Part 1

Overview

The first five lectures give an overview of the course: We formulate the main result, the Geometrization Conjecture; we show how it implies the Poincaré Conjecture; we then introduce the Ricci flow and Ricci flow with surgery and give an indication of how these are used to prove the Geometrization Conjecture.

Lecture 1

In this course, we focus on **compact** and **orientable** 3-manifolds. We ask the following fundamental questions:

What do all 3-manifolds look like? Can we classify them?

Consider the case of closed orientable surfaces. They are characterized by the genus. The case $g = 0$ is the 2-sphere. One can equip it with the round metric. The

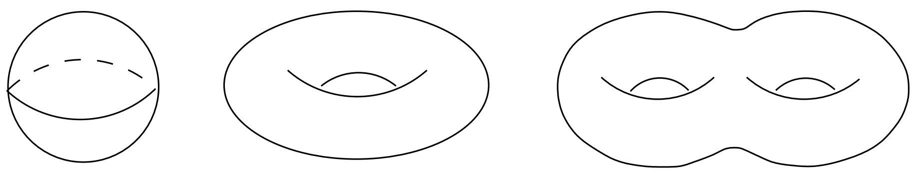

FIGURE 1. Surfaces with genus 0, 1 and 2

case $g = 1$ corresponds to the two dimensional torus $\cong \mathbb{R}^2/\Gamma$, where Γ is a lattice subgroup of $\mathbb{R}^2$. It admits a naturally induced flat metric from $\mathbb{R}^2$. For $g \geq 2$ we can equip the surface Σ_g of genus g with the hyperbolic metric induced from the Poincaré disk model of $\mathbb{H}^2$; that is to say there is a discrete, torsion-free subgroup Γ_g of the isometries of $\mathbb{H}^2$ with $\Sigma_g \cong \mathbb{H}^2/\Gamma_g$.

Geometric manifolds

DEFINITION (Homogeneous metric). *Let (M, g) be a Riemannian manifold. The metric g is called **homogeneous** if the action $\mathrm{Isom}(M) \times M \to M$ is transitive.*

DEFINITION (Locally homogeneous metric). *A Riemannian metric g on a manifold M is called **locally homogeneous** if its lifted metric $\tilde{g}$ on the universal cover $\tilde{M}$ is homogeneous.*

The round metric of the 2-sphere (i.e. $g = 0$) is homogeneous, but the hyperbolic metric of the genus-2 Riemann surface is not. However, the latter is locally homogeneous.

DEFINITION (Geometric manifolds). *A manifold is called **geometric** if it admits a finite volume complete locally homogeneous Riemannian metric.*

Here is the list of Geometric 3-manifolds by type (see [**19**]):

(1) Round: S^3 and its finite quotients – lens spaces, dodecahedron spaces – classification was completed by Hopf.
(2) Flat: T^3 and its finite quotients – these are completely classified.

(3) Hyperbolic: $\mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is hyperbolic 3-space and Γ is a torsion-free lattice group acting cocompactly on $\mathbb{H}^3$ - these are not classified.
(4) Round×$\mathbb{R}$: $S^2 \times S^1$ and $\mathbb{RP}^3 \sharp \mathbb{RP}^3$.
(5) Hyperbolic×$\mathbb{R}$: $\Sigma_g \times S^1$ and manifolds finitely covered by these – these are completely classified.
(6) Nil(3): Quotients of the Heisenberg group with a left-invariant metric by cocompact subgroups – these are completely classified and each is finitely covered by a non-trivial circle bundle over the two-torus. The Heisenberg group is:

$$\left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

(7) Solv(3): The Solv(3) group is $= \mathbb{R}^2 \rtimes \mathbb{R}^+$ with a left-invariant metric where $t \in \mathbb{R}^+$ acts by $\mathrm{diag}(t, t^{-1})$ on $\mathbb{R}^2$: its finite volume quotients are finitely covered by T^2-bundle over S^1 with Anasov monodromy.
(8) $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$: Finite volume quotients are finitely covered by circle bundles over hyperbolic surfaces – these are completely classified in terms of 2-dimensional orbifolds.

Remark: It is known that there is no noncompact examples of geometric 3-manifolds in the Round, Flat, Nil(3), Solv(3) and Round×$\mathbb{R}$ cases.

Also, any noncompact geometric 3-manifold is the interior of a compact 3-manifold M with boundary ∂, where $\partial = \coprod T^2$ where the tori T^2 are incompressible, i.e. $\iota_* : \pi_1(T^2) \hookrightarrow \pi_1(M)$ is injective.

Thurston manifolds

Definition (Thurston manifolds). *A* ***Thurston manifold*** *is one constructed as follows:*
Let Γ be a finite connected graph. Each vertex $v \in Vert(\Gamma)$ is in correspondence with a compact manifold M_v^3, where ∂M_v^3 is a disjoint union of incompressible tori and int M_v^3 is either geometric or a twisted I-bundle over the Klein bottle. To each vertex v, the edge set $E(v)$ is in bijection with boundary components of M_v^3; the boundary component associated with $e \in E(v)$ is denoted $\partial_e M_v$. Each edge e connecting vertices v_1 and v_2 is associated to an orientation-reversing diffeomorphism α_e of $\partial_e M_{v_1}^3$ and $\partial_e M_{v_2}^3$.

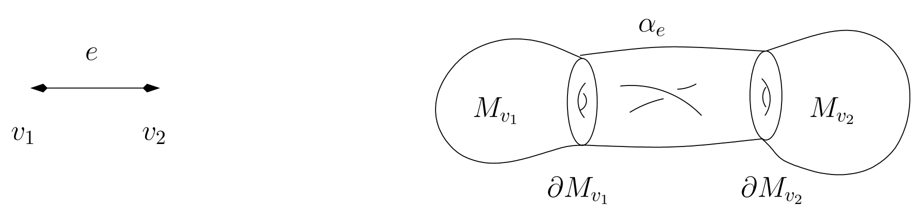

Figure 2. an example of a Thurston manifold

Note: This decomposition is unique if we do not allow α_e to match S^1-fibers up to isotropy.

LEMMA. *M^3 is Thurston if and only if there exists $\mathcal{J} \subset M^3$ where $\mathcal{J} = \coprod_i T_i^2 \coprod_j K_j^2$, each T_i^2 and K_j^2 is incompressible and such that each component of $M^3 - \mathcal{J}$ is geometric (K^2 is the Klein bottle).*

DEFINITION (Prime manifold). *A 3-manifold P^3 is called **prime** if it is not S^3 and every separating $S^2 \subset P^3$ (i.e $P^3 - S^2$ is not connected) bounds a 3-ball. In other words, if $P^3 = M_1 \sharp M_2$, then exactly one of M_1 or M_2 is diffeomorphic to S^3.*

Notice $S^2 \times S^1$ is prime.

It is interesting to note that the only Thurston manifold which is not prime is $\mathbb{RP}^3 \sharp \mathbb{RP}^3$.

THEOREM (Prime decomposition). *Every 3-manifold M^3 is a connected sum of prime 3-manifolds. The decomposition is unique up to the order of factors.*

For a proof see [**12**].

The theorems

Here is the *main theorem* of our course:

CONJECTURE (Thurston's Geometrization Conjecture). *Every closed, orientable 3-manifold is a connected sum of Thurston manifolds, or equivalently every prime closed 3-manifold admits a disjoint union $\mathcal{J}$ of incompressible tori and Klein bottles so that every connected component of the complement is geometric.*

One can easily see the Geometrization Conjecture implies the Poincaré Conjecture:

CONJECTURE (Poincaré Conjecture). *Every closed, orientable and simply connected 3-manifold is homeomorphic to S^3.*

PROOF OF GEOMETRIZATION CONJECTURE ⇒ POINCARÉ CONJECTURE: Suppose M^3 is a closed orientable 3-manifold with $\pi_1(M^3) = 1$. Prime decomposition asserts that if M^3 is not homeomorphic to S^3, then $M^3 = P_1 \sharp \cdots \sharp P_N$ where P_i are all prime. It follows that $1 = \pi_1(M^3) = \pi_1(P_1) * \cdots * \pi_1(P_N)$, hence $\pi_1(P_i) = 1$ for every i. By the Geometrization Conjecture, each P_i is a connected sum of Thurston manifolds. As P_i is prime, P_i is itself Thurston. Thus, P_i contains $\mathcal{J}$ a disjoint union of incompressible tori and Klein bottles such that $P_i - \mathcal{J}$ is geometric. Since P_i is simply-connected, such incompressible tori or Klein bottles cannot exist. Therefore P_i is geometric. By the classification of geometric 3-manifolds, the only compact simply-connected geometric 3-manifold is S^3, so $P_i \cong S^3$ for every i. This proves the Poincaré Conjecture. □

One way to recognize a round metric is to look at the sectional curvature:

LEMMA. *If M^3 admits a metric of constant sectional curvature $+1$, then the universal covering of M^3 is isometric to S^3 and in particular M^3 is geometric.*

PROOF. Here we give the sketch of proof. For more details see [**24**]. By a result of Riemann, such M^3 is locally isometric to S^3. We write $M^3 = \bigcup_i U_i$ where each U_i is connected and isometric to an open subset of S^3. Lift the local isometries to the universal cover $\widetilde{M^3}$. We claim the local isometries can be made to glue to a global isometry. Consider two overlapping open sets $\widetilde{U_1}, \widetilde{U_2} \subset \widetilde{M^3}$ which are locally isometric to S^3, i.e. there exist isometries $\phi_i \colon \widetilde{U_i} \to V_i, (i = 1, 2)$, where

$V_i, (i = 1, 2)$, are open subsets of S^3. The isometries ϕ_i's may not agree on the overlap $\widetilde{U_1} \cap \widetilde{U_2}$. However, by the homogeneousness of the round 3-sphere, one can find an isometry $\Phi\colon S^3 \to S^3$ such that ϕ_1 and $\Phi \circ \phi_2$ agree on $\widetilde{U_1} \cap \widetilde{U_2}$ (See the figure below). We replace ϕ_2 by $\Phi \circ \phi_2$. By composing a suitable isometry of S^3 to each local isometries between $\widetilde{M^3}$ and S^3, the local isometries can be made to agree on their overlap, so they glue to a global map from $\widetilde{M^3}$ and S^3 which is a local isometry and hence is an immersion. This means the map from $\widetilde{M}$ to S^3 is a covering projection. Since $\widetilde{M}$ is connected and S^3 is simply-connected, this covering projection is a diffeomorphism, but a local isometry that is also a diffeomorphism is a global isometry. This proves that $\widetilde{M}$ with the induced metric is isometric to S^3. Therefore, M^3 is a spherical space-form. □

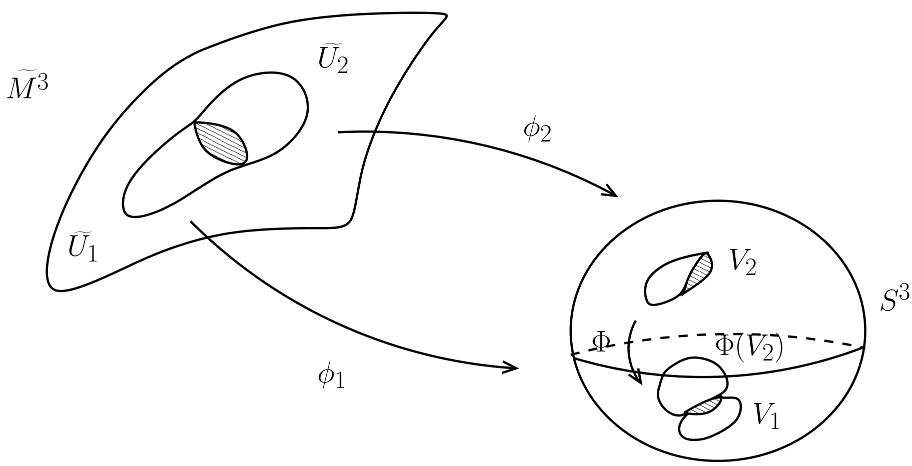

FIGURE 3. gluing local isometries to a global isometry

The idea behind using Ricci flow to prove the Geometrization Conjecture is that, starting with any Riemannian metric, the Ricci flow should smooth out the curvature and hence in the limit the curvature will become evenly distributed and the resulting metric is then homogeneous.

Lecture 2

Lectures 2 – 5 give an outline of the proof of the Geometrization Conjecture using the Ricci flow which was first introduced by Hamilton, see [**5**].

Our approach to these results is through studying evolving one-parameter families of Riemannian metrics on a given 3-manifold.

Basics of Riemannian geometry

We first review some basic Riemannian geometry and define the notations of curvatures for the course.

Let (M, g) be a Riemannian manifold, there exists a unique torsion-free and metric connection ∇ (call the **Levi-civita connection**). Let $(x^1, \cdots, x^n)$ be local coordinates and $\{\partial_1, \cdots, \partial_n\}$ be the local basis of TM given by $\partial_i = \frac{\partial}{\partial x^i}$. The connection ∇ is determined by the Christoffel's symbols Γ_{ij}^k defined by:

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k.$$

Note that the torsion-free condition is equivalent to saying $\Gamma_{ij}^k = \Gamma_{ji}^k$ for any i, j, k. Using the metric compatibility of ∇, one can derive a local expression of Γ_{ij}^k in terms of g_{ij} and its first derivatives. To carry this out, one can consider

$$\partial_i \langle \partial_j, \partial_k \rangle_g = \langle \nabla_{\partial_i}\partial_j, \partial_k \rangle_g + \langle \partial_j, \nabla_{\partial_i}\partial_k \rangle_g,$$

and so we have (with, as always, the Einstein summation convention)

$$\partial_i g_{jk} = \Gamma_{ij}^l g_{lk} + \Gamma_{ik}^l g_{jl}.$$

By cyclic permuting indices i, j, k one can easily derive the following local expression:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

Next we define the Riemann curvature:

Denote $\mathrm{Rm}_{ij} \doteq \nabla_{\partial_i} \circ \nabla_{\partial_j} - \nabla_{\partial_j} \circ \nabla_{\partial_i}$. Then the Riemann curvature tensor is defined as

$$R_{ijkl} = \langle \mathrm{Rm}_{ij}(\partial_l), \partial_k \rangle_g .$$

One can show that R_{ijkl} is skew-symmetric in (ij) and (kl), i.e. $R_{ijkl} = -R_{jikl} = -R_{ijlk}$, and satisfies $R_{ijkl} = R_{klij}$. Therefore, the Riemann curvature tensor can be viewed as a symmetric bilinear form on $\wedge^2 TM$.

In dimension 3, every orthonormal basis for $\wedge^2 TM$ is of the form $\{e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2\}$ where $\{e_1, e_2, e_3\}$ is an orthonormal basis of TM.

Regard the Riemann curvature tensor as a symmetric bilinear form on $\wedge^2 TM$. For any $x \in M$, one can find orthonormal basis $\{e_1, e_2, e_3\}$ for $T_x M$ such that Rm_x

is diagonalized with respect to basis $\{e_2 \wedge e_3,\, e_3 \wedge e_1, e_1 \wedge e_2\}$:

$$\mathrm{Rm}_x = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \text{ where } \lambda \le \mu \le \nu.$$

Now we define the Ricci curvature tensor as $R_{ik} = g^{jl} R_{jilk}$. Under the above orthonormal basis $\{e_1, e_2, e_3\}$ on $T_x M$, the Ricci tensor is also diagonalized:

$$\mathrm{Ric}_x = \begin{bmatrix} \mu+\nu & 0 & 0 \\ 0 & \lambda+\nu & 0 \\ 0 & 0 & \lambda+\mu \end{bmatrix}$$

Thus in dimension 3, Rm and Ric can be simultaneously diagonalized and Ric determines Rm. That makes Ricci flow particularly useful in three dimension.

Basics of Ricci flow

Now we turn to the partial differential equation that generates the one-parameter families of metrics we study.

As defined by Hamilton in [**5**], the **Ricci flow equation** is:

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}(g(t)).$$

A smooth one-parameter family of metrics $g(t)$ satisfying the Ricci flow equation is called a Ricci flow. We will mostly deal with initial value problems of Ricci flow, that is we require $g(0) = g_0$ where g_0 is a given Riemannian metric on M.

Hamilton proved the following existence and uniqueness theorem on Ricci flow:

THEOREM (Hamilton ([**5**]). *If M^n is compact and g_0 is C^∞, then there exists $T_{\max} \in (0, +\infty]$ depending on g_0 and a Ricci flow solution $g(t)$, defined for $t \in [0, T_{\max})$, such that $g(0) = g_0$. Moreover, any solution $\tilde{g}(t)$ defined on $t \in [0, T)$ with $\tilde{g}(0) = g_0$ must have $T \le T_{\max}$ and $\tilde{g}(t) = g(t)$ for any $t \in [0, T)$.*

Remark on existence and uniqueness:

The Ricci flow equation is NOT strictly parabolic because of diffeomorphism invariance of the Ricci tensor, so the short-time existence and uniqueness do not follow from standard parabolic theory. DeTurck in [**4**] simplified Hamilton's original proof of short-time existence and uniqueness by modifying the flow by modding out its diffeomorphism invariance. The modified flow is strictly parabolic so its short-time existence follows from standard theory. Then the solution of the Ricci flow is recovered by pulling-back the solution of the modified flow. This trick is now well-known as DeTurck's trick.

Here are some examples of Ricci flow solutions:

(1) (M^n, g_0) is Einstein, i.e. $\mathrm{Ric}(g_0) = \lambda_0 g_0$, where $\lambda_0 \in \mathbb{R}$.
Let

$$g(t) = \lambda(t) g_0.$$

Assuming it satisfies the Ricci flow equation, we will have

$$-2\mathrm{Ric}(g(t)) = \frac{\partial g(t)}{\partial t} = \lambda'(t) g_0.$$

As the Ricci tensor is scale-invariant, i.e. $\mathrm{Ric}(\lambda(t) g_0) = \mathrm{Ric}(g_0)$ so we have

$$-2\lambda_0 g_0 = -2\mathrm{Ric}(g_0) = \lambda'(t) g_0.$$

Solving for $\lambda(t)$, we get

$$g(t) = (1 - 2\lambda_0 t)g_0.$$

Note that the solution becomes singular at $t = 1/2\lambda_0$.

Therefore, under the Ricci flow, the sphere (S^n, g_{round}) equipped with the round metric such that $\text{Ric} = g_{\text{round}}$ will shrink homothetically along the flow and become singular at $t = 1/2$. In contrast, the surface of genus $g \geq 2$ equipped with hyperbolic metric with $\text{Ric} = -g$ will inflate forever, and the Ricci flow solution exists for all $t > 0$. Obviously, the flat metric will be stationary along the flow.

(2) Let us consider $(S^2, h_0)) \times \mathbb{R}$ with product metric $g_0 = h_0 \times ds^2$, where h_0 is the round metric of S^2 such that $\text{Ric}(h_0) = \lambda_0 h_0$. As the Ricci tensor preserves the product structure, so does the Ricci flow solution. Therefore, the Ricci flow solution with this initial metric g_0 is given by

$$g(t) = (1 - 2\lambda_0 t)h_0 \times ds^2.$$

In a nutshell, the flow will deform the S^2-cylinder by shrinking the radius of the S^2 and after some finite time, the manifold converges to an infinite line.

The above are examples of Ricci flow solutions that can be written down explicitly. In a more general setting, Hamilton proved in 1982 the following result on closed 3-manifolds with positive Ricci curvature:

Theorem (Hamilton ([**5**]). *Suppose (M^3, g_0) is a closed 3-manifold with positive Ricci curvature, then $T_{\max} < \infty$ and the following holds:*

(1) $\lim_{t\to T_{\max}} \text{diam}(M, g(t)) = 0$
(2) $\lim_{t\to T_{\max}} \dfrac{\textit{maximum sectional curvature of } g(t)}{\textit{minimum sectional curvature of } g(t)} = 1$
(3) *The rescaled metric $\text{Rm}_{\min}(t)g(t)$ converges smoothly to a limiting round metric.*

The above theorem therefore asserts that any 3-manifold with positive Ricci curvature behaves asymptotically along the Ricci flow like a round manifold. In particular, every such 3-manifold admits a round metric, hence topologically it is S^3/Γ where Γ is an isometry group acting freely on S^3.

Hamilton also classified the case for nonnegative Ricci curvature:

Theorem (Hamilton, [**6**]). *Again (M^3, g_0) is a closed 3-manifold, but now we suppose $\text{Ric}(g_0) \geq 0$, the one of the following holds:*

(1) $\text{Ric}(g(t)) > 0$ *for any $t > 0$, then the previous theorem applies and so M^3 is a spherical space-form.*
(2) $\text{Ric}(g_0) \equiv 0$, *or equivalently M^3 is flat.*
(3) $\text{Ric}(g(t))$ *never becomes strictly positive nor identically zero, then locally (M^3, g_0) splits into $(\Sigma^2, h) \times \mathbb{R}$ where (Σ^2, h) is a surface with positive curvature.*

The main ingredient of the proof of the above theorems is the maximum principle argument. To illustrate how to apply maximum principles to get estimates, let us consider the evolution equations of various curvature quantities:

We define the scalar curvature $R \doteq \mathrm{tr}(\mathrm{Ric})$. In dimension 3, $R = 2(\lambda + \mu + \nu)$ where λ, μ, ν are eigenvalues of Rm. Hamilton derived the evolution equation of Rm is of the form

$$\frac{\partial}{\partial t}\mathrm{Rm} = \Delta \mathrm{Rm} + Q(\mathrm{Rm}),$$

where $Q(\mathrm{Rm})$ is a quadratic expression of components of Rm.

The evolution equation of the scalar curvature R in dimension n is given by

$$\frac{\partial}{\partial t}R = \Delta R + 2|\mathrm{Ric}^0|^2 + \frac{2}{n}R^2,$$

where Ric^0 is the traceless part of Ric, i.e. $\mathrm{Ric} - \frac{R}{n}g$.

Denote $R_{\min}(t)$ to be the minimum scalar curvature at time t, from above we have

$$\frac{d}{dt}R_{\min} \geq \frac{2}{n}R_{\min}(t)^2.$$

Here we have used the fact that the Laplacian is nonnegative at a minimum point.

Applying the scalar maximum principle, we get two consequences:

(1) $R_{\min}(t)$ is a nondecreasing function of t.
(2) If $R_{\min}(0) > 0$, $R_{\min}(t)$ blows-up in finite time.

We have just demonstrated the use of scalar maximum principle. In order to derive estimates on tensors like Rm and Ric, Hamilton used some more sophisticated maximum principle machinery, namely maximum principle on tensors which we will state explicitly later.

There are two invariances of the Ricci flow: rescaling and time shifting of a Ricci flow solution $g(t)$

(1) (Rescaling) If $g(t)$ is a Ricci flow solution, so is $h(t') = \lambda g(\lambda^{-1}t')$ for any constant $\lambda > 0$.
(2) (Time shifting) If $g(t)$ is a Ricci flow solution, then so is $g(t - t_0)$ for any fixed t_0.

We shall often work with Ricci flows with normalized initial conditions, meaning:

(1) $\mathrm{Rm}(x, 0)$ has eigenvalues between 1 and -1.
(2) $\mathrm{Vol}(B(x, 1)) \geq \frac{1}{2}\mathrm{Vol}B_{\mathbb{R}^3}(1)$ for every $x \in M$.

Provided the manifold is compact, by rescaling of the metric, one can normalize its initial conditions.

While the Ricci flow produces important one-parameter families of metrics, these are not sufficient for the topological applications. We shall need more general families, called Ricci flows with surgery.

The key to constructing these and understanding them is the notion of:

Canonical Neighborhoods

Our next goal is to describe, at a somewhat qualitative level, Ricci flow with surgery. In order to do this, we must first introduce canonical neighborhoods. It turns out these neighborhoods, introduced in [**15**], have many special topological and geometric properties that we make use of in both defining Ricci flow with surgery and proving the existence of these for all positive time. We shall not give all the structure of these neighborhoods at once, but rather introduce more and more structure as it is needed. Thus, as we proceed in the argument we will refine what we mean by a canonical neighborhood. Here is the first definition:

First we fix an $\epsilon > 0$, there are essentially 3 types of ϵ-canonical neighborhoods:

(1) (ϵ-neck) - a neighborhood $N_\epsilon \subset M^3$ diffeomorphic to $S^2 \times (-\epsilon^{-1}, \epsilon^{-1})$ under diffeomorphism $\varphi \colon S^2 \times (-\epsilon^{-1}, \epsilon^{-1}) \to N_\epsilon$, such that the rescaled pull-back metric $R(x,t)\varphi^* g(t)$ on $S^2 \times (-\epsilon^{-1}, \epsilon^{-1})$ is within ϵ in $C^{[\epsilon^{-1}]}$-topology to the product of the round metric on S^2 with $R = 1$ with the usual metric on $(-\epsilon^{-1}, \epsilon^{-1})$.
(2) (ϵ-cap) - topologically B^3 or a punctured real projective 3-space $\mathbb{RP}^3_0$, and whose end is a ϵ-neck.
(3) connected component of positive sectional curvature.

A point $x \in M$ is said to **have an ϵ-canonical neighborhood** if it lies in the central two-sphere of an ϵ-neck, lies in an ϵ-cap in the complement of the ϵ-neck forming the end of the cap, or lies in a component of positive sectional curvature.

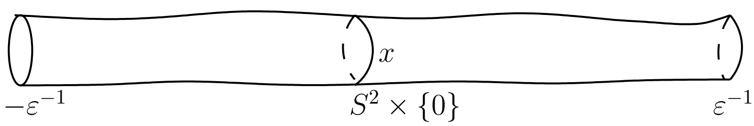

FIGURE 4. an ϵ-neck

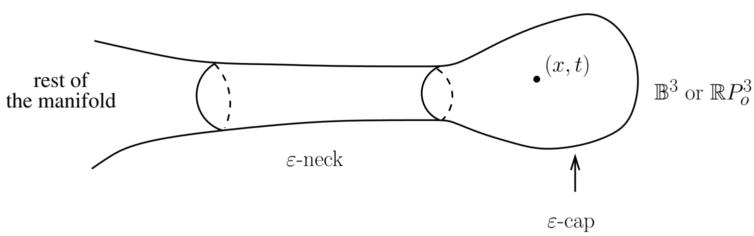

FIGURE 5. an ϵ-cap connected to an ϵ-neck

The following results by Perelman and Hamilton illustrate the importance of all these canonical neighborhoods:

THEOREM (Perelman, [**15**]). *Fix $\epsilon > 0$, then there exists a non-increasing function $r(t) > 0$ with $\lim_{t\to\infty} r(t) = 0$, such that for any normalized Ricci flow solution $(M^3, g(t))$, $t \in [0,T)$, any point (x,t) with $R(x,t) \geq r^{-2}(t)$ has an ϵ-canonical neighborhood.*

THEOREM (Hamilton, [**5**]). *If $(M^3, g(t))$ is a Ricci flow solution with $T_{\max} < \infty$, then*

$$\limsup_{t\to T_{\max}^-} R_{\max}(t) = \infty.$$

It follows from the results that all finite-time singularities are contained in regions covered by ϵ-canonical neighborhoods.

Lecture 3

More on Canonical Neighborhoods

In the last lecture we defined three types of canonical neighborhoods, namely ϵ-necks, ϵ-caps and connected components of positive curvature and we stated a result to the effect that the regions in a 3-manifold where finite-time singularities develop are covered by ϵ-canonical neighborhoods. Here is a theorem that characterizes subsets covered by canonical neighborhoods:

THEOREM. *Fix $\epsilon > 0$ sufficiently small. Suppose $X \subset M^3$, X is connected and every $x \in X$ has a ϵ-canonical neighborhood. Then one of the following holds:*

(1) *X is contained in a connected component of positive curvature.*
(2) *X is contained in an ϵ-tube or a circular ϵ-tube. An ϵ-tube is a submanifold diffeomorphic to $S^2 \times (a, b)$, is a union of ϵ-necks and the S^2-factors in the ϵ-necks separate the ends of the tube. A circular ϵ-tube is an S^2-bundle over S^1, is a union of ϵ-necks and the S^2-factors in the ϵ-necks are homotopic to the fibers of the fibration structure.*
(3) *X is contained in a capped or doubly-capped ϵ-tube, which is the union of an ϵ-tube with an ϵ-cap attached to one or both ends.*

In fact, this result extends to non-connected subsets $X \subset M$.

COROLLARY. *Fix $\epsilon > 0$. Suppose every $x \in X$ has a ϵ-canonical neighborhood. Then X is contained in a disjoint union of subsets as in 1, 2, and 3 in the previous theorem.*

This gives us topological control over regions containing the finite-time singularities, but more delicate geometric and analytic properties of these neighborhoods (which we have not yet introduced) are also crucial.

Here is one of the main results we shall need to define surgery over Ricci flow.

THEOREM (Perelman, [**15**]). *Assume M^3 is compact and $(M^3, g(t))$ is a Ricci flow defined on $0 \leq t < T_{\max} < \infty$ and fix $\epsilon > 0$ sufficiently small. We define a subset $\Omega \subset M^3$ by*

$$\Omega = \{x \in M^3 : \liminf_{t \to T_{\max}^-} R(x, t) < \infty\}.$$

Then,

(1) *Ω is an open set, which may be empty.*
(2) *$g(t)|_\Omega$ converges smoothly, uniformly on compact subsets, to a limiting metric $g(T_{\max})$ on Ω, and the scalar curvature $R_{T_{\max}} : \Omega \to \mathbb{R}$ of $g(T_{\max})$ is a proper function and is bounded below.*
(3) *For any connected component Ω^0 of Ω, every end of Ω^0 is an ϵ-horn, i.e. a neighborhood of the end is diffeomorphic to $S^2 \times (a, b)$ and this neighborhood is a union of ϵ-tubes.*

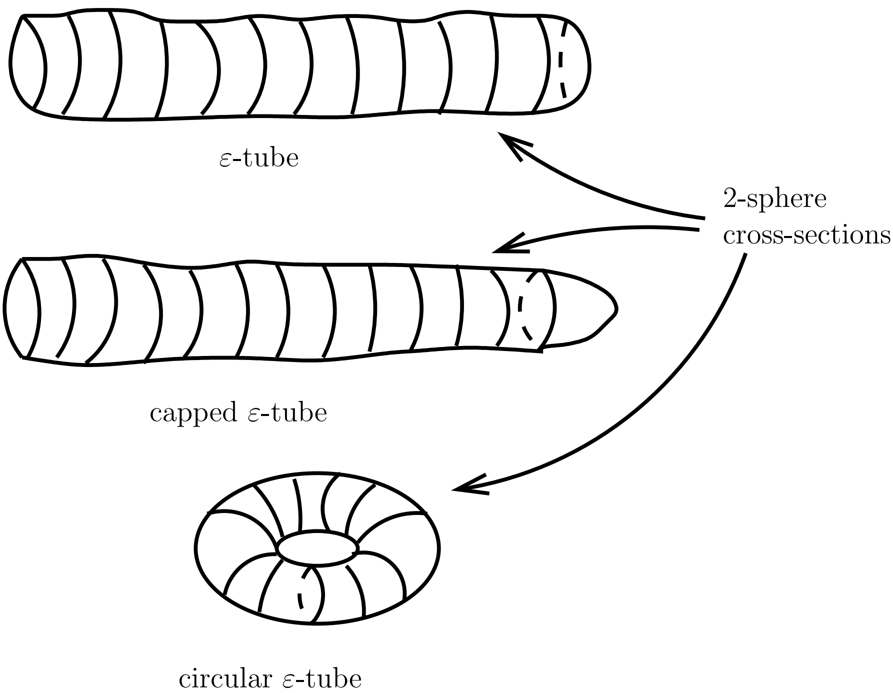

FIGURE 6. ϵ-tubes

(4) *There is a compact subset $K \subset \Omega$ such that for every t sufficiently close to $T_{\max}$, $M - K$ is covered by a finite disjoint union of ϵ-canonical neighborhoods in $(M, g(t))$.*

Remark: We do not know in general if Ω has finitely many components or not.

Surgery on Ricci flow

All of this control on the regions of high curvature and the limiting object at the singular time allows us to define a Ricci flow with surgery. We start with a closed Riemannian 3-manifold (M^3, g_0) and run the Ricci flow $g(t)$. Suppose for this flow $T_{\max} < \infty$. Fix $\rho \in (0, r(T_{\max}))$ where $r(t)$ is the canonical neighborhood threshold function. Define $\Omega(\rho) \subset \Omega$ as:

$$\Omega(\rho) \doteq \{x \in \Omega : R(x, T_{\max}) \leq \rho^{-2}\}.$$

We perform the following operations:

(1) Remove $M \setminus \Omega = R_{T_{\max}}^{-1}(\infty)$ from M
(2) Remove all connected components of Ω that do not meet $\Omega(\rho)$.
(3) In each ϵ-horn end of each component of Ω^0 of Ω meeting $\Omega(\rho)$ we fix a central 2-sphere of an ϵ-neck in the ϵ-horn and remove the part of the ϵ-horn outside that 2-sphere.

Removing this open set of M leaves a compact 3-manifold M_0 with ∂M_0 being a disjoint union of 2-spheres. M_0 has the limiting metric $g(T_{\max})$. We cap off the manifold by gluing the interior of a 3-ball to the ends of the remaining manifold using the partition of unity.

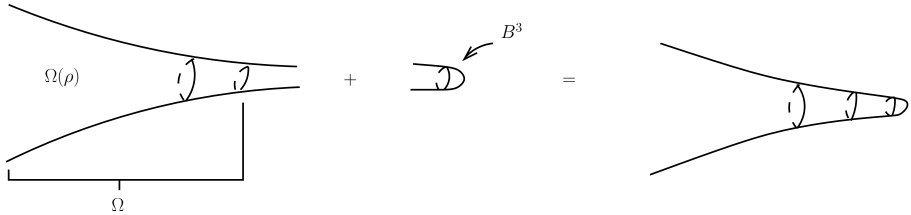

FIGURE 7. gluing a cap

(The metric on the B^3 is invariant under SO(3) and is isometric to S^2 times an interval near the boundary.)

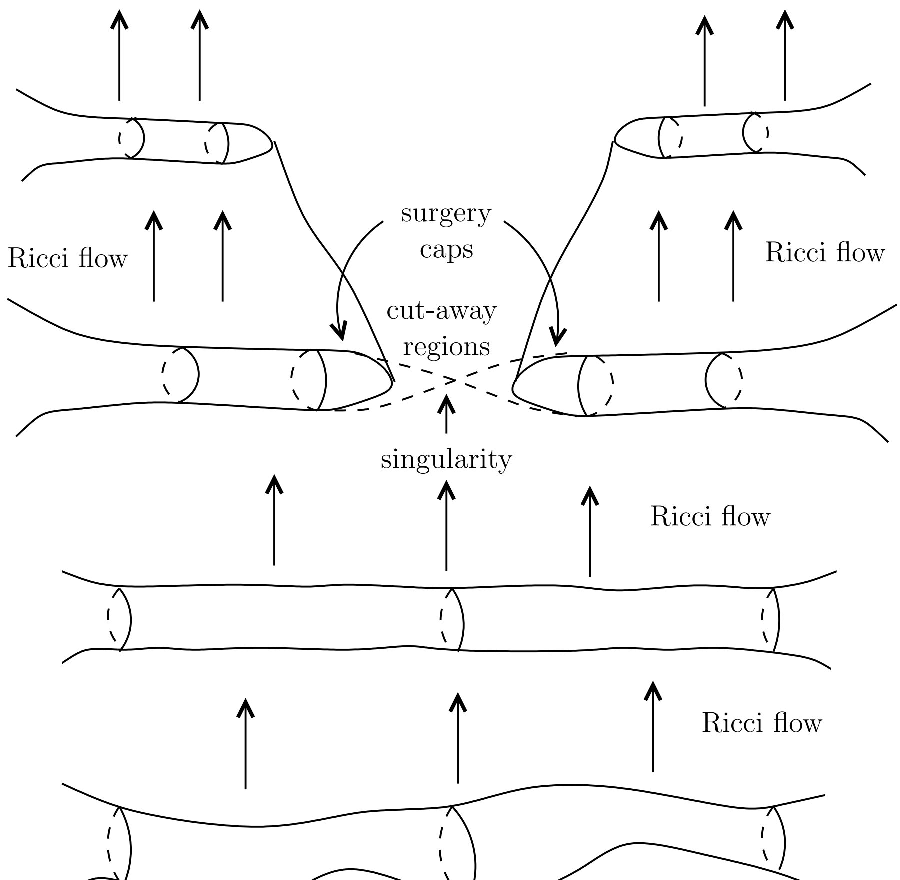

FIGURE 8. Ricci flow with surgery

Then we get a new compact manifold, call it M_1, with a new metric $g_1(T_{\max})$. Restart the Ricci flow on this manifold until singularity occurs again and repeat the process. In doing so, we have constructed a sequence of manifolds $(M_i, g_i(t)), t \in [t_i, t_{i+1})$ where $0 = t_0 < t_1 < t_2 < \cdots$, where M_{i+1} is obtained by the aforesaid removal of parts from M_i and gluing of 3-balls. We call these t_i's the *surgery times*.

Here is a theorem about the nature of surgery times under Ricci flow with surgery:

THEOREM. *The surgery times* $0 = t_0 < t_1 < t_2 < \cdots$ *satisfies either one of the following conditions*

(1) $t_k = \infty$ *for some* k*; or*
(2) $\{t_i\}$ *form a discrete subset of* $[0, \infty)$.

The main ingredient of the proof of the above theorem is a volume estimate. Under the Ricci flow, the volume evolves according to the following equation:

$$\frac{d}{dt}\mathrm{Vol}(M, g(t)) = -\int_M R(x,t)dV_{g(t)},$$

and so $\frac{d}{dt}\mathrm{Vol}(t) \leq -R_{\min}(t)\mathrm{Vol}(t)$.

Since the evolution equation of R in dimension 3 is $\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}^0|^2 + \frac{2}{3}R^2$, a maximum principle argument shows that $R_{\min}(t)$ is a nondecreasing function. After normalizing the initial metric one can assume $1 \geq \mathrm{Rm}(x, 0) \geq -1$ for any $x \in M$. Hence, $R_{\min}(t) \geq -6$ for any $t \in [0, T_{\max})$. The inequality is preserved under each surgery because only pieces with nonnegative scalar curvature are added to the manifold. Therefore, the volume satisfies

$$\mathrm{Vol}(t) \leq \mathrm{Vol}(0)e^{6t}.$$

In each 2-sphere surgery we remove half of an ϵ-tube (and more) and glue in a 3-disk. Direct examination shows that for $\epsilon > 0$ sufficiently small we remove at least $R(x, T_{\max})^{-3/2} \times \frac{\epsilon^{-1}}{2}$ where $(x, T_{\max})$ is a point in the S^2 that we cut along. Since $\epsilon > 0$ is fixed, as long as we have an upper bound on every finite time interval to the scalar curvature or equivalently a positive lower bound to the diameter of these 2-spheres we cut along, it follows that we can only do finitely many surgeries in each finite time interval. It turns out we can do surgeries with these bounds though we must allow them to decay as time goes to infinity.

Topological effects of surgery

Here is the crucial result for using Ricci flow with surgery to obtain topological results.

THEOREM. *Suppose* M_1 *is obtained from* M_0 *by a surgery on a Ricci flow. Then topologically* M_1 *is obtained from* M_0 *by:*

(1) *connected-sum decompositions, and*
(2) *removal of connected components with semi-positive locally homogeneous metric*

COROLLARY. *If* M_1 *satisfies the Geometrization Conjecture, then so in* M_0.

We shall prove this theorem in the next lecture.

Lecture 4

More structure (geometric and analytic) of Canonical Neighborhoods

Recall there are three types of canonical neighborhoods, namely ϵ-tubes, ϵ-caps and connected components of positive curvature. While we have already stated some of the properties of ϵ-neighborhoods, there are more. By definition, the scalar curvature and the diameter of the canonical neighborhoods satisfy the following estimates.

There exists $C = C(\epsilon) > 0$ such that:

(1) diameter of the ϵ-canonical neighborhood $\leq CR(y,t)^{-1/2}$ for any y in the neighborhood.
(2) $\frac{R(y,t)}{R(x,t)} \leq C$ for any x, y in the canonical neighborhood.
(3) $|\frac{\partial R}{\partial t}(x,t)| \leq CR^2(x,t)$ at every (x,t) in the canonical neighborhood.
(4) $|\nabla R(x,t)| \leq CR^{3/2}(x,t)$ at every (x,t) in the canonical neighborhood.

These estimates can be easily verified for the ϵ-necks. The other cases follow from compactness results. Using the last two estimates, one can show the fact we claimed last time that $\Omega = \{x \in M : \liminf_{t \to T_{\max}} R(x,t) < \infty\}$ is an open subset of M.

Proof of openness of Ω*:*

Pick up $x \in \Omega$, there exists $K < \infty$ and $t_n \to T_{\max}$ such that $R(x,t_n) < K$ for every n.

Take n sufficiently large such that $T_{\max} - \delta < t_n < T_{\max}$ where δ is to be chosen later. Using $|\nabla R(x,t)| \leq CR^{3/2}(x,t)$, one can find $r = r(K) > 0$ such that $R \leq 2K$ on $B(x,t_n,r)$.

Using $|\frac{\partial R}{\partial t}| \leq CR^2$ one can choose δ small enough (depending on K) such that $R(y,t) \leq 4K$ for any $y \in B(x,t_n,r)$ and $t_n < t < T_{\max}$). Therefore, $B(x,t_n,r) \subset \Omega$ and so Ω is open. □

Now let us prove the theorem stated at the end of the previous lecture. Fix $0 < \rho < r/2$. Recall that at $T_{\max}$ we remove:

(1) X $= M - \Omega$, i.e. points $x \in M$ where $R(x,t) \to \infty$ as $t \to T_{\max}$;
(2) components of Ω disjoint from $\Omega(\rho)$;
(3) ends of ϵ-horns in the remaining components of Ω.

Let us denote $\mathcal{C}$ be the set of points we removed from M. For any point on $\mathcal{C}$, we have $R_{T_{\max}} > \rho^{-2}$. Using the third estimate for R, we can prove for $t' < T_{\max}$ but sufficiently close to $T_{\max}$ we have $R|_{\mathcal{C}\times t'} \geq \frac{1}{4}\rho^{-2}$. Thus by the theorem on canonical neighborhoods stated at the end of Lecture 2, every $(y,t') \in \mathcal{C} \times \{t'\}$ has a canonical neighborhood, and so there exists disjoint union of ϵ-tubes, capped ϵ-tubes and components of positive curvature in $(M, g(t'))$ containing $\mathcal{C} \times \{t'\}$. To finish the proof, we need a little more about the surgery 2-spheres:

In fact the surgery 2-spheres are the centers of δ-necks where $\delta \ll \min(\epsilon, C^{-1}(\epsilon))$. By the diameter estimate of ϵ-caps, the size of δ-neck is bigger than the diameter of an ϵ-cap, so the surgery 2-spheres all lie far from the caps. Therefore, all the surgery spheres lie in ϵ-tubes and are parallel in that ϵ-tube to the S^2-factor.

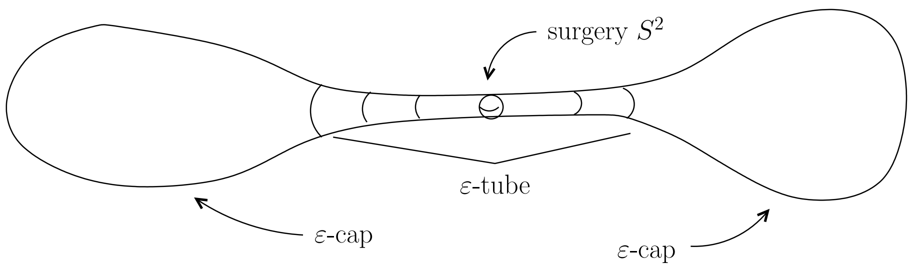

FIGURE 9. position of surgery 2-sphere

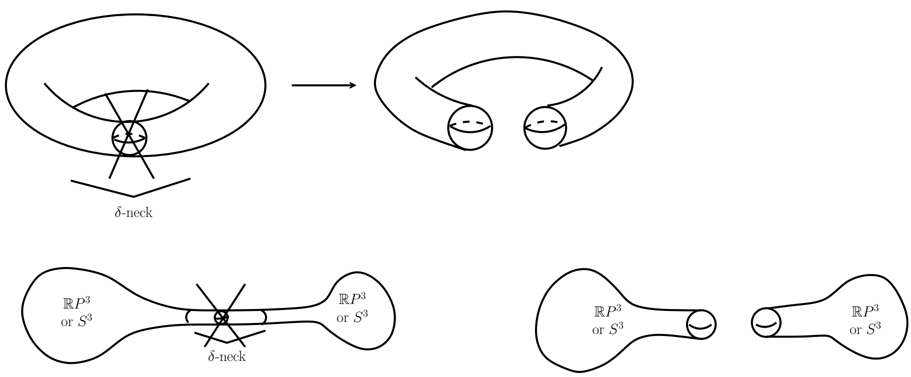

FIGURE 10. resulting manifolds after surgery

We cap off the boundary by 3-balls. Thus, the topological effect of surgery is a finite connected-sum decomposition (possibly trivial) followed by removal of connected components of positive curvature.

COROLLARY. *If the manifold after surgery satisfies the Geometrization Conjecture, so does the original manifold.*

Finite-time extinction

Inductively, after finitely many surgeries if the resulting manifold satisfies the Geometrization Conjecture, so does the starting manifold. One is concerned about whether the manifold could completely disappear after finitely many times of surgeries. The following theorem by Perelman answers this question affirmatively, assuming certain conditions on the fundamental group of M.

THEOREM (Perelman, [**17**]). *Suppose $\pi_1(M)$ is a free product of finite groups and infinite cyclic groups. Then the Ricci flow with surgery constructed with (M, g_0) as initial condition becomes* ***extinct*** *at a finite time. In other words, $M_n = \emptyset$ for some large n.*

COROLLARY. *Under the same assumption on $\pi_1(M)$ as in the above theorem, M is a finite connected-sum of manifolds with semi-positive, locally homogeneous metric g, i.e. connected-sum of spherical space-forms and $S^2 \times S^1$.*

It also follows from the arguments in [**17**] that:

THEOREM. *In general, given any (M^3, g_0) and $T_1 < \infty$, there exists $T_2 < \infty$ such that every component of the time-slice of Ricci flow with surgery beginning with (M^3, g_0) is either aspherical, or S^3 and if it is an S^3 then it is created by a trivial surgery at time $\geq T_1$.*

To complete the proof of the Geometrization Conjecture we must also consider the complementary case when the Ricci flow with surgery exist for all times. Clearly in this case we must study the limits as $t \to \infty$. That we discuss in the next lecture.

Lecture 5

In this lecture we complete the outline of the proof of the Geometrization Conjecture by indicating how one shows it holds for the large time-slices of a Ricci flow with surgery.

Let us first recall our notations for Ricci flow with surgery. Let (M^3, g_0) be a compact Riemannian 3-manifold, $0 = t_0 < t_1 < t_2 < \cdots$ are surgery times and for each n, $(M_n, g_n(t))$ is a Ricci flow for $t \in [t_n, t_{n+1})$ where M_{n+1} is made from M_n by surgery. Also, we will denote $(M_t, g(t))$ to be the t time-slice in the Ricci flow with surgery.

We mentioned the following two important results:

(1) If M_t satisfies the Geometrization Conjecture for some t, then so does $M_{t'}$ for any $t' < t$. In particular, so does $M = M_0$.

(2) For $t \gg 1$, every connected component of M_t is either an S^3 or aspherical.

For the rest of this lecture, we fix a Ricci flow with surgery $(M_t, g(t))$. To finish the proof of the Geometrization Conjecture, we need to understand the nature of $(M_t, g(t))$ for sufficiently large t. This involves the notion of geometric limits.

Geometric limits

Definition (Geometric limit). *(M_n, g_n, x_n) is said to be* ***converging to*** *$(M_\infty, g_\infty, x_\infty)$* ***geometrically*** *if for every compact $K \subset M_\infty$, $x_\infty \in K$, for all n sufficiently large, there are embeddings $\phi_n\colon (K, x_\infty) \to (M_n, x_n)$ such that the pull-back metric $\phi_n^* g_n$ converges to $g_\infty|_K$ in C^∞-topology.*

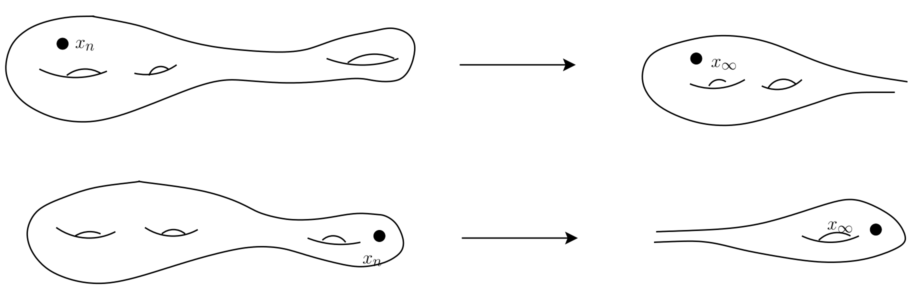

Figure 11. Geometric limit with respect to different marked points

Define the function $\rho\colon M_t \to (0, \infty)$ by setting $\rho(x,t)$ equal to the maximum number such that $\mathrm{Rm}|_{B(x,t,\rho(x,t))} \geq -\rho^{-2}(x,t)$. Fix $w > 0$, we define

$$M_{t,\mathrm{thick}}(w) = \{(x,t) \in M_t : \mathrm{Vol}(B(x,t,\rho(x,t))) \geq w\rho^3(x,t)\}.$$

The structure of $M_{t,\mathrm{thick}}(w)$ follows from the convergence theorem:

THEOREM (Hamilton [**6**]; Perelman [**16**]). *Let (x_n, t_n) be a sequence of points in $M_{t_n,\text{thick}}(w)$ with $t_n \to \infty$, then after passing to a subsequence, the rescaled sequence $(M_{t_n}, \frac{1}{t_n} g(t_n), (x_n, t_n))$ converges geometrically to $(\mathbb{H}, g_{\text{hyp}}, x)$ which is a complete, finite volume hyperbolic 3-manifold with constant curvature $-\frac{1}{4}$.*

A heuristic explanation of the constant $-\frac{1}{4}$ is by the evolution equation of the scalar curvature:

$$\frac{\partial R}{\partial t} = \Delta R + 2|\text{Ric}^0|^2 + \frac{2}{3}R^2.$$

Since hyperbolic manifolds have constant scalar curvature and are Einstein, we have $\frac{\partial R}{\partial t} = \frac{2}{3}R^2$. Solving the equation we have

$$R(t) = \frac{3}{\frac{3}{R(0)} - 2t}.$$

Note that $R(\frac{1}{t_n} g_n(t)) = t_n R(g_n(t))$, so the scalar curvature of the rescaled metric is asymptotically converging to $-\frac{3}{2}$ and thus on the limiting hyperbolic manifold, we have sectional curvature equals $-\frac{1}{4}$.

Hyperbolic limits

There is a stronger result that gives a more global picture around the non-collapsed parts:

THEOREM (Hamilton [**6**]; Perelman [**16**]). *There exists a finite set $\mathcal{H} = \coprod_{i=1}^{k} \mathbb{H}_i$ of complete, finite volume hyperbolic 3-manifolds with curvature $-\frac{1}{4}$ such that the following holds:*

Fix $w > 0$ sufficiently small, let $\overline{\mathcal{H}}(w/2)$ be the truncation of $\mathcal{H}$ at horospherical tori of area $\frac{w}{2}$. For each $t \gg 1$, there exists an embedding $\phi_t \colon \overline{\mathcal{H}}(w/2) \hookrightarrow (M_t, g(t))$ such that $\phi_t^*(\frac{1}{t} g(t))$ converges smoothly to $g_{\text{hyp}}|_{\overline{\mathcal{H}}(w/2)}$ and the image $\phi_t(\overline{\mathcal{H}}(w/2)) \supset M_{t,\text{thick}}(w)$.

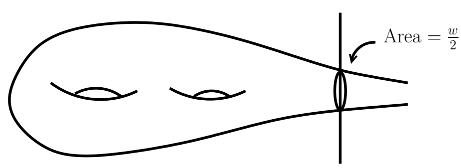

FIGURE 12. truncation on a hyperbolic space

Furthermore, the following result by Hamilton tells us the boundary tori are incompressible:

THEOREM (Hamilton [**6**]). *The image $\phi_t(\partial\overline{\mathcal{H}}(w/2))$ consists of incompressible tori for any t sufficiently large.*

The thin part

The non-collapsed part $M_{t,\text{thick}}(w)$ is pictorially the "thick" part of the manifold. For the "thin" part, we define

$$M_{t,\text{thin}}(w) = M_t - \phi_t(\text{Int}(\overline{\mathcal{H}}(w/2)).$$

By the previous theorems, we know that $M_{t,\text{thin}}(w)$ is a compact 3-manifold with boundary and $\partial M_{t,\text{thin}}(w)$ is a finite disjoint union of incompressible tori which are convex in the induced metric.

When restricted to this locally collapsed part, the function ρ defined before will satisfy $\text{Rm}|_{B(x,t,\rho(x,t))} \geq -\rho^{-2}(x,t)$ and $\text{Vol}(B(x,t,\rho(x,t))) < w\rho^3(x,t)$ for any $(x,t) \in M_{t,\text{thin}}(w)$. We say the part $M_{t,\text{thin}}(w)$ is locally volume collapsed on the negative curvature scale.

Take $w_n \to 0$. For each n take t_n sufficiently large depending on w_n and such that $t_n \to \infty$. Let $(N_n, g_n) = (M_{t_n,\text{thin}}(w_n), \frac{1}{t_n} g(t_n))$ be a sequence of compact manifolds, which are locally volume collapsed on negative curvature scale and which have convex boundary consisting of incompressible tori. Then for any sequence $x_n \in N_n$, we have

$$\frac{\text{Vol}(B(x_n, t_n, \rho(x_n, t_n)))}{\rho(x_n, t_n)^3} \to 0 \text{ as } n \to \infty$$

The following theorem tells us the structure of these N_n:

Theorem. *Given (N_n, g_n, w_n) as before, for all sufficiently large n, there exists a disjoint union of 2-tori in N_n, denoted $\mathcal{J}_n \subset N_n$, such that every complementary component is Seifert fibered.*

In fact, an easy topological argument (using the fact that each component of N_n is either S^3 or aspherical) allows us to modify $\mathcal{J}_n$ and show that there exists a disjoint union $\hat{\mathcal{J}}_n$ of incompressible tori cross I and twisted I-bundles over Klein bottles so that each component of $N_n - \hat{\mathcal{J}}_n$ has a geometric structure.

Alexandrov spaces

In order to prove the previous theorem we must study manifolds and more general metric spaces with curvature bounded below.

Here is a classical theorem that characterizes manifolds with curvature bounded below by -1:

Theorem. *(M, g) has sectional curvature ≥ -1 if and only if the following holds:*

Given $a, b, x \in M$, find $\tilde{a}, \tilde{b}, \tilde{x}$ in hyperbolic plane with the same pairwise distance, e.g. $d(a,x) = d(\tilde{a}, \tilde{x})$. Define $\tilde{\angle} axb = \angle \tilde{a}\tilde{x}\tilde{b}$, then for any four points $a, b, c, x \in M$ we have $\tilde{\angle} axb + \tilde{\angle} bxc + \tilde{\angle} cxa \leq 2\pi$.

We use this to motivate the following definition. An **Alexandrov space with curvature** ≥ -1 is a complete metric space X with the property that any two points are the endpoints of an isometric embedding of an interval into X (such spaces are **length spaces**) and such that the above inequality holds for all quadruples of points $\{x; a, b, c\}$ in X. There is an analogous definition for an Alexandrov space with curvature $\geq k$ for any k; one replaces the hyperbolic plane by the complete simply-connected surface of constant curvature k.

One defines the dimension of an Alexandrov space as its Hausdorff dimension. This turns out to be either an integer or $+\infty$. If it is an integer N then the Alexandrov space has an open dense set that is a topological N-manifold.

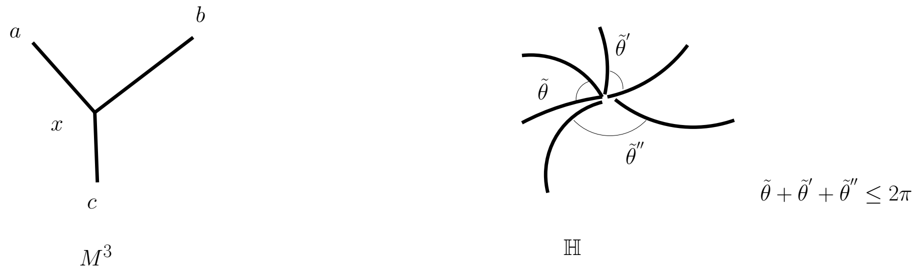

FIGURE 13. angle comparison

We have the following the convergence result for manifolds with curvature bounded below by -1:

LEMMA (Burago-Gromov-Perelman [**2**]). *If (M_n, g_n, x_n) is a sequence of complete N dimensional Riemannian manifolds with sectional curvature ≥ -1, then after passing to a subsequence they limit in the Gromov-Hausdorff topology to an Alexandrov space of curvature ≥ -1 with of dimension $\le N$.*

We would like to understand the locally collapsed part of the manifold as $t \to \infty$. We apply the above lemma to the sequence of rescaled balls $\frac{1}{\rho(x_n)} B(x_n, \rho(x_n))$ where $x_n \in N_n$. By our choice of ρ, the rescaled balls have curvature bounded below by -1. The limit of these balls is an Alexandrov space so understanding the locally collapsed part will follow from classifying all possible Alexandrov spaces that occur as limits.

The possible Gromov-Hausdorff limit of Alexandrov spaces must have dimension 0, 1, or 2 (dimension 3 is ruled out by the fact that the volumes of these balls are converging to 0).

In case when the limit is of dimension 0, the limit is a point, i.e. $\frac{\mathrm{diam}}{\rho_n} \to 0$. The rescaled balls $\sqrt{\frac{\rho_n}{\mathrm{diam}}} B_n$ therefore are complete and have diameter 1 and curvature $\ge -\frac{\mathrm{diam}}{\rho_n} \to 0$, so that the metric converges to a flat metric. Here, we use parabolic regularity to get smooth convergence of the metric.

In case when the limit is of dimension 1, possible limits are intervals $[a, b), (a, b)$ or $[a, b]$ and S^1. If the limit is either an open interval or the circle then the corresponding part of N_n is a fibration over this base with fiber either S^2 or T^2. In the case of endpoints we add in either a solid torus or a twisted I-bundle over the Klein bottle (when the generic fiber is a torus) or a 3-ball or a puncture $\mathbb{R}P^3$ (when the generic fiber is a two-sphere) for each endpoint.

In case when the limit is of dimension 2, the limit is a topological manifold which may contain some non-smooth parts like a cone. In this case the corresponding part of N_n is Seifert fibered. One can then glue these local models together in order to prove the theorem.

Summary of Part 1

This completes PART 1 of these notes - namely the overview of the argument. We have discussed the existence of Ricci flow with surgery defined for all $0 \leq t < \infty$ beginning with any (normalized) initial compact Riemannian 3-manifold. We have examined the topological effect of surgery (connected sum decomposition and removal of topologically standard components). As a consequence, we saw that if any time-slice in a Ricci flow with surgery satisfies the Geometrization Conjecture then so does the initial manifold. This reduces the proof of the Geometrization Conjecture to the study of the time-slices for sufficiently large t. We then outlined the fact that for initial conditions $(M^3, g(0))$ where $\pi_1(M^3)$ is a free product of finite groups and infinite cyclic groups, the Ricci flow with surgery becomes extinct in finite time. Consequently, these manifolds are connected sums of manifolds admitting positive and semi-positive metrics (i.e. those modeled on S^3 and $S^2 \times \mathbb{R}$).

Finally, we examined the nature of the t time-slice, for t sufficiently large, in general. We indicated that this time-slice decomposed along incompressible tori into thick and thin parts, and on the thick parts the metric of the Ricci flow is (after rescaling by $\frac{1}{t}$) becoming hyperbolic with constant curvature $-\frac{1}{4}$. Lastly, we examined the nature of the thin pieces and indicated that by metric/topological arguments one could show that they decompose and incompressible tori and Klein bottles into pieces admitting complete locally homogeneous metrics of finite volume. This then will complete the proof of the Geometrization Conjecture.

The rest of these lectures will examine in turn all of these issues, beginning with non-collapsing results which we examine in Part 2.