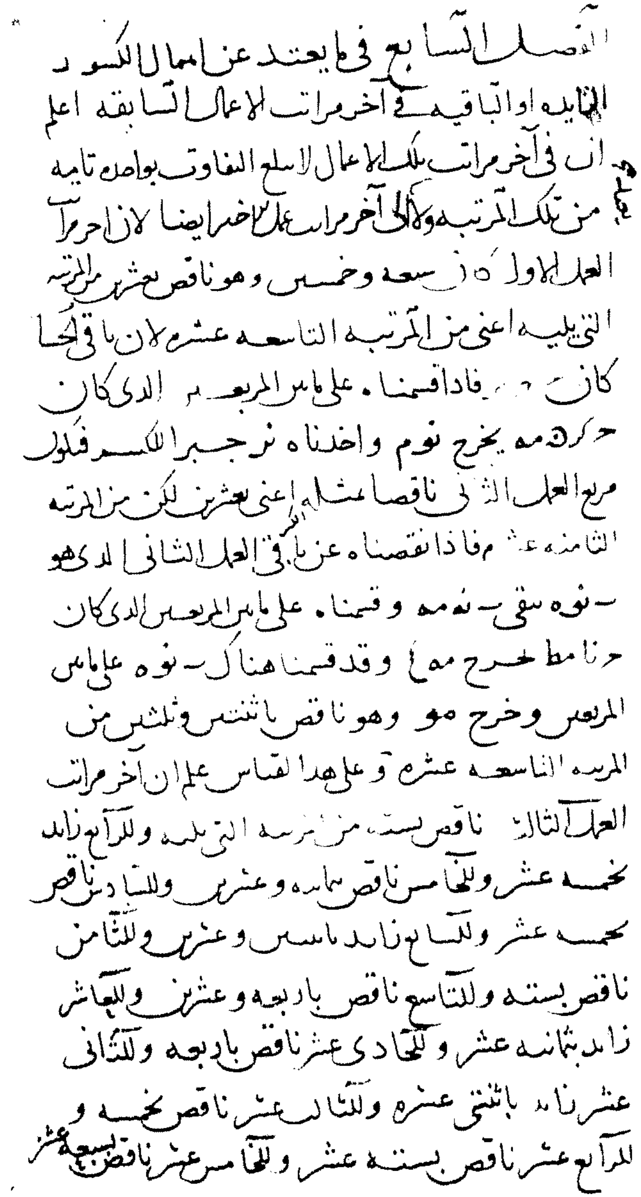
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 46



وللسادس عشر زائد بتسعه وللسابع عشر زائد بسته عشر
وللثامن عشر زائد بثمانيه وللتاسع عشر ناقص بثلثه وللعشرين
ناقص بسبعه عشر وللحادى والعشرين ناقص باربعه عشر وللثانى
والعشرين زائد بسبعه وعشرين وللثالث والعشرين بثلثه وللرابع
والعشرين بواحد وللخامس والعشرين بثمانيه عشر وللسادس والعشرين
بخمسه عشر وللسابع والعشرين بعشره وللثامن والعشرين
بسته وعشرين [illegible] السبعه [illegible] المقادير فيكون
مربع عمل الثامن والعشرين زائدا بعشره لكن من المرتبه الثامنه عشره
فيكون آخر مراتب فضل مربع القطر عليه اعنى [illegible] ناقصا
بعشره من المرتبه الثامنه عشر فيكون آخر مراتب جذره الذى
اخذناه خمسه وعشرين ناقصا باثنين وخمسين من المرتبه الخامسه
عشر واخر مراتب حاصل الضرب اعنى المحيط الذى اخذناه سته و
اربعين من المرتبه التاسعه ناقصا باربعه وخمسين
من المرتبه العاشره والاولى ان ناخذ آخر مراتب الضلع اربعه
وعشرين وآخر مراتب المحيط فى الدايره خمسه واربعين وحينئذ
يكون الجبر خمس عشر تاسعه والحذف اربع عشره وقد اطنبت الكلام
فيه لتعلم ان اهمال الكسور الزايده او الناقصه فى آخر مراتب الاعمال
لا تخل الى تاسعه واحده تامه فى مقدار المحيط وقد وضعنا هذه
المقادير فى الجدول ايضا لئلا يقع غلط النساخ [illegible]

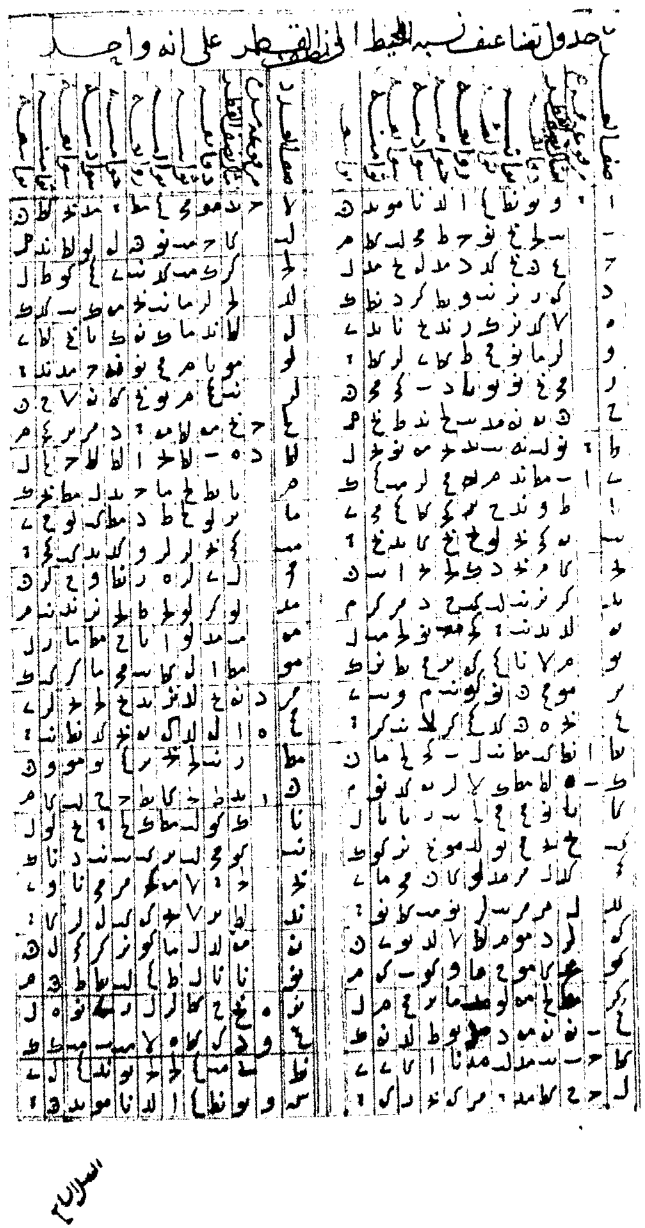
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 45

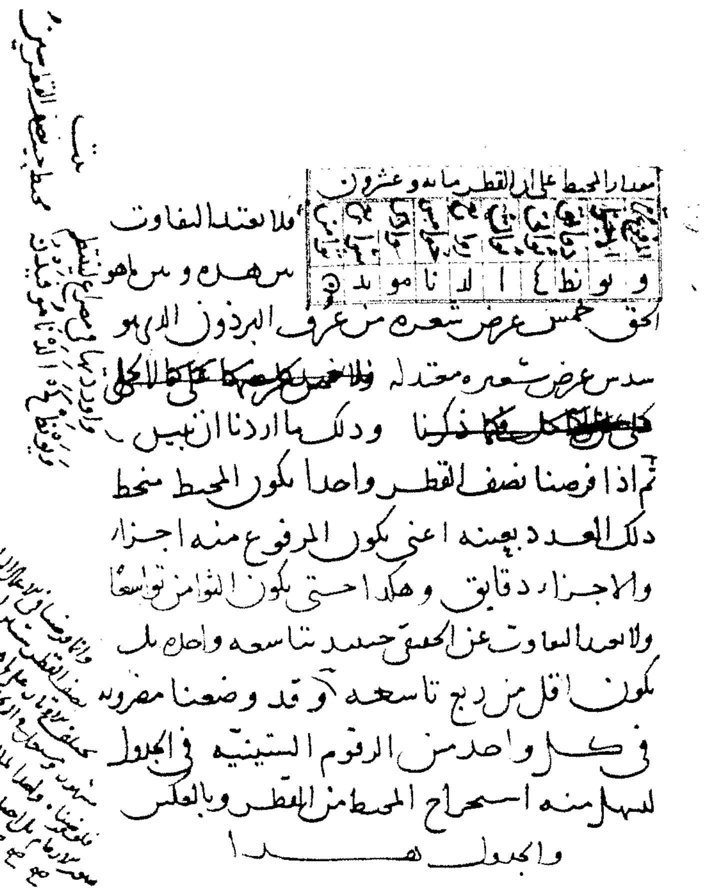


الفصل السابع فى ما يعتد عن اهمال الكسور
الزائده او الباقيه فى آخر مراتب الاعمال السابقه اعلم
ان فى آخر مراتب تلك الاعمال لا ضلع التفاوت بواحد تامه
من تلك المرتبه ولا الى آخر مراتب عمل [illegible] ايضا لان آخر مراتب
العمل الاول كان سبعه وخمسين وهو ناقص بعشرين من المرتبه
التى يليه اعنى من المرتبه التاسعه عشره لان باقى [illegible]
كان [illegible] فاذا قسمنا على ما من المربع [illegible] الذى كان
[illegible] يخرج نوم واخذناه نرجبرا للكسر فيكون
مربع العمل الثانى ناقصا بمثله اعنى بعشرين لكن من المرتبه
الثامنه عشره فاذا نقصناه عن باقى العمل الثانى الذى هو
[illegible] قسمنا على ما من المربعين الذى كان
[illegible] وقد قسمنا هناك [illegible] على ما من
المربعين وخرج مو وهو ناقص باثنين وثلثين من
المرتبه التاسعه عشره وعلى هذا القياس علم ان آخر مراتب
العمل الثالث ناقص بسته من المرتبه التى تليه وللرابع زائد
بخمسه عشر وللخامس ناقص بثمانيه وعشرين وللسادس ناقص
بخمسه عشر وللسابع زائد بسبعه وعشرين وللثامن
ناقص بسته وللتاسع ناقص باربعه وعشرين وللعاشر
زائد بثمانيه عشر وللحادى عشر ناقص باربعه وللثانى
عشر زائد باثنى عشره وللثالث عشر ناقص بخمسه و
للرابع عشر ناقص بسته عشر وللخامس عشر ناقص بسبعه عشر

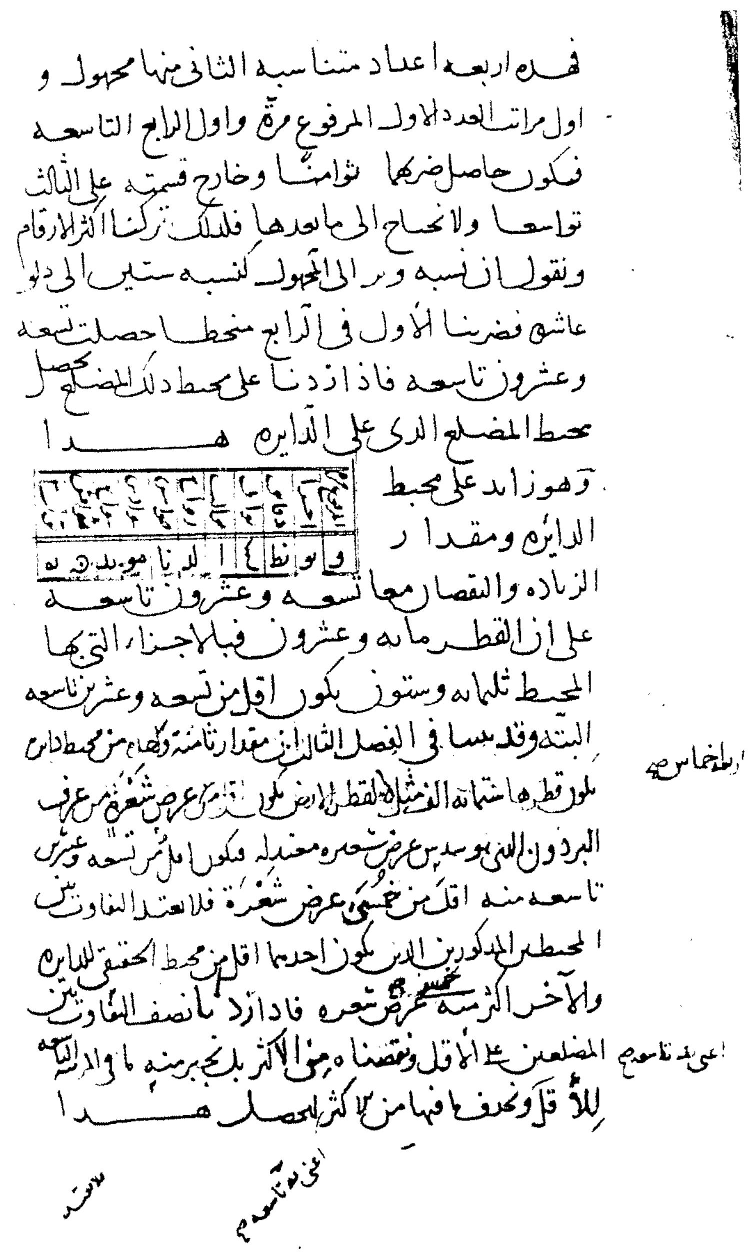
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 44


جدول تضاعف نسبة المحيط الى نصف القطر على انه واحد

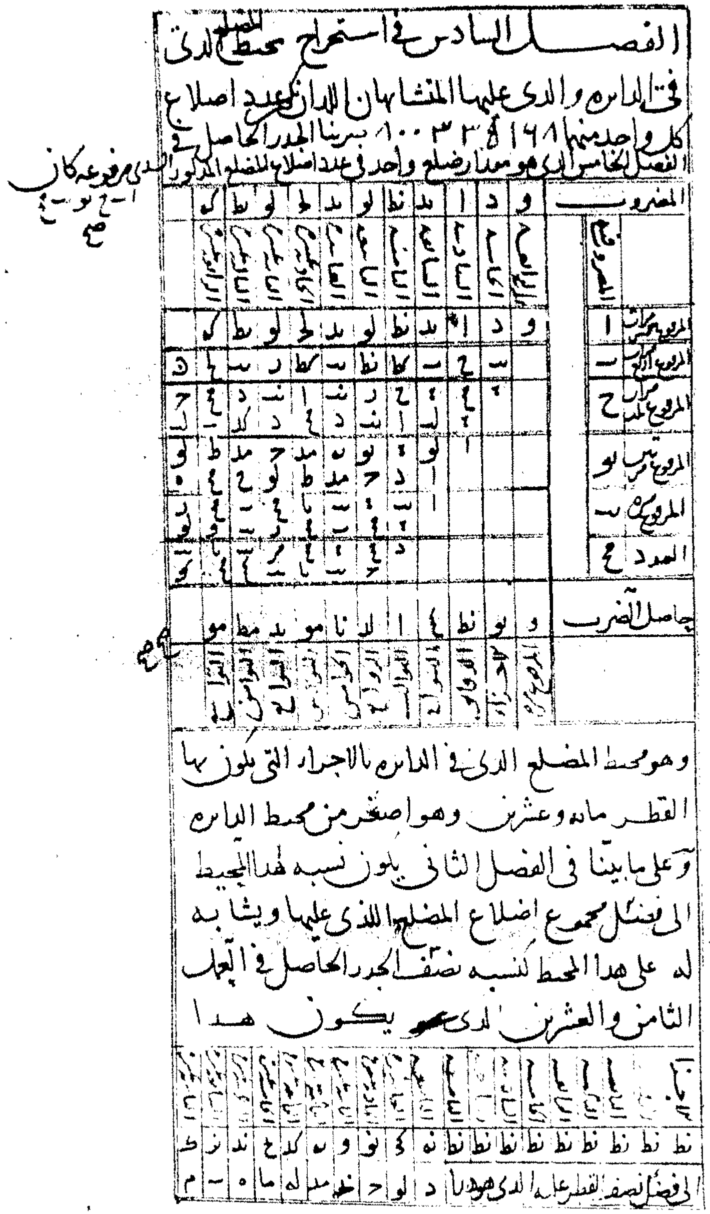
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 43


ولا نعتد التفاوت
الحق خمس عرض شعرة من عرق البرذون الذي هو
سدس عرض شعرة معتدلة
... ذكرنا وذلك ما اردنا ان نبين
ثم اذا فرضنا نصف القطر واحدا يكون المحيط منحط
ذلك العدد بعينه اعني يكون المرفوع منه اجزاء
والاجزاء دقايق وهكذا حتى يكون التوامن تواسعا
ولا نعتد التفاوت عن الحقيق حينئذ تاسعة واحدة بل
يكون اقل من ربع تاسعة وقد وضعنا مضروبه
في كل واحد من الرقوم الستينية في الجدول
ليسهل منه استخراج المحيط من القطر وبالعكس
والجدول هذا

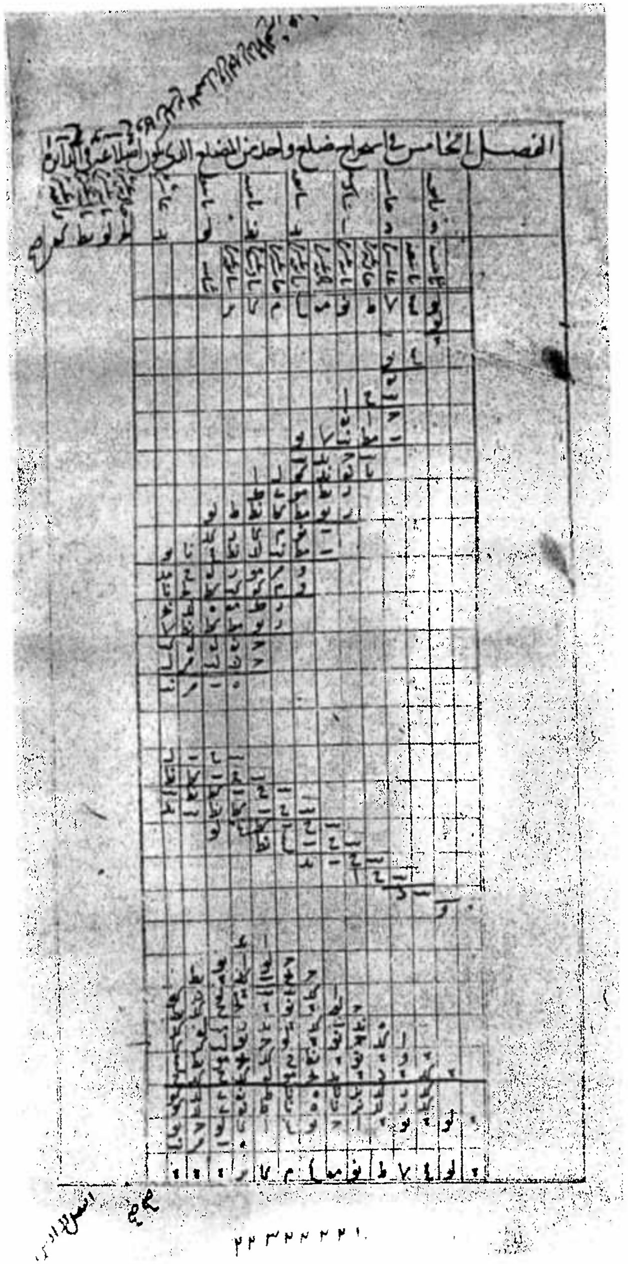
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 42



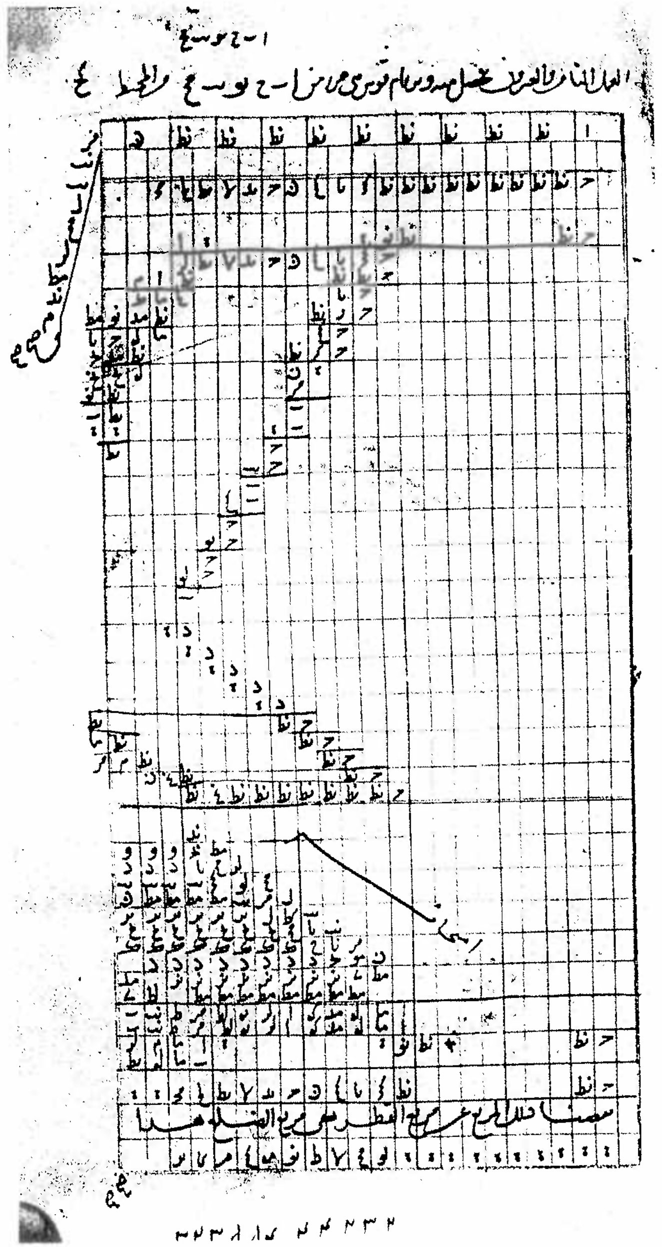
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 41



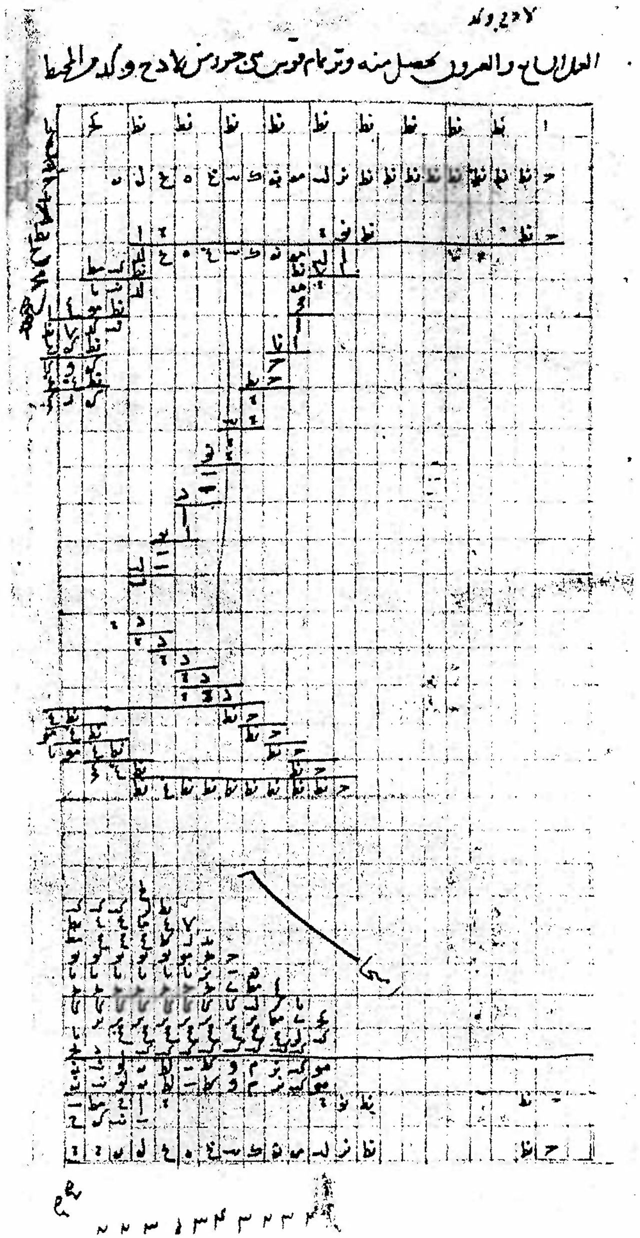
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 40



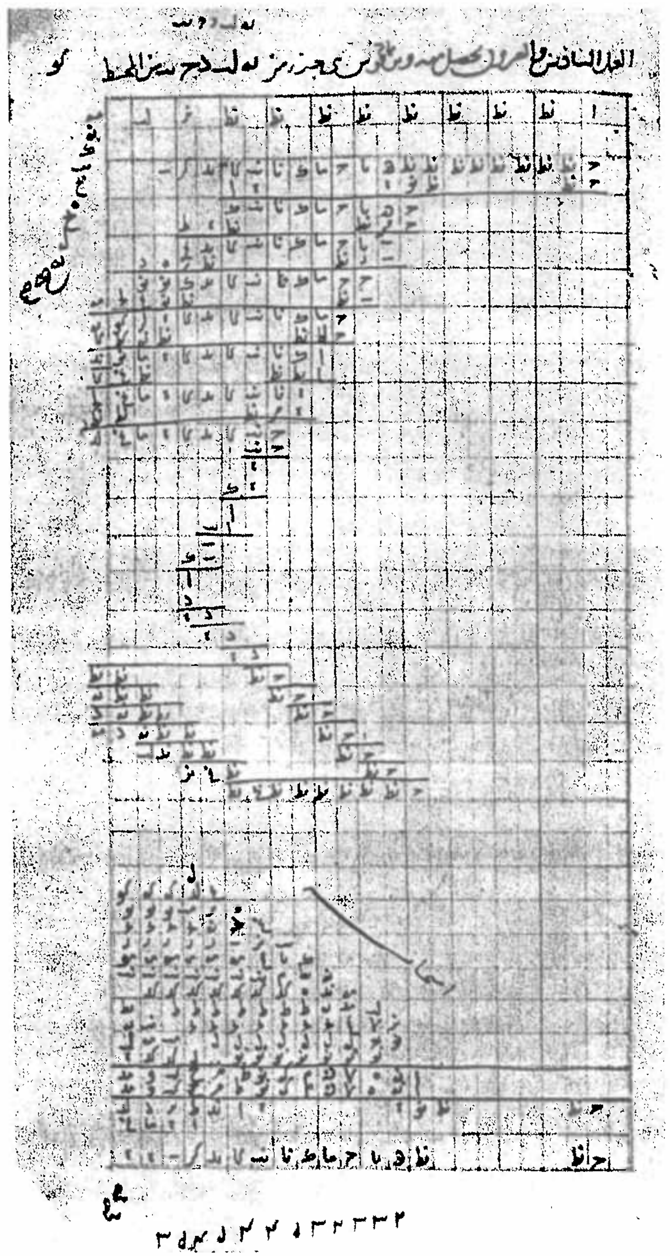
Al-Kâshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 39



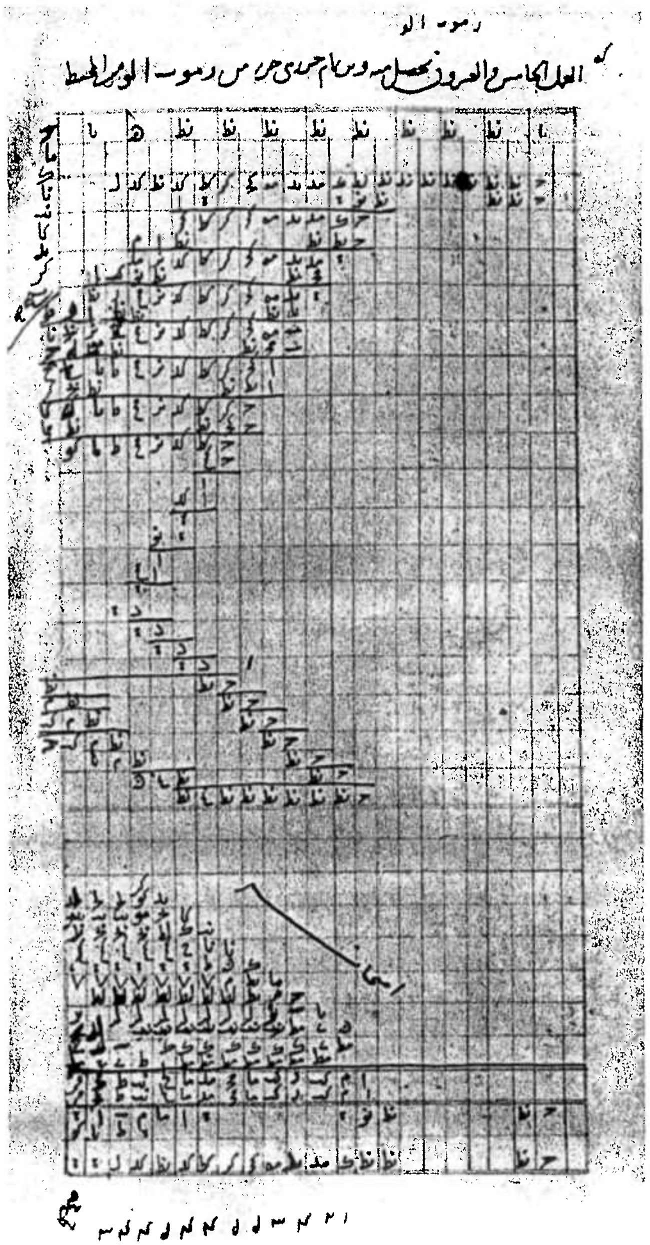
Al-Kāshi, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 38



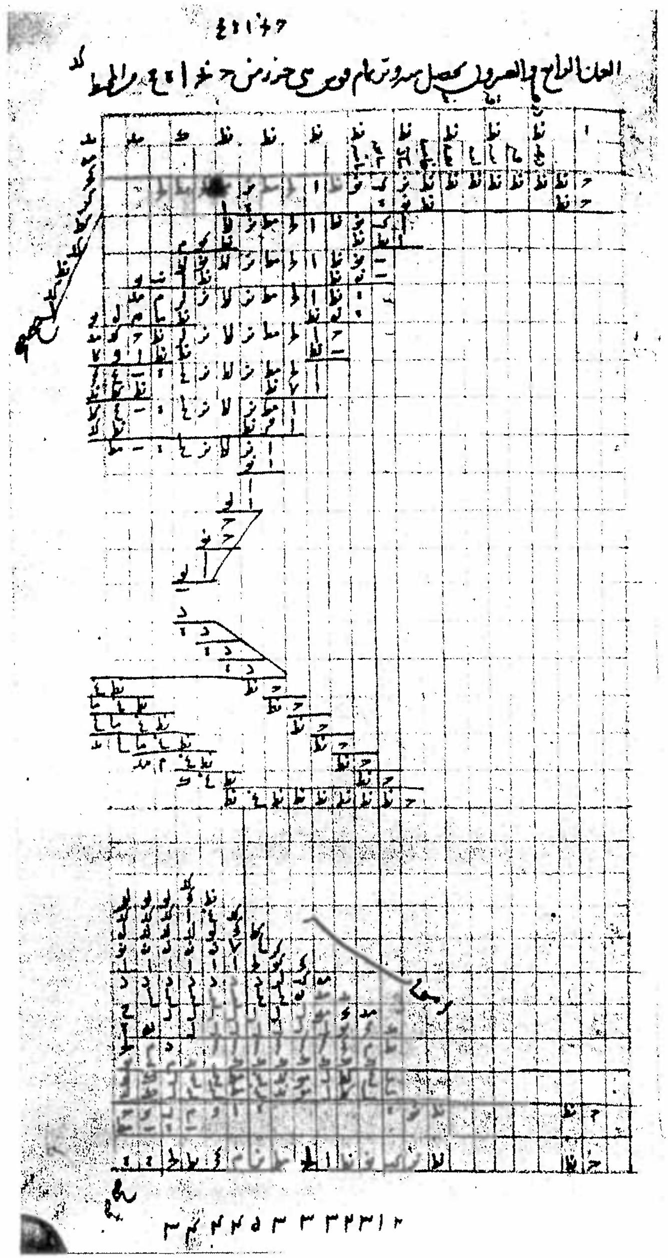
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 37



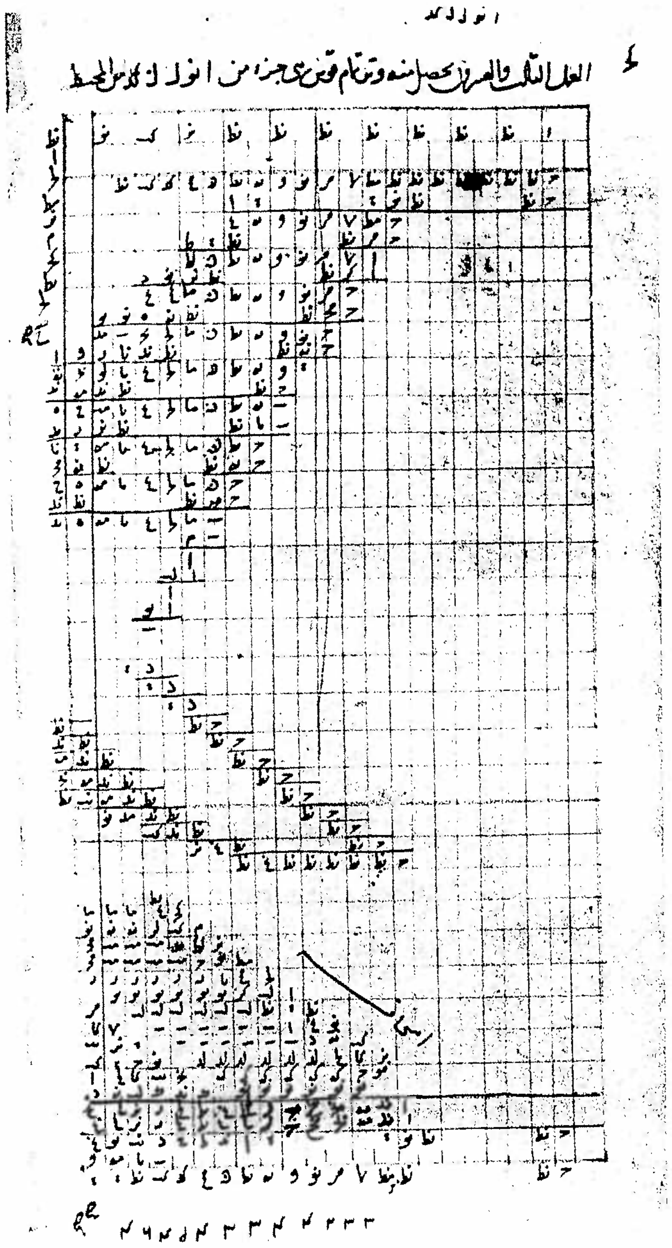
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 36



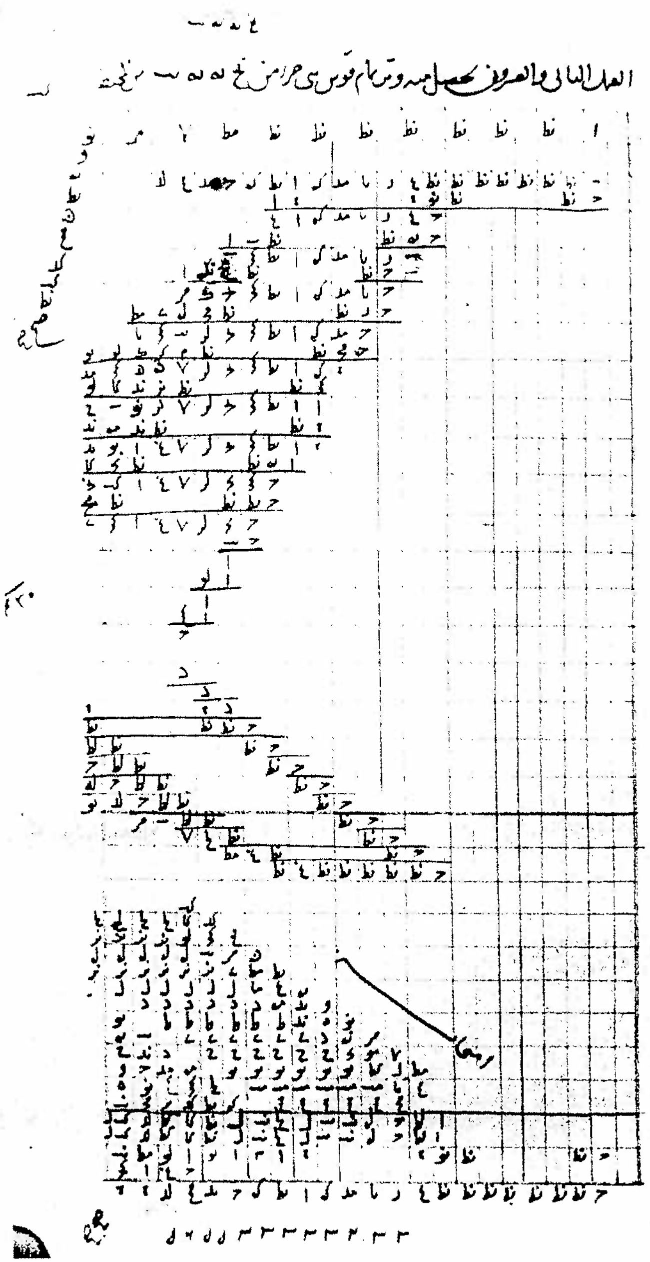
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 35



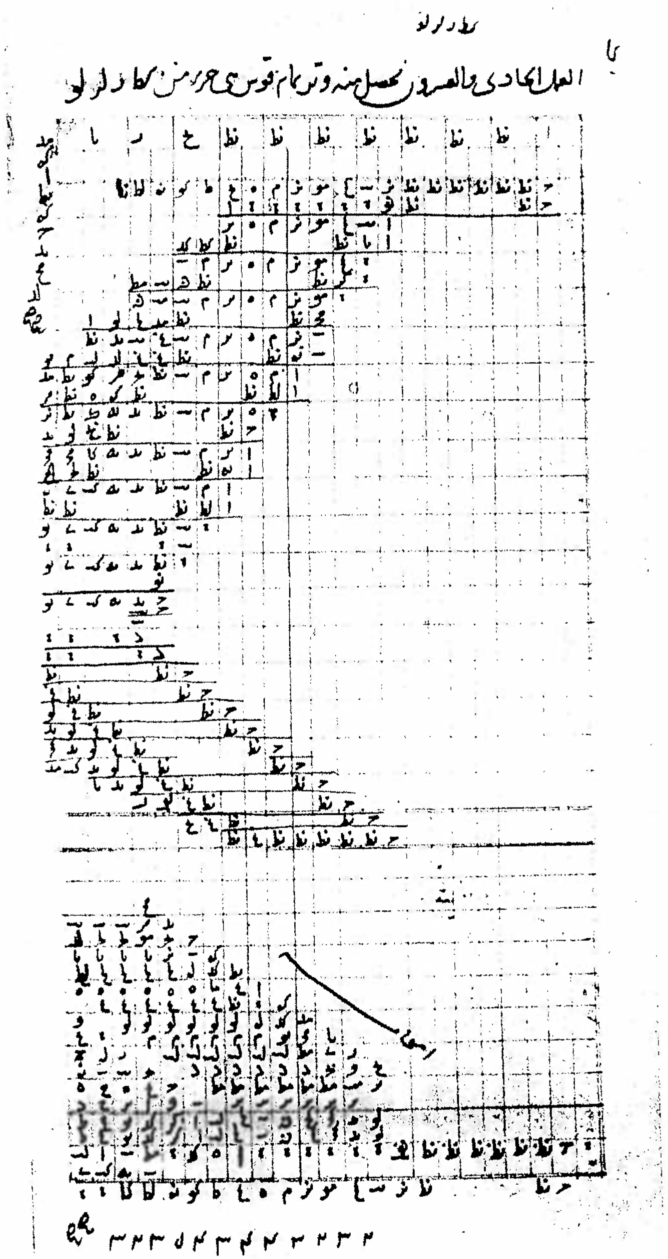
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 34



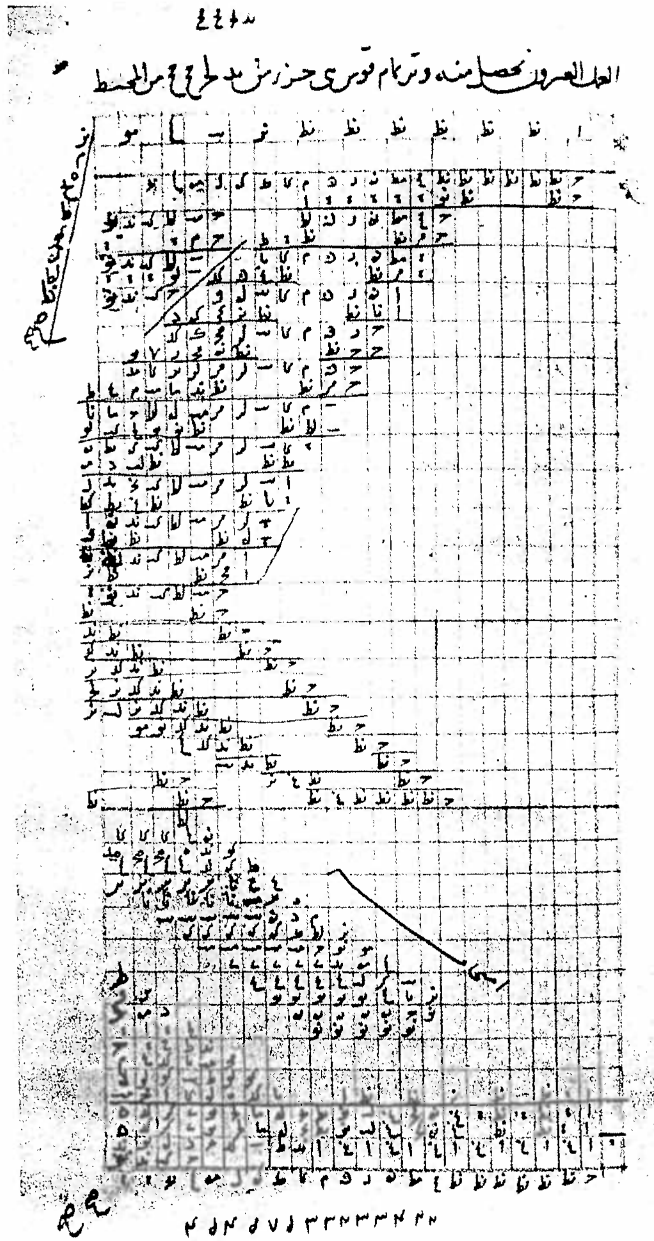
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 33



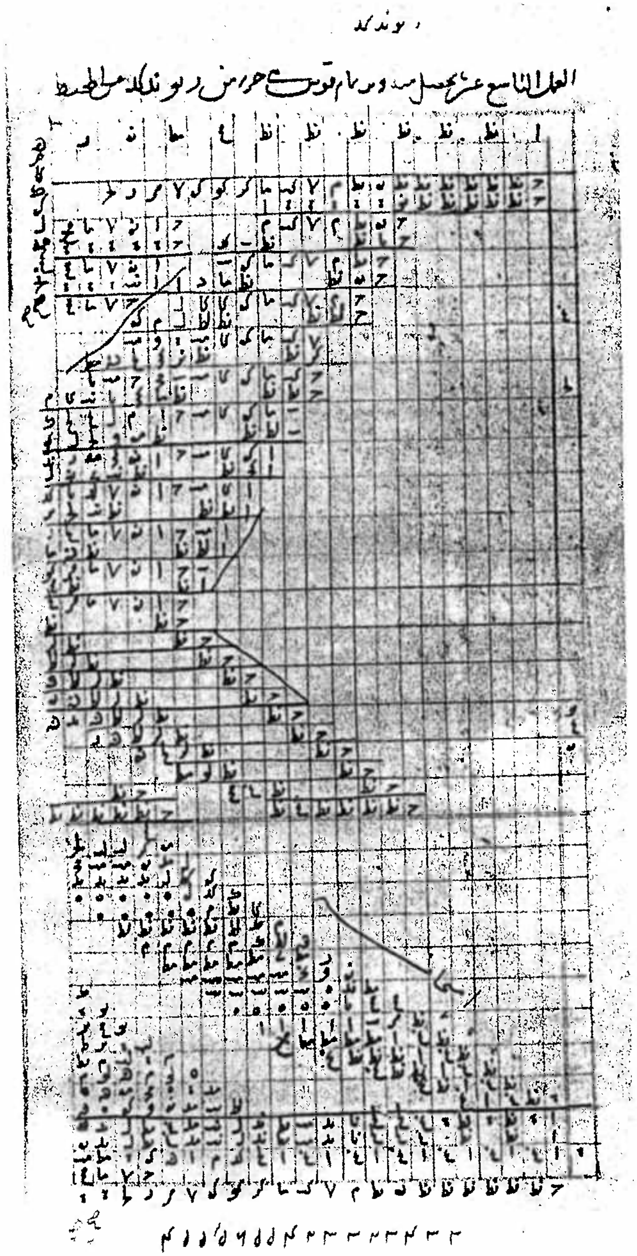
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 32



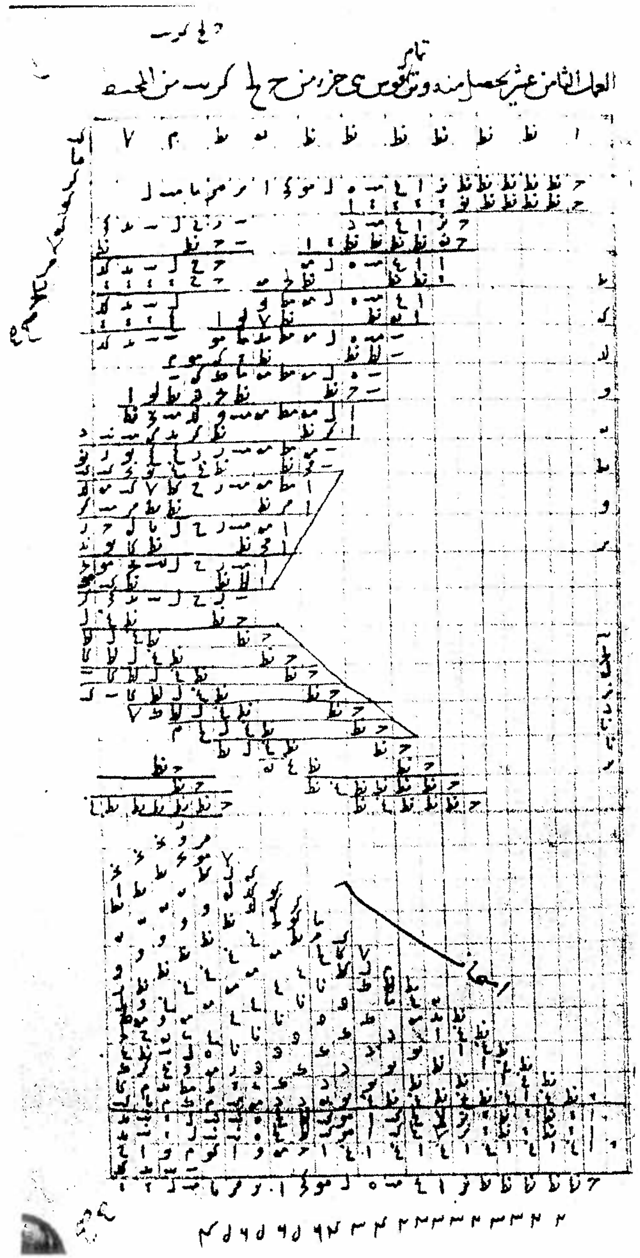
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 31



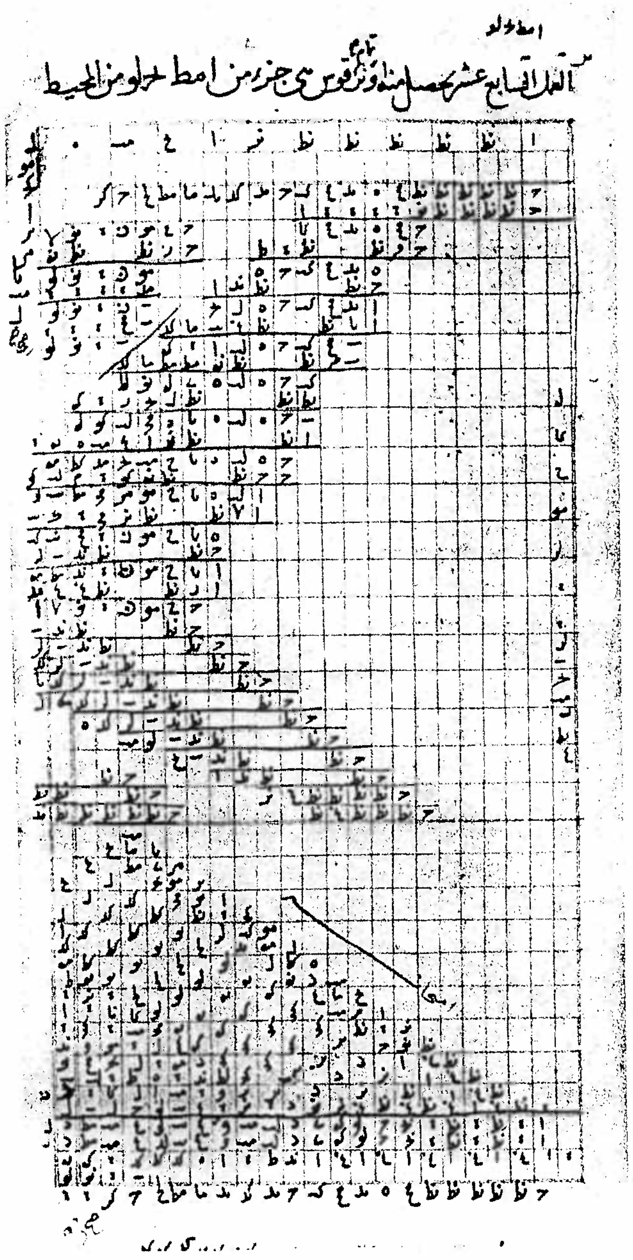
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 30



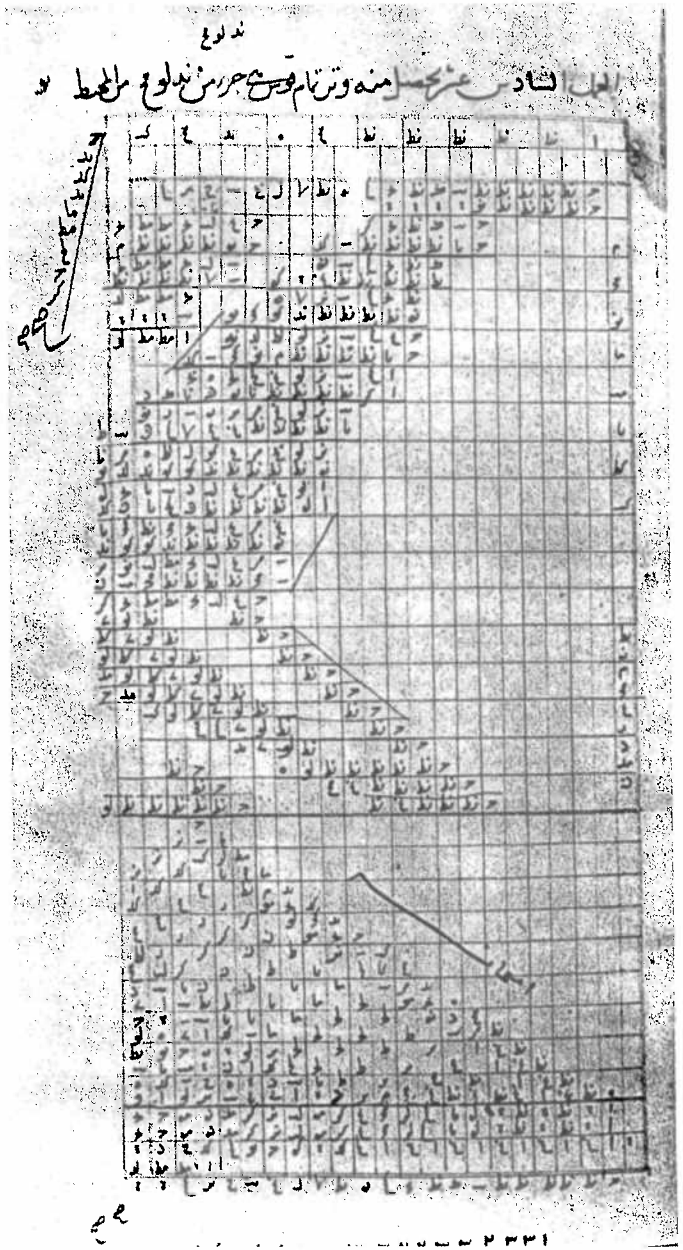
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 29



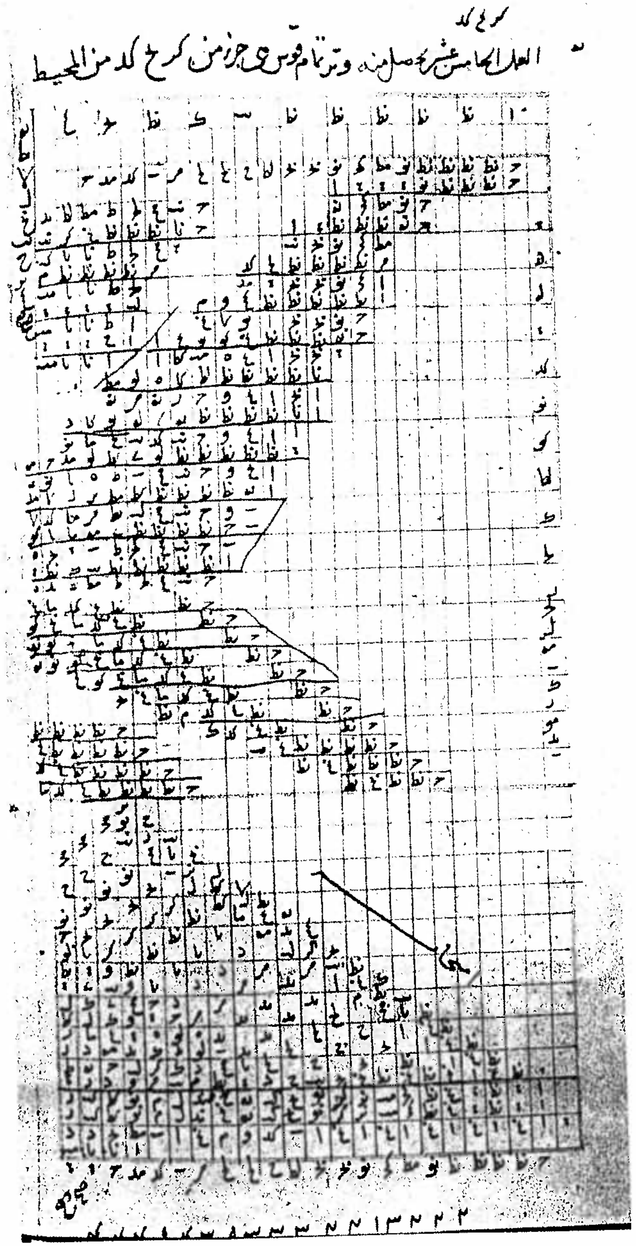
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 28



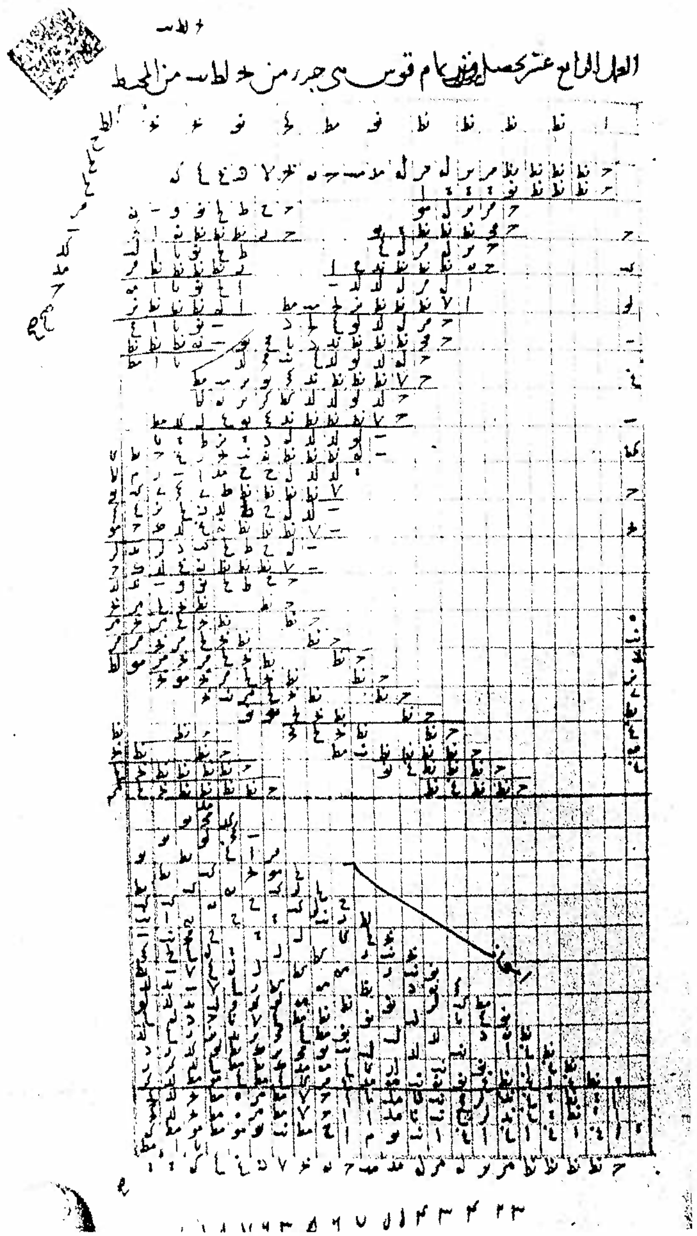
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 27



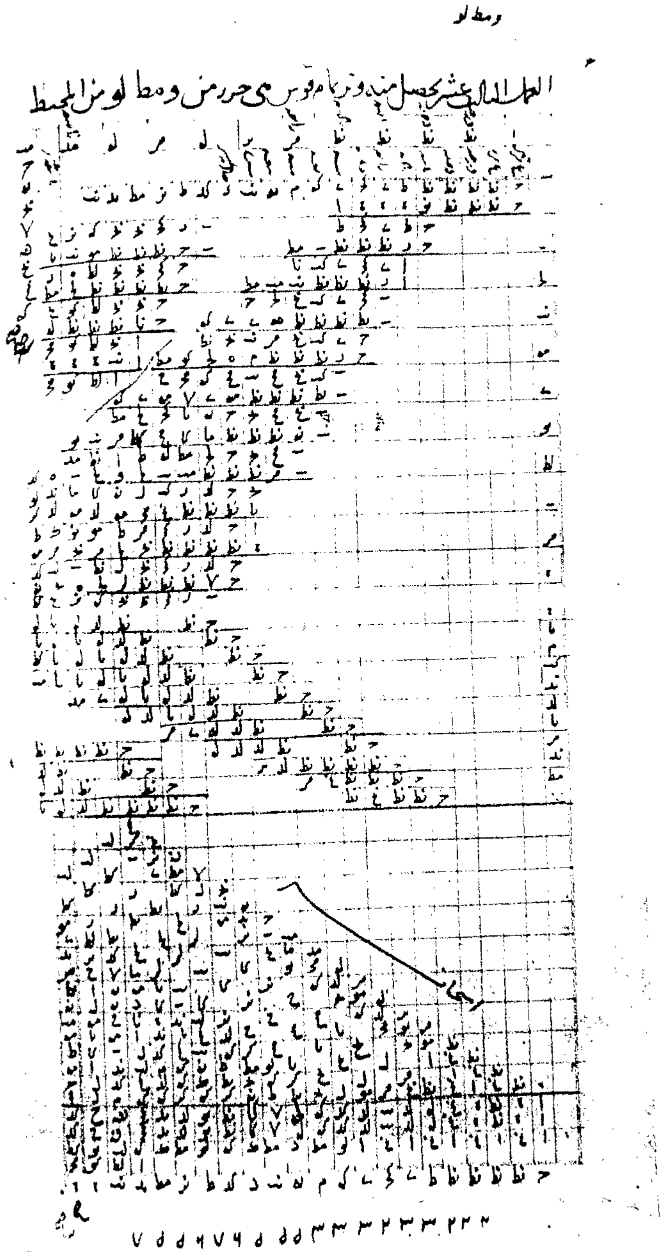
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 26



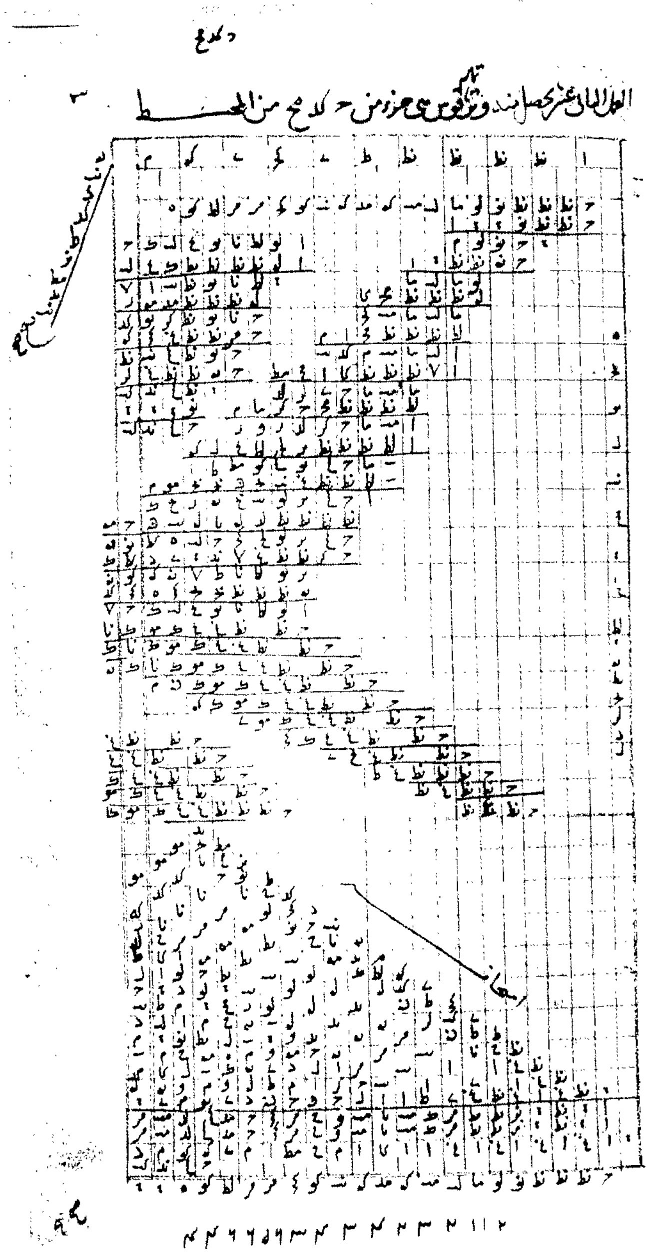
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 25



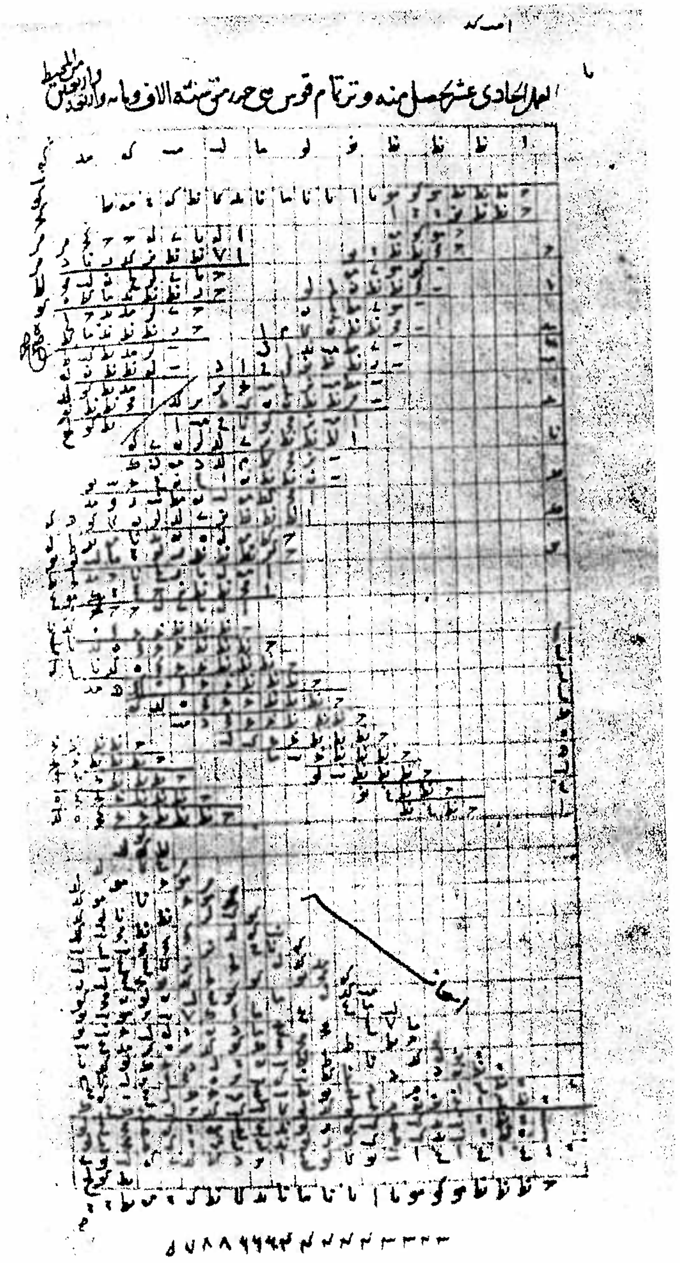
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 24



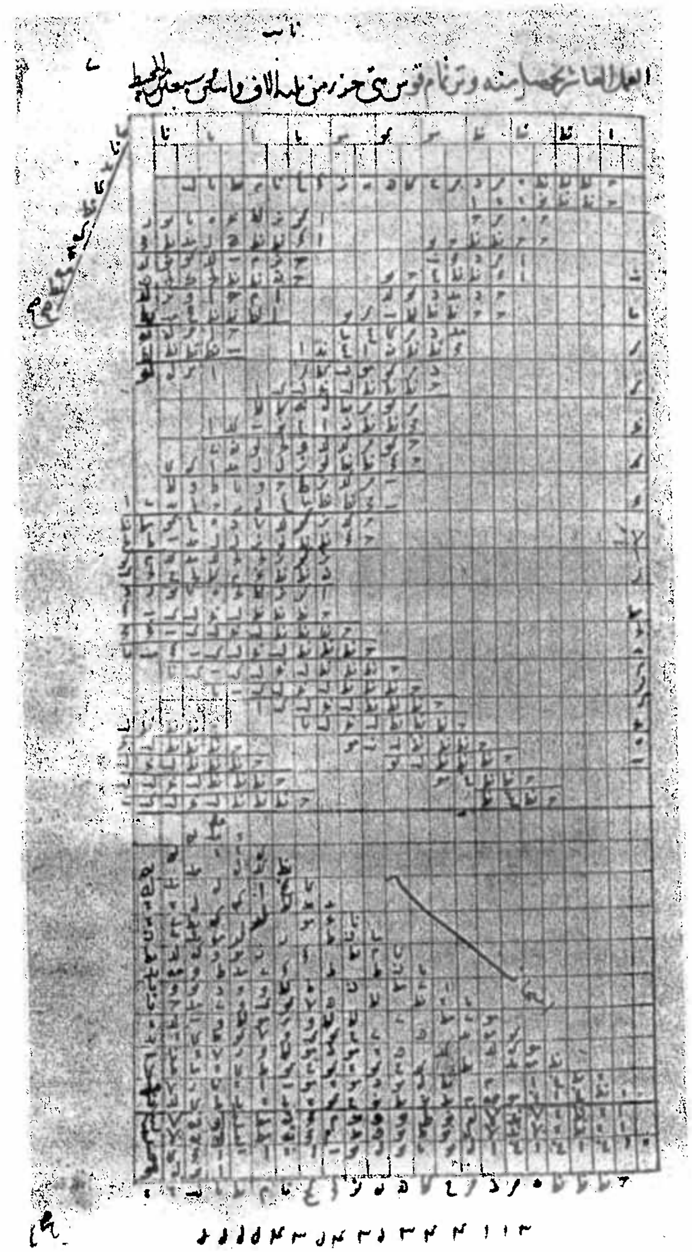
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 23



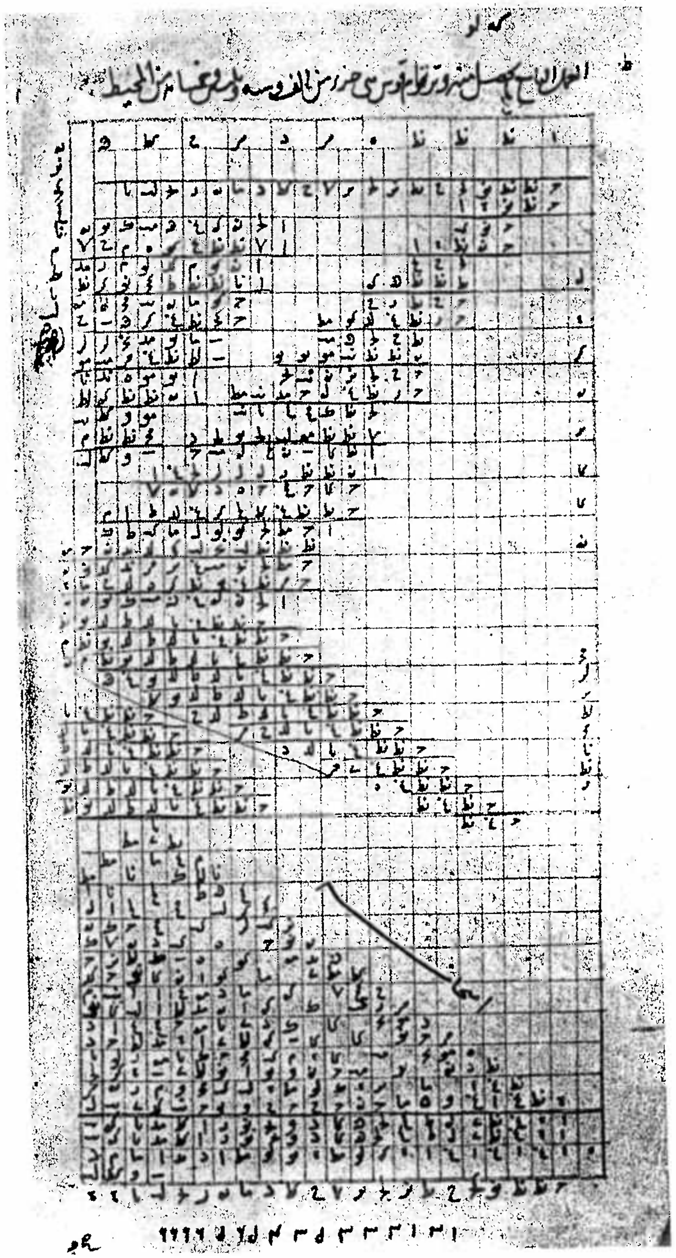
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 22



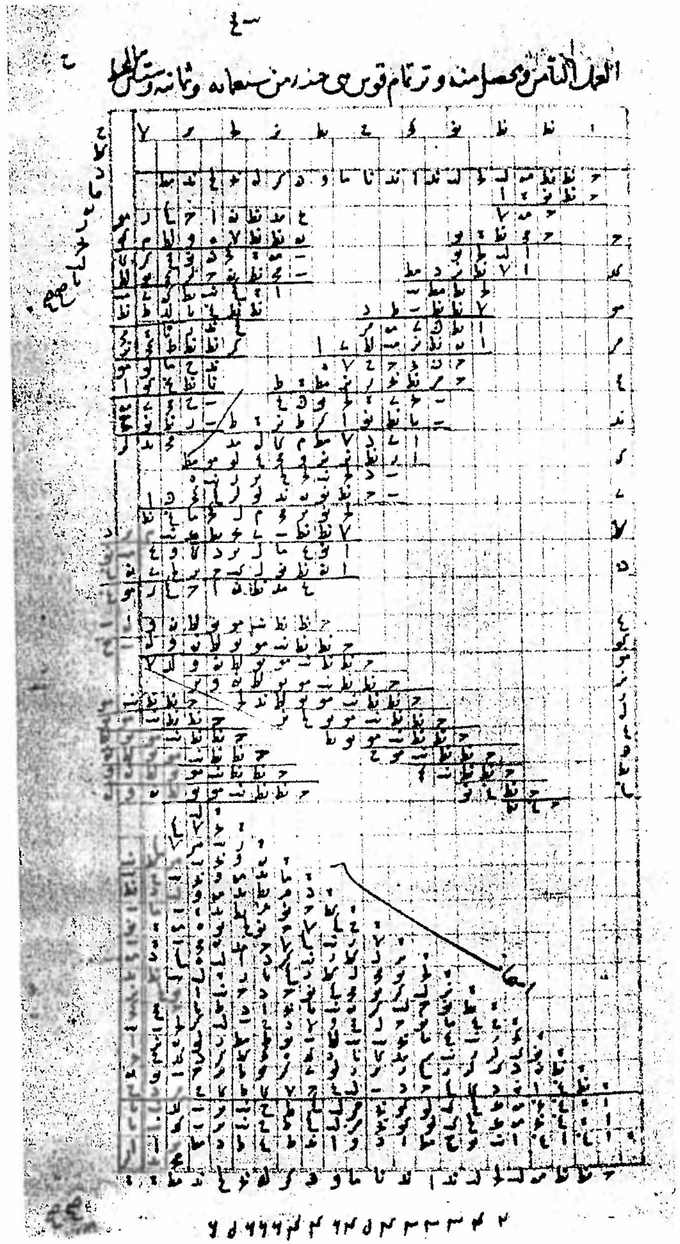
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 21



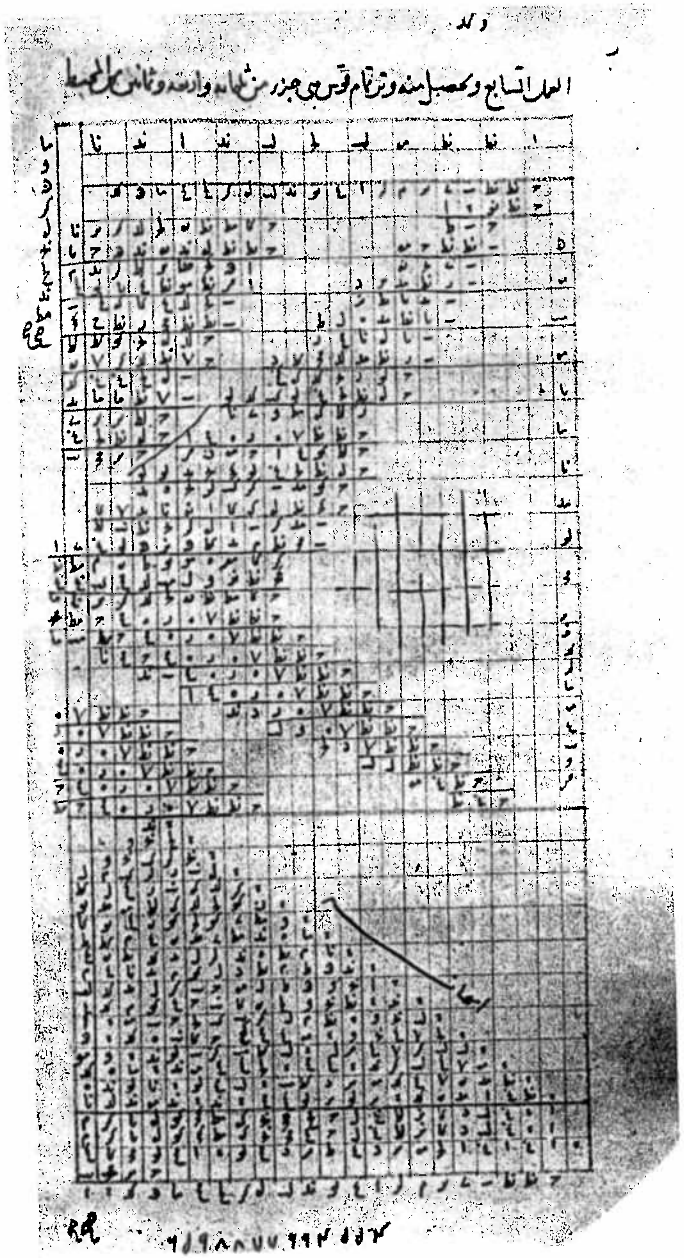
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 20



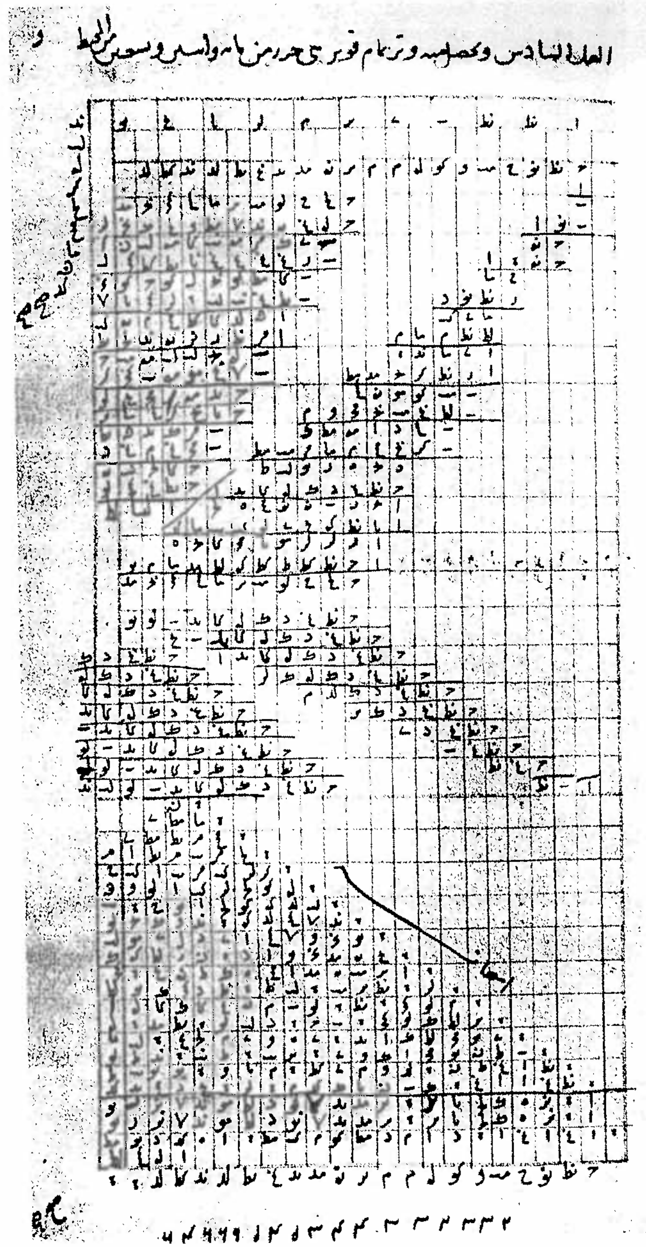
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 19



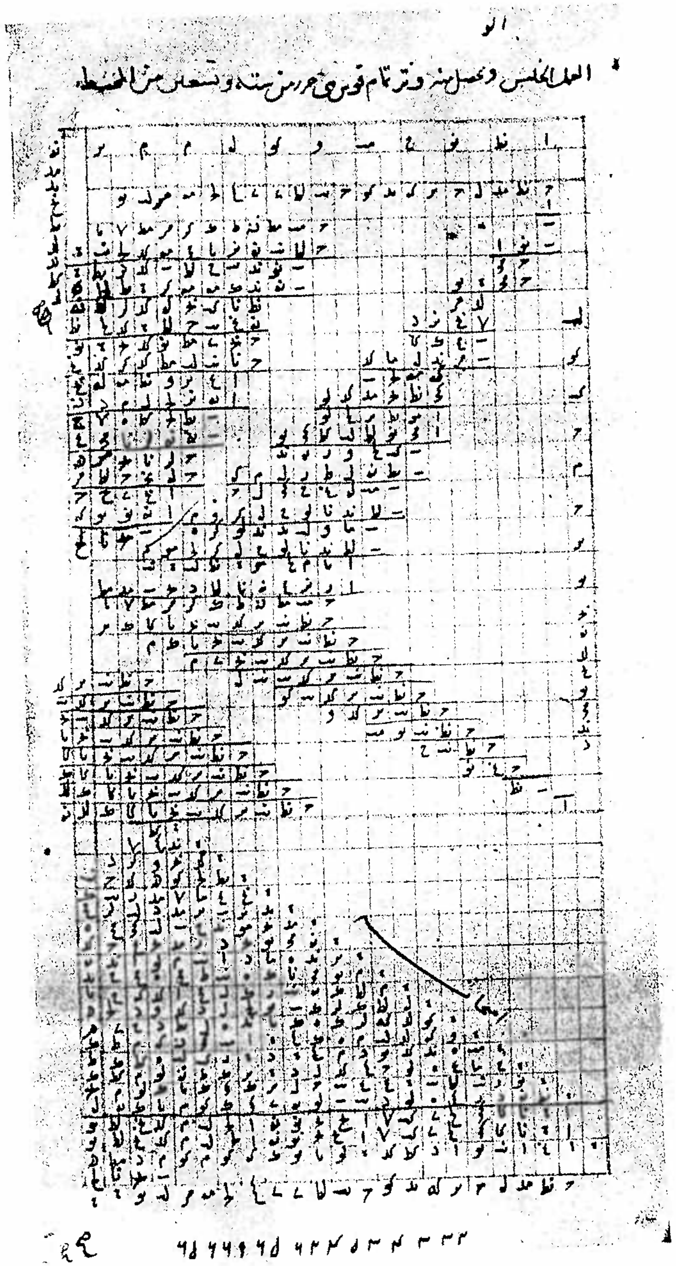
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 18



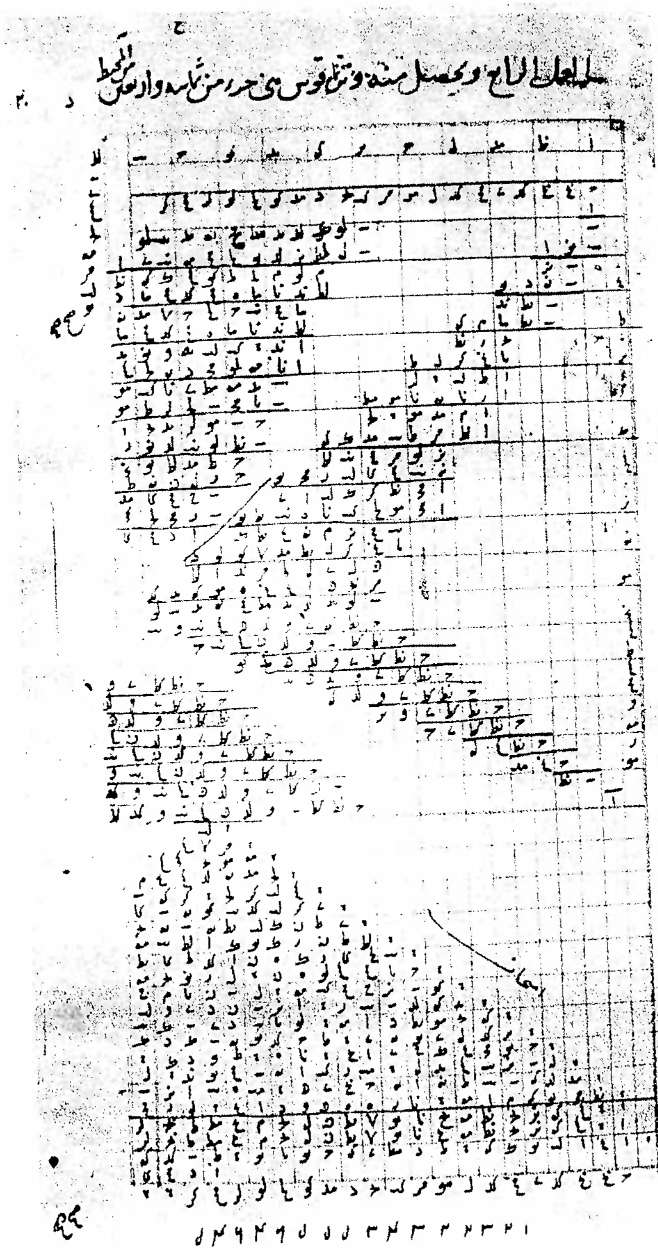
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 17



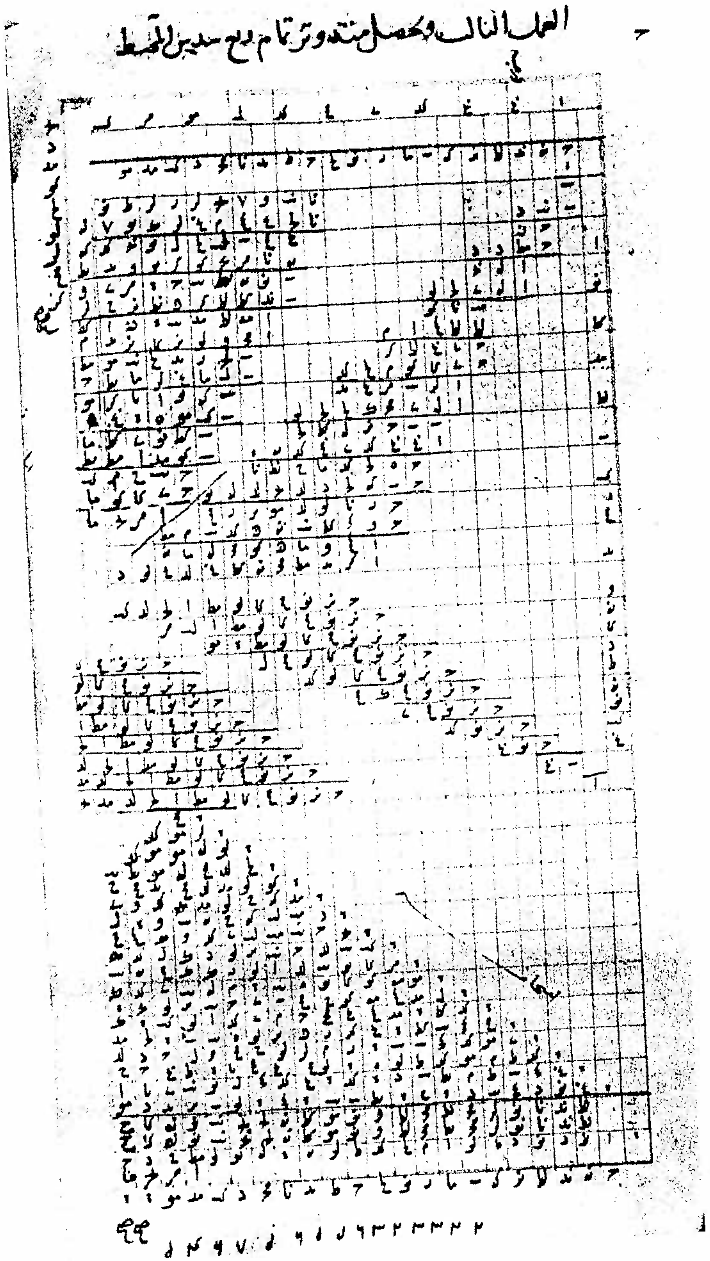
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 16



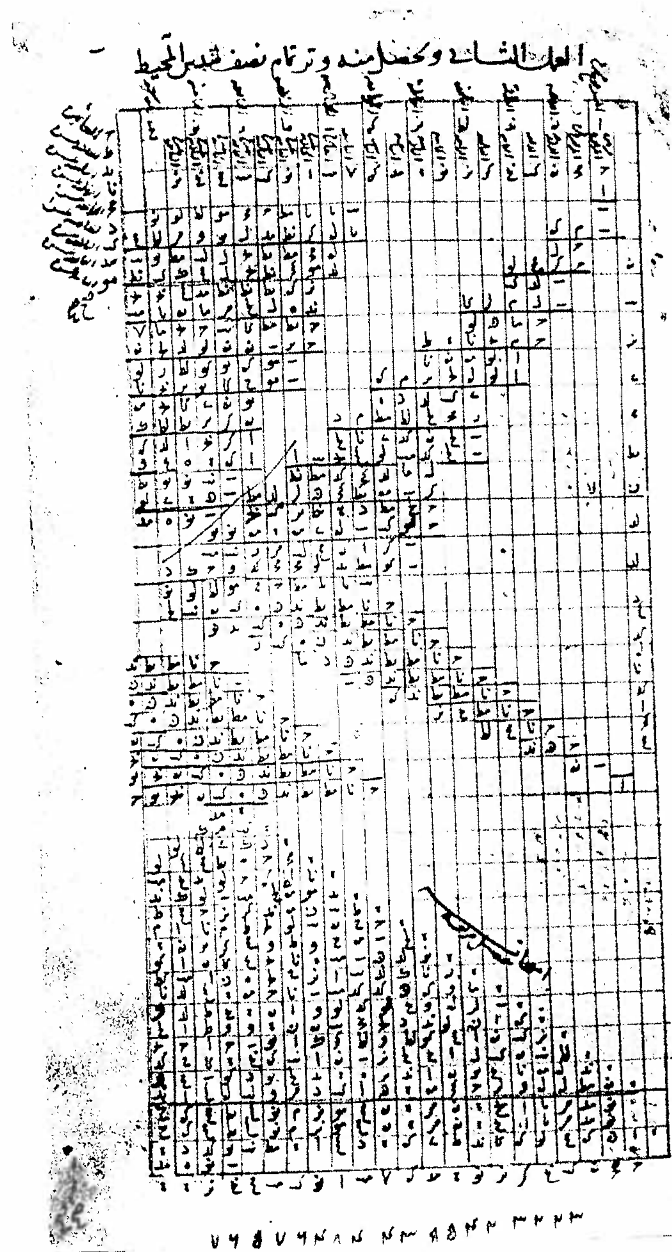
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 15



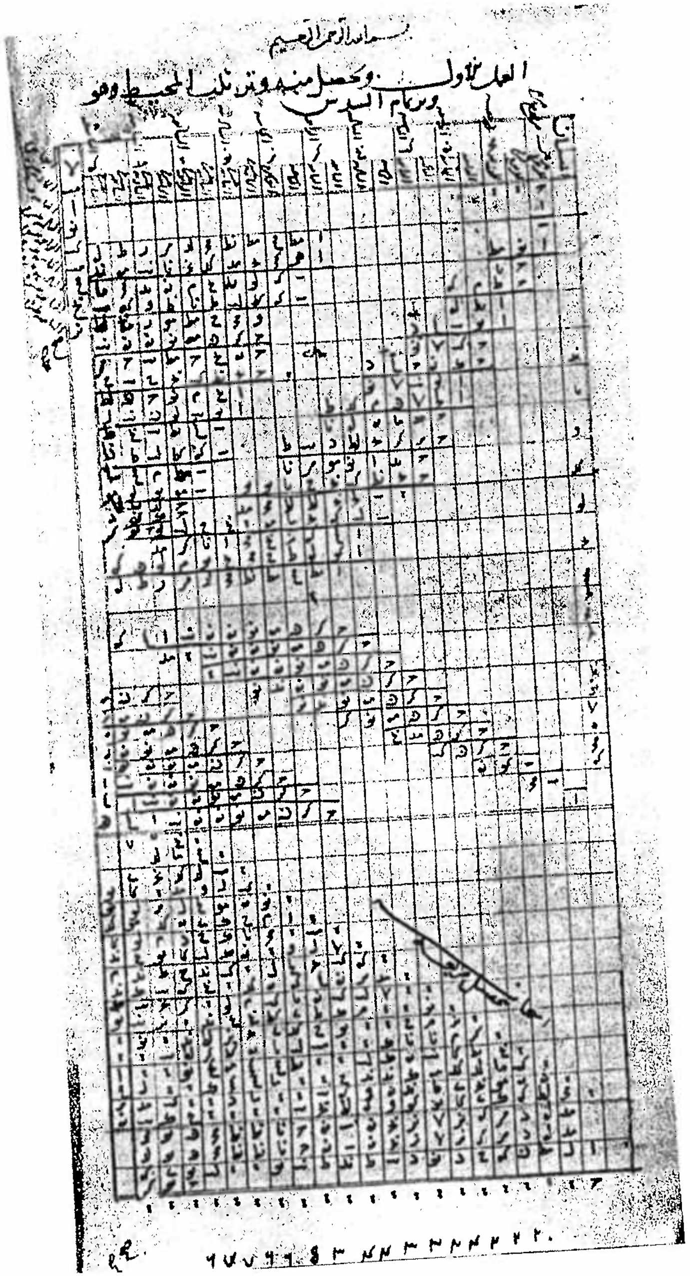
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 14


العمل الثالث وحصل منه وتر تمام ربع سدس المحيط

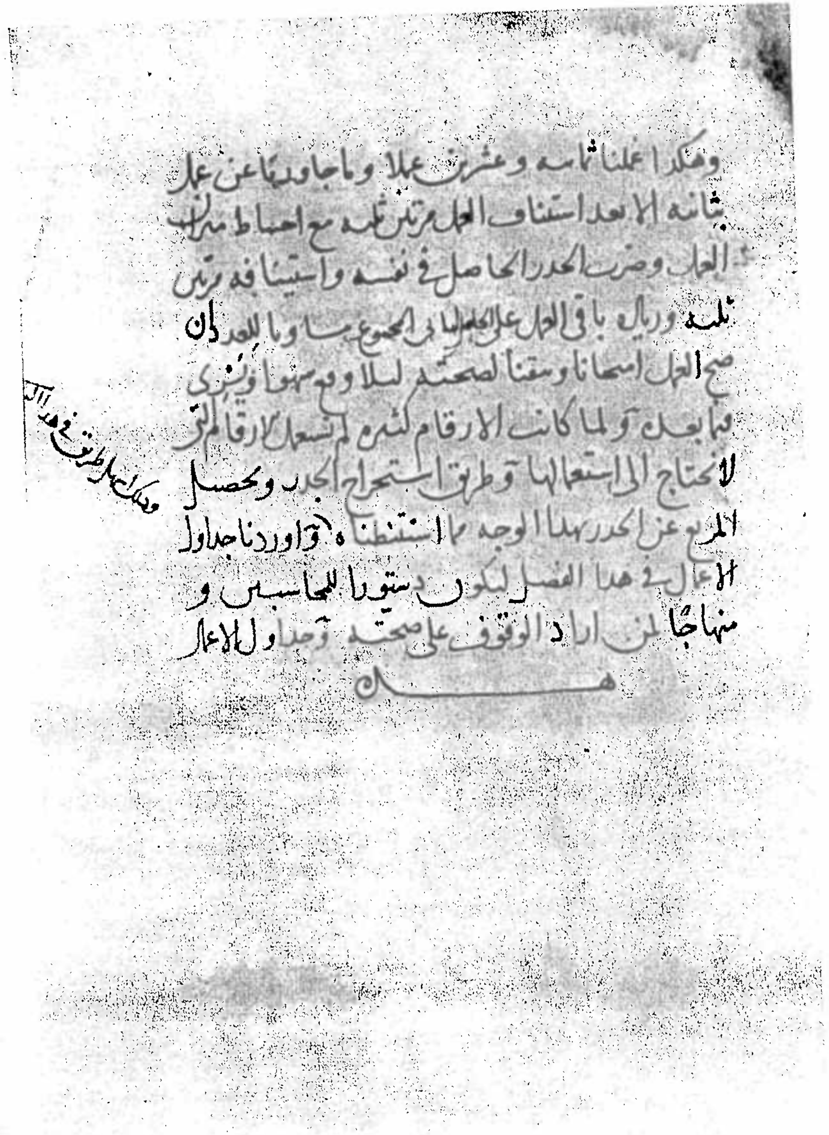
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 13


العمل الثاني وحصل منه وتر تمام نصف سدس المحيط

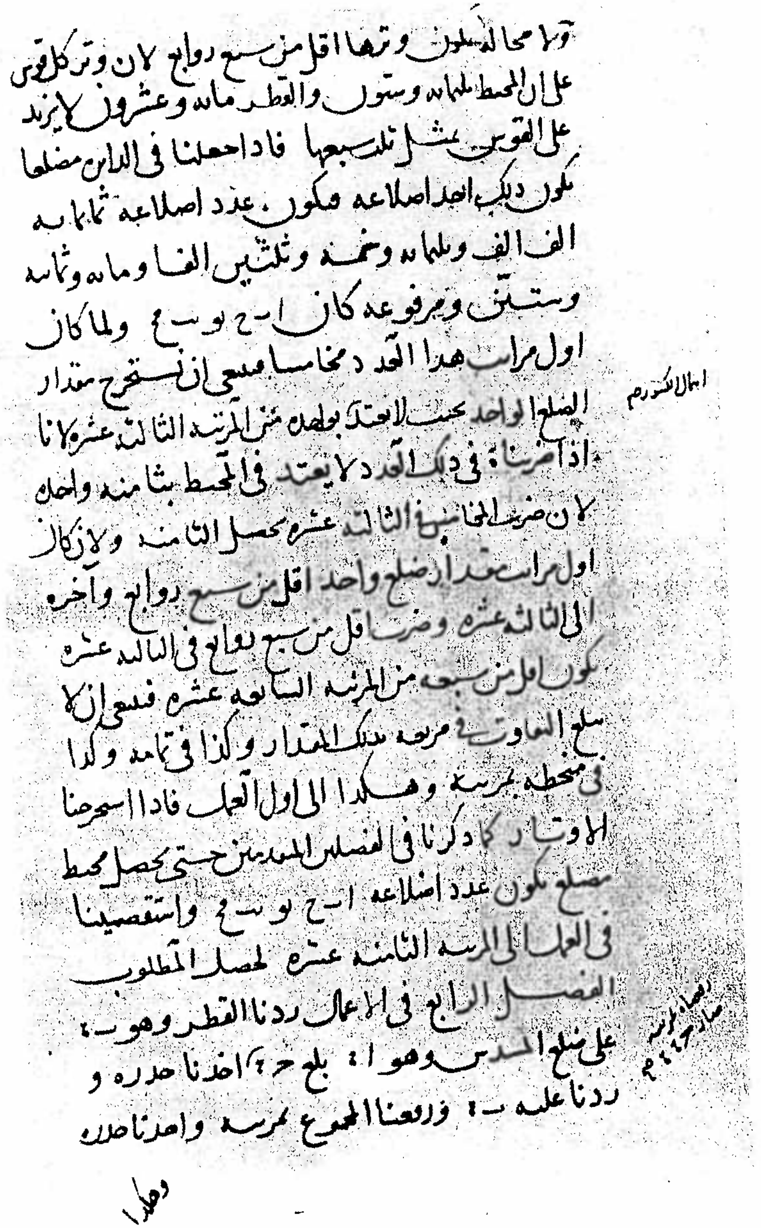
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 12


بسم الله الرحمن الرحيم

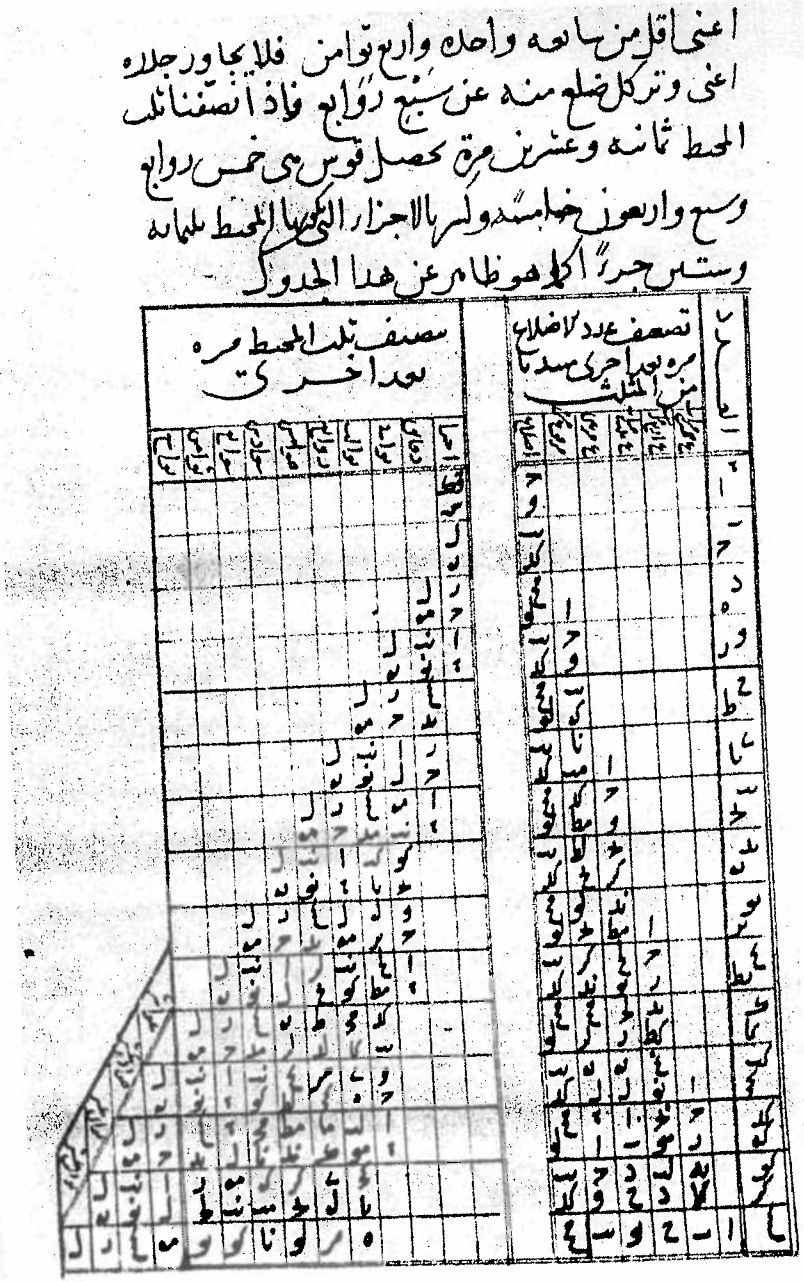
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 11


وهكذا عملنا ثمانية وعشرين عملا وما جاوزناها عن عمل
ثمانية الا بعد استيناف العمل مرتين ثلثة مع احتياط مراتب
العمل وضرب الجذر الحاصل في نفسه واستينافه مرتين
ثلثة وزيادة باقي العمل على الحاصل الى المجموع مساويا للعدد ان
صح العمل امتحانا ووثقنا لصحته لئلا وقع سهوا ويسرى
فيما بعده ولما كانت الارقام كثيرة لم نستعمل الارقام التي
لا نحتاج الى استعمالها وطريق استخراج الجذر وتحصيل
المربع عن الجذر بهذا الوجه مما استنبطناه واوردنا جداول
الاعمال في هذا الفصل ليكون دستورا للمحاسبين و
منهاجا لمن اراد الوقوف على صحته وجداول الاعمال
هذه

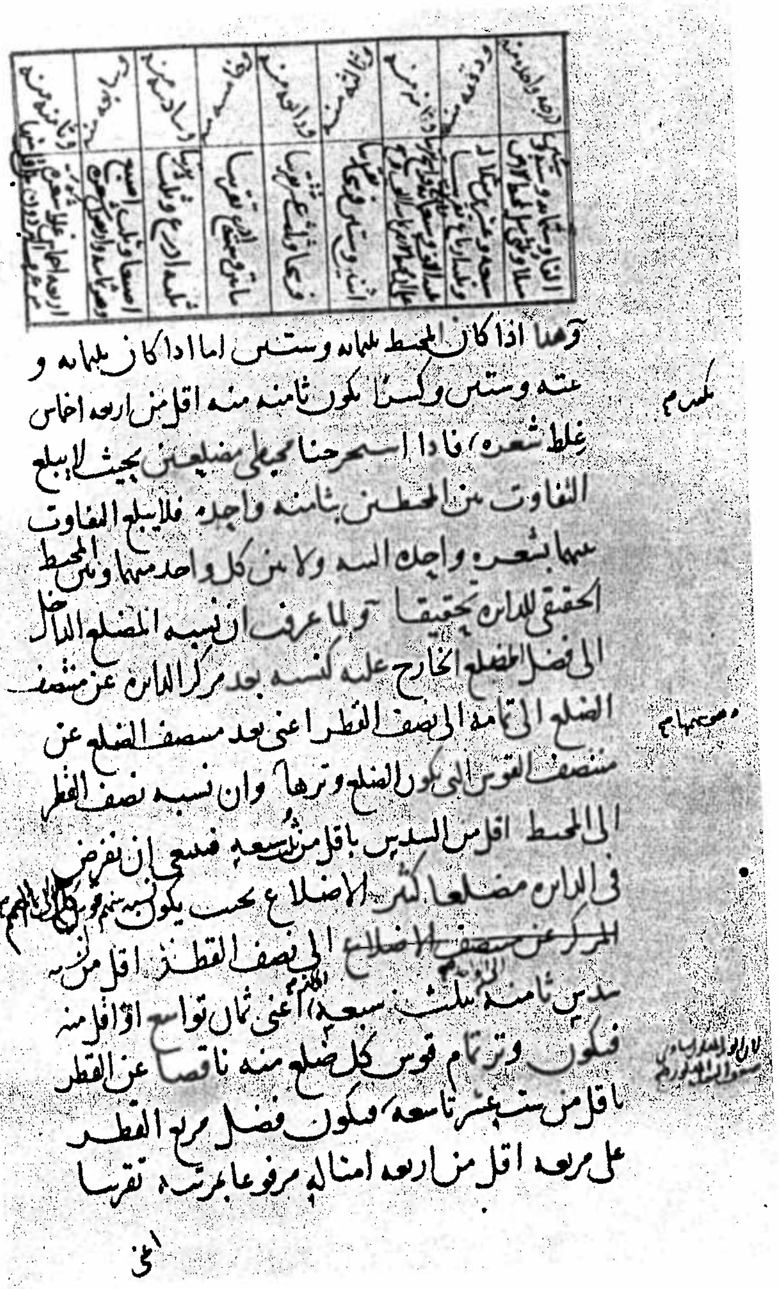
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 10


تولا محاله للمضلون وترها اقل من سبع روابع لان وتر كل قوس
على ان المحيط ثلثمائه وستون والقطر مائه وعشرون لا يزيد
على القوس بمثل ثلث سبعها فاذا جعلنا في الدائرة مضلعا
يكون ذلك احد اضلاعه فيكون عدد اضلاعه ثمانمائه
الف الف وثلثمائه وثمنية وثلثين الفا ومائه وثمانيه
وستين ومرفوعه كان اح يو سح ولما كان
اول مراتب هذا العدد مخامسا فينبغي ان نستخرج مقدار
الضلع الواحد بحيث لا يتعدى بواحد من المرتبة الثالثة عشرة لانا
اذا ضربنا في ذلك العدد لا يتعدى في المحيط بثامنة واحدة
لان ضرب الخامسة في الثالثة عشرة يحصل الثامنة ولان كان
اول مراتب مقدار ضلع واحد اقل من سبع روابع وآخره
الى الثالثة عشرة وضربنا اقل من سبع روابع في الثالثة عشرة
يكون اقل من سبعة من المرتبة السابعة عشرة فينبغي ان لا
يبلغ التفاوت في مربعه تلك المقدار وكذا في تامه وكذا
في منحطه بمرتبة وهكذا الى اول العمل فاذا استخرجنا
الاوتار كما ذكرنا في الفصل المتقدم حتى يحصل محيط
مضلع يكون عدد اضلاعه اح يو سح واستقصينا
في العمل الى المرتبة الثامنة عشرة لحصل المطلوب
الفصل الرابع في الاعمال زدنا القطر وهو ب
على ضلع المسدس وهو ا بلغ ح اخذنا جذره و
زدنا عليه ب وربعنا المجموع بمرتبة واخذنا جذره
وهكذا

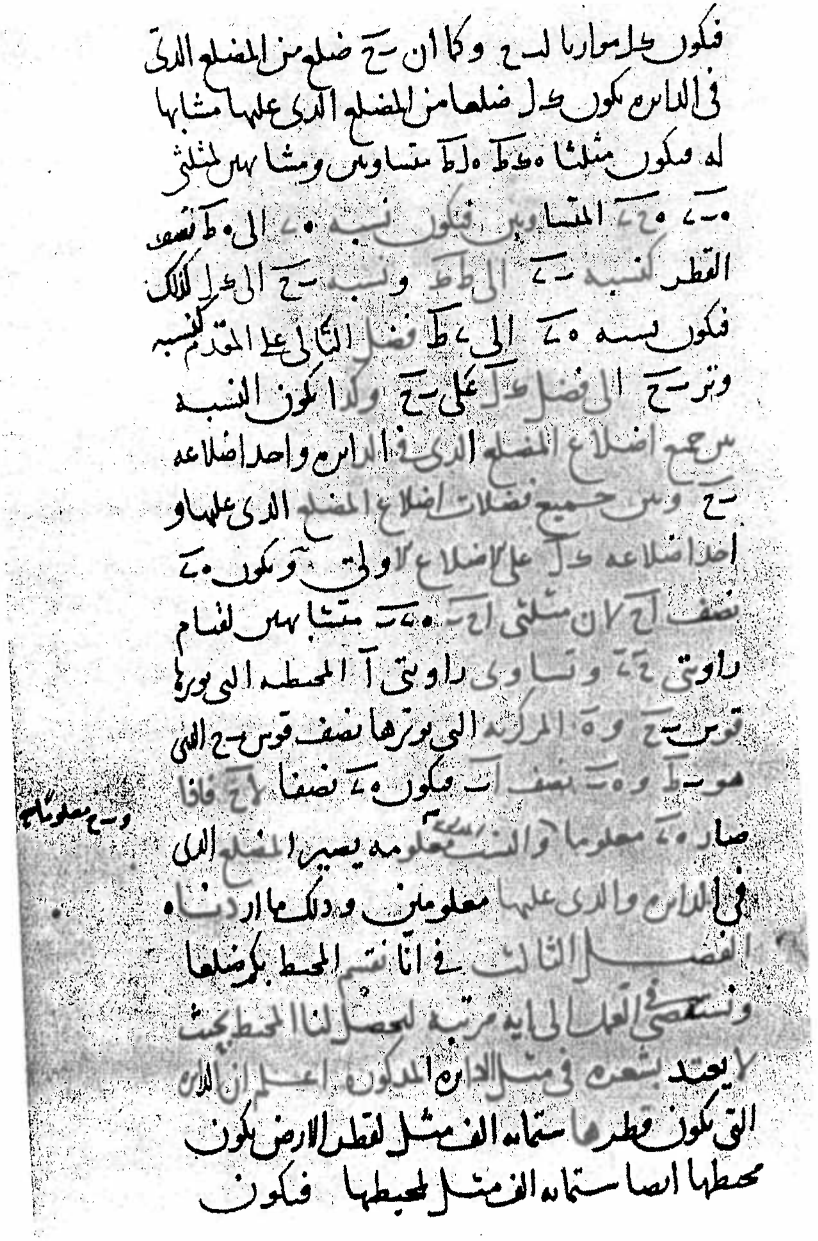
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 9


اعني اقل من سابعه واحد واربع ثوامن فلا يجاوز جذره
اعني وتر كل ضلع منه عن سبع روابع فاذا نصفنا ثلث
المحيط ثمانيه وعشرين مرة يحصل قوس هي خمس روابع
وسبع واربعون خامسة ولنرها بالاجزاء التي بها المحيط ثلثمائه
وستين جزءًا كما هو ظاهر عن هذا الجدول

Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 8

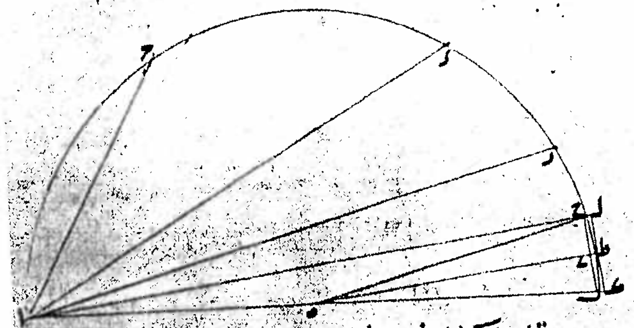

وهنا اذا كان المحيط ثلثمائة وستين اما اذا كان ثلثمائة و
عشرة وستين وكسرا يكون ثامنه منه اقل من اربعة اخماس
غلط شعرة فاذا استخرجنا محيطي مضلعين بحيث لا يبلغ
التفاوت بين المحيطين ثامنة واحدة فلا يبلغ التفاوت
بينهما بشعرة واحدة البتة ولا بين كل واحد منهما وبين المحيط
الحقيقي للدائرة تحقيقا ولما عرفنا ان نسبة المضلع الداخل
الى فضل المضلع الخارج عليه كنسبة بعد مركز الدائرة عن منتصف
الضلع الى تمامه الى نصف القطر اعني بعد منتصف الضلع عن
منتصف القوس التي يكون الضلع وترها وان نسبة نصف القطر
الى المحيط اقل من السدس باقل من تسعه فيجب ان نفرض
في الدائرة مضلعا كثير الاضلاع بحيث يكون نسبة
~~المركز عن منتصف الاضلاع~~ الى نصف القطر اقل من نسبة
سدس ثامنة ثلث سبعة اعني ثمان تواسع او اقل منه
فيكون وتر تمام قوس كل ضلع منه ناقصا عن القطر
باقل من ست عشر تاسعة فيكون فضل مربع القطر
على مربعه اقل من اربعة امثاله مرفوعا بمرتبة تقريبا

Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 7


فيكون كل موازيا لـ ح وكما ان ح ضلع من المضلع الذي
في الدائرة يكون كل ضلعا من المضلع الذي عليها متشابها
له فيكون مثلثا ه ط ه ل ط متساويين ومتشابهين لمثلثي
ح ه ح المتساويين فيكون نسبة ه الى ه ط نصف
القطر كنسبة ح الى ط ط ونسبة ح الى كل لذلك
فيكون نسبة ه الى ط فضل التالي على المقدم كنسبة
وتر ح الى فضل كل على ح وكذا يكون النسبة
بين جميع اضلاع المضلع الذي في الدائرة واحد اضلاعه
ح وبين جميع فضلات اضلاع المضلع الذي عليها و
احد اضلاعه كل على الاضلاع الاولى ويكون ه
نصف آح لان مثلثي اح متشابهين لتمام
زاويتي وتساوي زاويتي آ المحيطة التي يوترها
قوس ح و المركزية التي يوترها نصف قوس ح التي
هو نصف فيكون ه نصف ح فاذا
صار ه معلوما والنسبة معلومة يصير المضلع الذي
في الدائرة والذي عليها معلومين وذلك ما اردناه
الفصل الثالث في انا نقسم المحيط بكم ضلعا
ونستقصي العمل الى اية مرتبة ليحصل لنا المحيط بحيث
لا يبعد بشعرة في مثل الدائرة الذي يكون اعظم من الدائرة
التي يكون قطرها ستمائة الف مثل لقطر الارض يكون
محيطها ايضا ستمائة الف مثل لمحيطها فيكون

Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 6

في الدائرة ومحيط المضلع الذي عليها المشابه له نرسم
على قطر اب نصف دائرة اجب على مركز ه ونفرض
اج سدس المحيط فيكون وتر اج مساويا لاه نصف
القطر بالشكل الخامس عشر من رابعة الاصول ثم ننصف



جب تمام اج الى نصف الدور على د ودب على ر و
رب على ح وهكذا الى حيث شئنا فعلى ما ذكرنا في الفصل
المتقدم يصير من اج اد معلوما ومنه يصير ار
معلوما ويصير منه اح معلوما وهكذا الى حيث اردنا
فاذا حصل لنا اح مثلا ونريد معرفة وتر بح
ننقص مربع اح عن مربع القطر يبقى مربع وتر بح لان
زاوية احب قائمة بالشكل الثلثين من ثالثة الاصول فيكون
مربع اب مساويا لمربعي اح حب بالشكل العروس ثم
ننصف قوس بح على ط ونصل ه ط فينصف الوتر
على ي ونجعل ه ك مماسا للدائرة على ط ثم نخرج من نقطة ط
على ك ه ونصل ه ح ونخرجه الى ل وكذا ه ب الى ع
فلو

عمود على ... ه ولا م

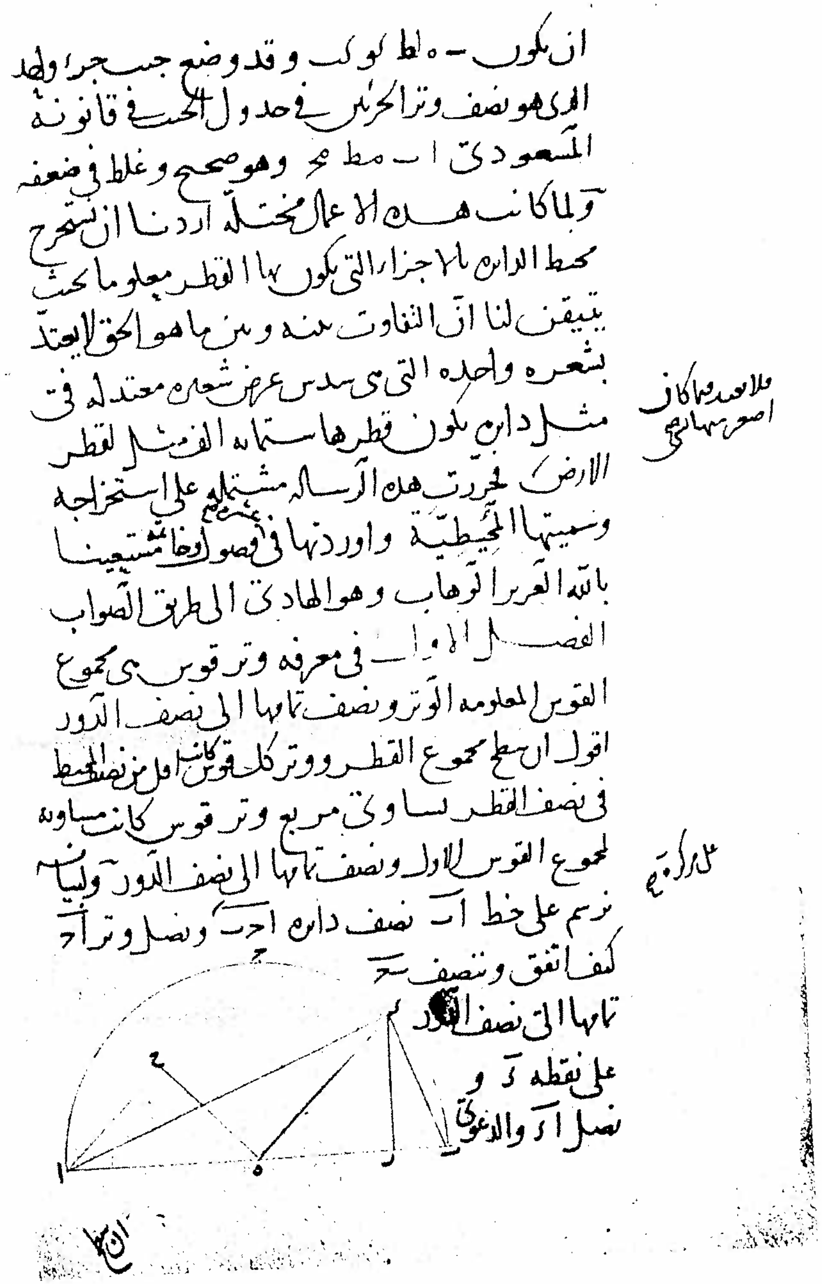
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 5

ان سطح نصف القطر في مجموع اب اح يساوي مربع
اد برهانه نصل بد فيكون زاوية ادب قائمة
بالشكل الثلثين من ثالثة الاصول ثم نخرج عن نقطة د عمود
دز على خط اب فيحدث مثلثا دبز داز متشابهين و
متشابهين لمثلث ادب بالشكل الثامن من سادسة الاصول
فيكون نسبة اب القطر الى اد كنسبة اد الى از فبالشكل
التاسع عشر من سابعة الاصول يكون سطح اب القطر في از
يساوي مربع اد ثم نخرج من نقطة ه عمود ه ح على
اح فيكون نقطة ح منتصف اح بالشكل الثالث من ثالثة
الاصول ونصل ه د ~~وهو موازي لاح~~ لان زاوية ب اح المحيطة
بعد نصف قوس حب وهو مقدار زاوية ب ه د فالزاويتان
متساويتان فيكون مثلثا اح ه ه ز د متساويين لتمام
زاويتي ح ز وتساوي زاويتي ه ا وضلعي ا ه ه د
فيكون ضلع ه ز مساويا لضلع اح الذي هو نصف اح
وكان سطح از اعني مجموع نصف القطر و ه ز بل نصف اح
في القطر مساويا لمربع اد فيكون سطح مجموع القطر و
ضعف ه ز اعني مجموع القطر و اح في نصف القطر يساوي
مربع اد وذلك ما اردناه فاذا كان اح معلوما بالاجزاء
التي يكون بها نصف القطر ستين ونزيد عليه القطر ونجعل
المجموع مرفوعا بمرتبة يكون الحاصل مربع اد

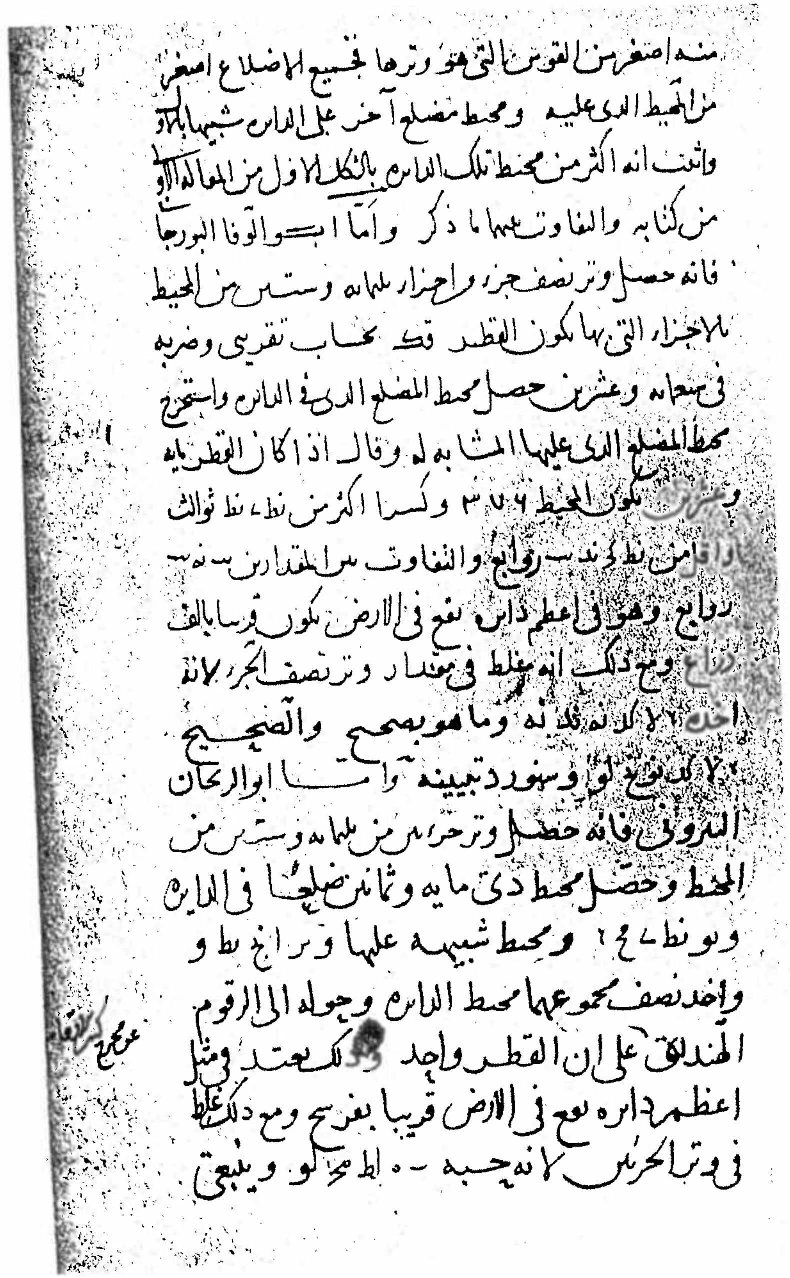
الفصل الثاني في معرفة محيط اي مضلع يكون

... نصف ... والاصلاع ...

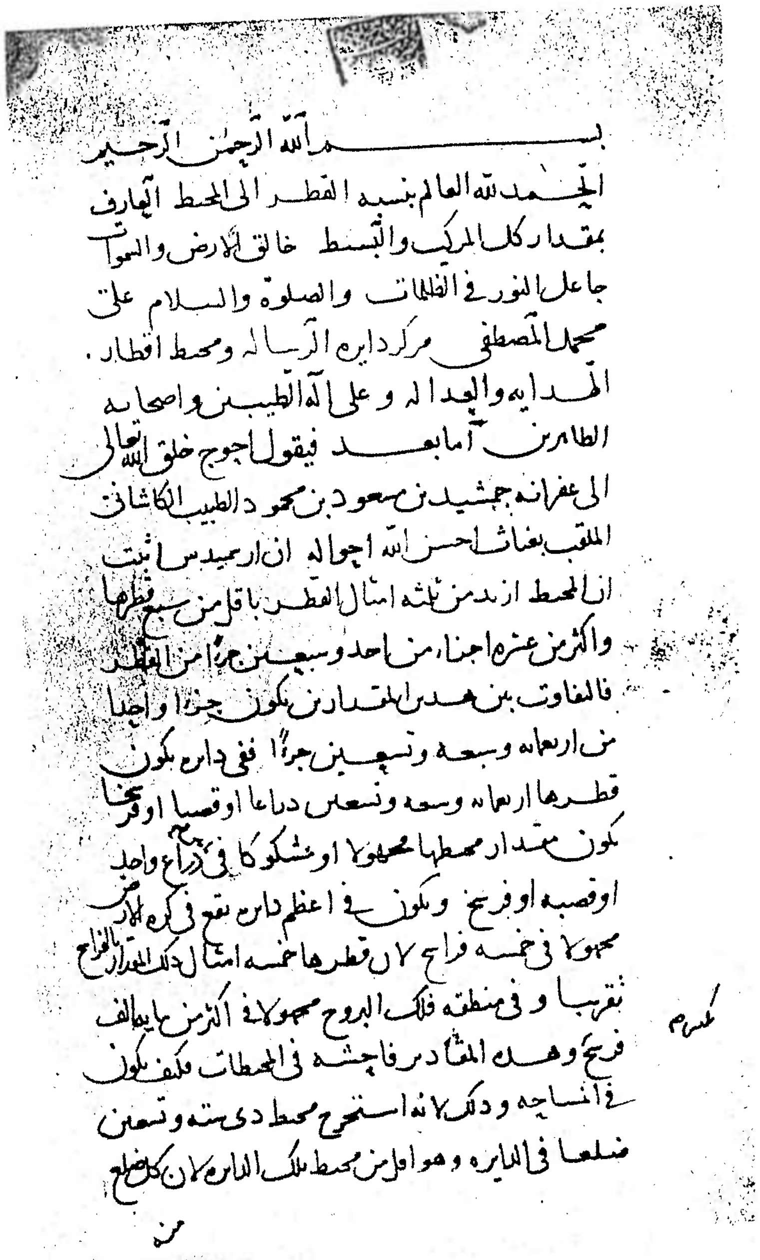
Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 4



Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 3



Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 2



بسم الله الرحمن الرحيم
الحمد لله العالم بنسبة القطر الى المحيط العارف
بمقدار كل المركب والبسيط خالق الارض والسموات
جاعل النور في الظلمات والصلوة والسلام على
محمد المصطفى مركز دائرة الرسالة ومحيط اقطار
الهداية والعدالة وعلى آله الطيبين واصحابه
الطاهرين اما بعد فيقول احوج خلق الله تعالى
الى غفرانه جمشيد بن مسعود بن محمود الطبيب الكاشي
الملقب بغياث احسن الله احواله ان ارشميدس بيّن
ان المحيط ازيد من ثلثة امثال القطر باقل من سبع قطرها
واكثر من عشرة اجزاء من احد وسبعين جزءا من القطر
فالتفاوت بين هذين المقدارين يكون جزءا واحدا
من اربعمائة وسبعة وتسعين جزءا ففي دائرة يكون
قطرها اربعمائة وسبعة وتسعين ذراعا او قصبا او فرسخا
يكون مقدار محيطها مجهولا او مشكوكا في ذراع واحد
او قصبة او فرسخ ويكون في اعظم دائرة تقع في كرة الارض
مجهولا في خمسة فراسخ لان قطرها خمسة امثال ذلك المقدار بالفراسخ
تقريبا وفي منطقة فلك البروج مجهولا في اكثر من مائتي الف
فرسخ وهذه المقادير فاحشة في المحيطات فكيف يكون
في المساحة وذلك لانه استخرج محيط ذي ستة وتسعين
ضلعا في الدائرة وهو اقل من محيط تلك الدائرة لان كل ضلع
منه

Al-Kāshī, *Treatise on the Circumference*,
Ms. Meshed, Holy Shrine Library, 5389, p. 1

ABSTRACT

This paper is concerned with the determination of to 16 decimals in the *Risāla Muḥīṭiyya* (Treatise on the Circumference) by Jamshšīd al-Kāshī (died A.H. 832/1429 CE). The Arabic text is presented in a facsimile edition of the manuscript Meshed, Holy Shrine Library no. 5389, which was once owned by Bahā' al-Dīn al-'Āmilī (A.H. 953-1031/1547-1622 CE). The facsimile edition includes all computations, including a series of 23 square-root extractions to 18 sexagesimal digits which have not been published before. In the introduction, the text by al-Kāshī is summarized and his computation is compared with a similar one in the Dutch work *Vanden Circkel* of Ludolph van Ceulen (1540-1610).

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