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Eine kleine Mathmusik 1
(2002)

Eine kleine Mathmusik 2
(2003)

For woodwind sextet

With explaining articles

Constructing Rhythmic Canons, *Perspectives of New Music*, 2003, 41(2), 66–94

Constructing Rhythmic Fugues (unpublished)

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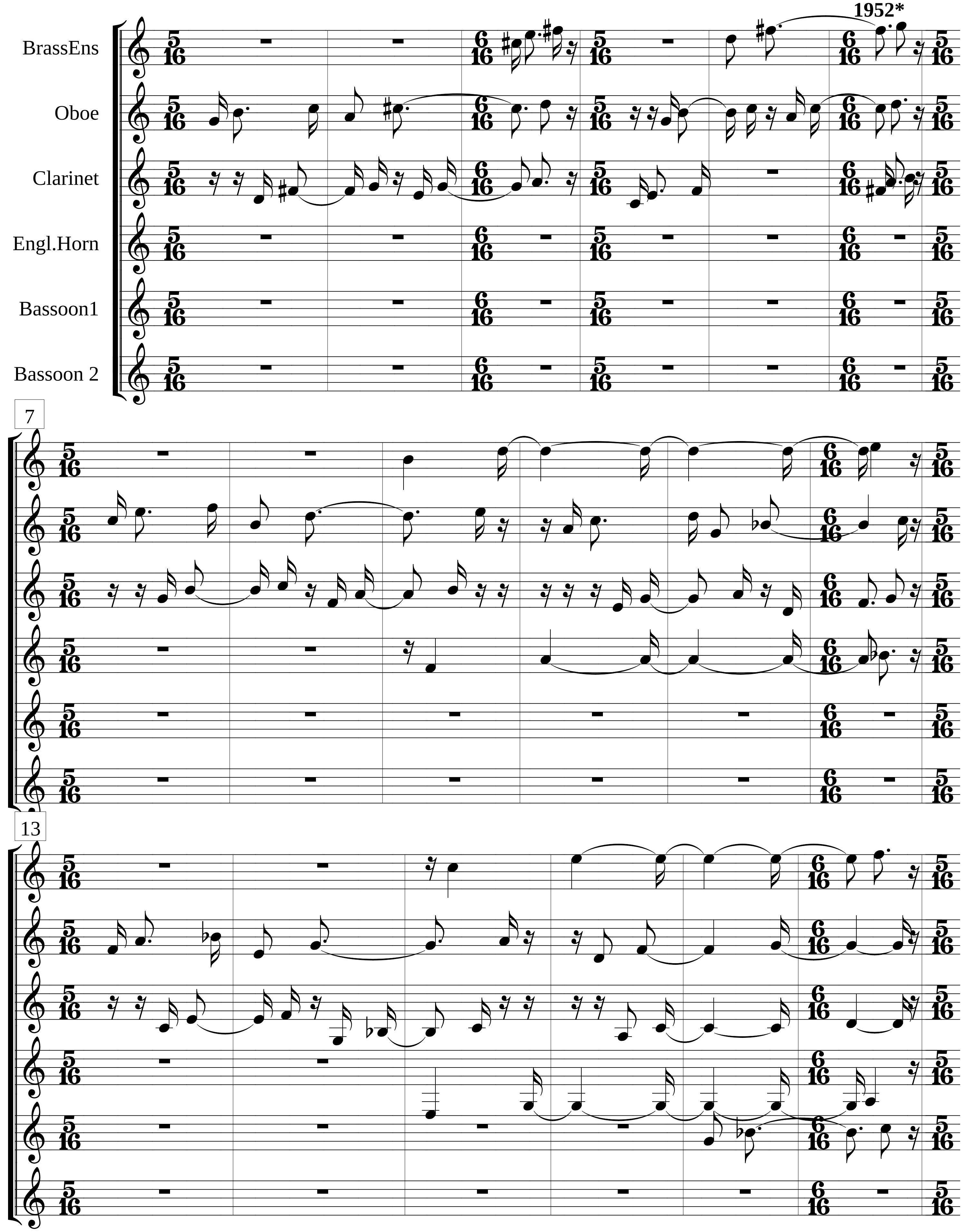
Eine kleine Mathmusik 1

5.2.2002

Herdecke, 5 February 2002

Andranik Tangian
1952*

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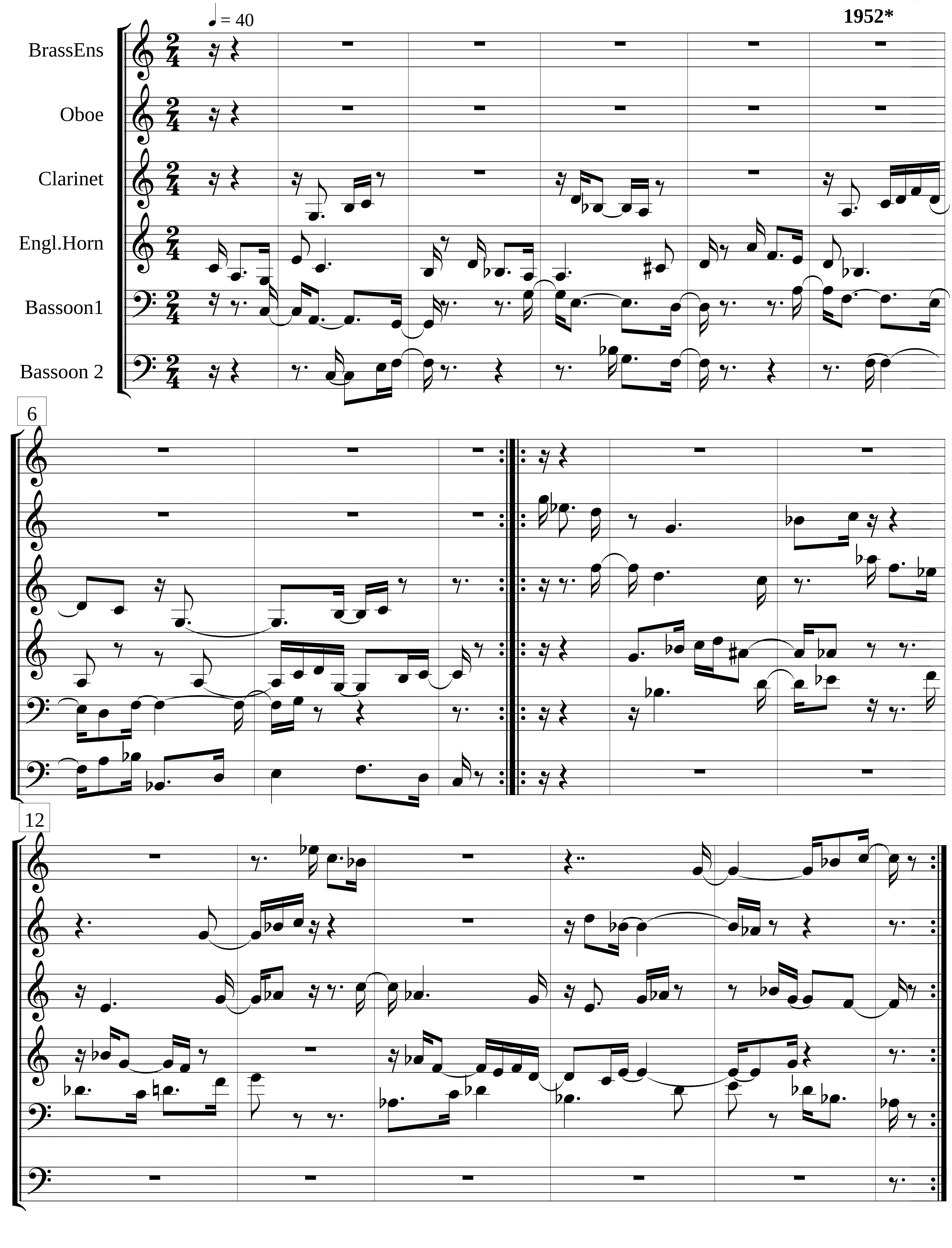
Eine kleine Mathmusik 2 (Romance)

23.1.2003

Herdecke, 23 January 2003

Andranik Tangian
1952*

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Perspectives of New Music, 2003, 41(2), 66-94

Constructing Rhythmic Canons

Andranik Tangian

1. Introduction

Recently a number of advanced mathematical models for music analysis and composition have appeared (e.g., Mazzola 2002). In particular, Vuza (1991–3, 1995) has developed a pioneering model for finding rhythms with special properties. The requirements for these rhythms were formulated in part by Vieru (1993).

Vieru's and Vuza's goal was to transfer Messiaen's (1944) *modes of limited transposition* to the domain of rhythm.[1] Recall that Messiaen considered a set of disjoint pitch classes with the same interval content which covers the twelve-tone tempered scale. For instance, four pitch classes {C, E♭, F♯, A} and two transpositions, by one and by two semitones, cover the twelve-tone scale and, consequently, meet this requirement. This is similar to what is called in mathematics *tiling*, that is, covering an area, e.g., a square, by disjoint equal figures.

Instead of the tempered scale, Vieru and Vuza considered a regular pulse train. By analogy with covering the scale by a few pitch classes and their transpositions, the pulse train was covered by a certain rhythmic pattern with different delays. The disjointedness of pitch classes implied no common beats in different instances of the rhythmic pattern. The circularity of pitch classes (= octave periodicity) corresponded to circular time (= beats in a measure).

Vieru and Vuza intended such "rhythms of limited transposition," or, better, "rhythms of limited delay," for constructing unending (= infinite, periodic) canons. Recall that a *canon* is a polyphonic piece whose voices lead the same melody with different delays. A *rhythmic canon* is one whose tone onsets result in a regular pulse train with no simultaneous tone onsets at a time. In that sense, a rhythmic canon tiles time, covering a regular pulse train by disjoint equal rhythms from different voices. Note that the established term "rhythmic canon" is somewhat misleading, and "disjoint rhythm canon" might be more exact.

From a musical standpoint, time-tiling is a technique of making polyphonic pieces from a single rhythmic/melodic pattern. It meets the principle of economy in both classical and twentieth-century music: recall long phrases built from the opening four-note motive in Beethoven's Fifth Symphony, twelve-tone composition, etc. On the other hand, in rhythmic canons the independence of the voices is maximal, since no two tones occur simultaneously, which is much appreciated in polyphony.

It is not surprising that time-tiling attracted the attention of music theorists (Amiot 2002–3), Andreatta et al. 2001, Fripertinger 2002, 2003). It turned out, however, that solutions to the time-tiling problem are mainly trivial and musically not interesting. A typical solution is a metronome rhythm entering with equal delays, e.g., a sequence of every fourth beat, entering at the first, at the second, and at the third beat, which is a rhythm analogy of the transpositions of pitch class {C, E♭, F♯, A}. Non-trivial solutions have been found by Vuza for a circular time with periods 72, 108, 120, . . ., meeting some factorization requirements.

As one can imagine, these solutions result in overcomplicated musical structures which are hard to hear as such. The effect is similar to the one in serial music, as described by Xenakis (1963):

> Linear polyphony destroys itself by its very complexity; what one hears is in reality nothing but a mass of notes in various registers. The enormous complexity prevents the audience from following the intertwining of the lines and has as its macroscopic effect an irrational and fortuitous dispersion of sounds over the whole extent of the

sonic spectrum. There is consequently a contradiction between the polyphonic linear system and the heard result, which is surface or mass.[2]

Johnson (2001) considered the time-tiling problem in a less strict way. In addition to a given rhythmic pattern he also used its augmentation, that is, the pattern with double durations, like in Bach's *The Art of the Fugue*. He heuristically constructed a simple finite canon (as opposed to unending canon) and asked for the existence of other solutions. Vuza's method was, however, adaptable neither to using augmentations of the theme, nor to linear time (as opposed to circular time).

This paper provides a numerical solution to the general problem. It introduces an algorithm for constructing rhythmic canons from several rhythmic patterns, in particular, from successive augmentations of the theme. As for an analytical solution, it is shown that the problem is equivalent to solving Diophantine equations in special polynomials (the dates of Diophante's life are not known exactly and are estimated 325–409 AD). For this purpose an isomorphism between rhythmic canons and these polynomials is established. Finally, an application of the method to algorithmic composition is described.

In Section 2, "Problem formulation," basic assumptions are introduced and illustrated with an example.

In Section 3, "Polynomial representation of rhythmic canons and some implications," an isomorphism is established between rhythms and 0–1 polynomials, that is, whose coefficients are zeros and ones, the same as for representing the structure of sound spectra (Tangian, formerly spelled Tanguiane, 1993, 1995, 2001). Then the problem of constructing rhythmic canons is reformulated as finding sums of products of 0–1 polynomials, which is analogous to Diophantine equations in 0–1 polynomials. Since no general solution is known for Diophantine equations already in integers, there is little hope to solve them in polynomials (polynomials generalize integers, containing them as polynomials of degree 0). Respectively, the question of analytically constructing rhythmic canons remains open.

Section 4, "Algorithm for constructing rhythmic canons," introduces a coding convention for rhythmic canons with no redundancy, and an enumeration algorithm. Its idea is similar to that of the sieve of Eratosthene (284–192 BC) for finding prime numbers. Some details on the algorithm implementation and processing are provided.

Section 5, "Example of application," describes the use of computer output for making the musical piece *Eine kleine Mathmusik*.

In Section 6, "Generalizations," some further extensions of the model are outlined, as using several basic patterns instead of one, fitting the patterns to a user-defined pulse train, or allowing simultaneous tone onsets.

In Section 7, "Summary," the main results of the paper are recapitulated.

The Appendix contains the rhythmic scores computed which were used in the composition of *Eine kleine Mathmusik*.

2. Problem Formulation

Consider Johnson's (2001) rhythm and its coding by zeros and ones with respect to a pulse train of sixteenths:

1 1 0 0 1

We are going to build rhythmic canons from this pattern and its augmentations shown in Example 1.

Pattern number	Musical Meaning	Progression of tone onsets and empty beats
1	Theme	11001
2	Theme in augmentation	101000001
3	Theme in double augmentation	10001000000000001

EXAMPLE 1: THREE RHYTHMIC PATTERNS CODED BY ONES AND ZEROS

To provide a homogeneous pulse train required in rhythmic canons, assume the following:

Assumption 1 (no gap). *The composite tone onsets result in a regular pulse (= no simultaneous zeros in all the voices).*

Assumption 2 (no double beat). *No tone onset occurs simultaneously in two or more voices (= no simultaneous ones in any of two voices).*

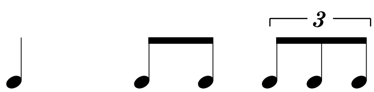
Example 2 depicts the score of a rhythmic canon (that is, the one which satisfies both assumptions).

Voice number	Pattern number	Beat number														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	0	0	1	.	.	.	.	.	.	.	.	.	.
2	1	.	.	1	1	0	0	1	.	.	.	.	.	.	.	.
3	2	.	.	.	.	.	1	0	1	0	0	0	0	0	1	0
4	1	.	.	.	.	.	.	.	.	1	1	0	0	1	.	.
5	1	.	.	.	.	.	.	.	.	.	.	1	1	0	0	1
Simultaneous onsets (pulse train)		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Example 2: a score of rhythmic canon 11211

The canon code "11211" is the succession of patterns as they enter in the canon given in the second column of the table in Example 2. In the score, ones are tone onsets, zeros denote sustained tones (tied notes), or, if the composer elects, sixteenth rests within the pattern, and periods denote sixteenth rests outside the pattern.

Coding a rhythm by a sequence of zeros and ones is feasible for all notationable rhythms, provided the reference pulse train is sufficiently dense, being a common divisor of the durations considered. For instance, a quarter note, two eighths, and three eighth triplets can be coded as follows.



1 0 0 0 0 0 1 0 0 1 0 0 1 0 1 0 1 0

3. Polynomial Representation of Rhythmic Canons and Some Implications

Define an isomorphism between rhythms and polynomials with coefficients 0 or 1. To be specific, represent the first pattern from Example 1 as follows:

$$P = 1\ 1\ 0\ 0\ 1 \longleftrightarrow p(x) = 1 + 1x + 0x^2 + 0x^3 + 1x^4.$$

If pattern P delays by 2 beats as in the second voice in Example 2, multiply $p(x)$ by x^2:

$$P_2 = 0\ 0\ 1\ 1\ 0\ 0\ 1 \longleftrightarrow p(x)x^2 = 0 + 0x + 1x^2 + 1x^3 + 0x^4 + 0x^5 + 1x^6.$$

No shift corresponds to the multiplication of $p(x)$ by the polynomial unit 1.

Generally speaking, if P is a rhythmic pattern represented by polynomial $p(x)$ then its delay P_k by k beats is represented by polynomial $p(x)x^k$:

$$P_k \longleftrightarrow p(x)x^k.$$

A superposition of rhythmic patterns corresponds to the sum of the associated polynomials. For instance, the superposition of P and P_2:

$$P + P_2 = 1\ 1\ 1\ 1\ 1\ 0\ 1 \longleftrightarrow p(x) + p(x)x^2 = p(x)(1 + x^2).$$

A double beat results in a coefficient 2 instead of 1 for a single beat:

$$P + P_3 = 1\ 1\ 0\ 1\ 2\ 0\ 0\ 1 \longleftrightarrow p(x) + p(x)x^3 = p(x)(1 + x^3).$$

Multiple superpositions of $P \leftrightarrow p(x)$ with delays correspond to polynomial products $p(x)q(x)$, where $q(x)$ represents multiple time delays. For instance, the superposition of P with delays by 2, 8, and 10 beats (sum of voices 1, 2, 4, and 5 in Example 2) corresponds to

$$p(x)q(x), \text{ where } q(x) = 1 + x^2 + x^8 + x^{10}.$$

Let voice delays in a rhythmic canon generated by pattern $P \leftrightarrow p(x)$ be represented by polynomial $q(x)$. Assumptions 1–2 mean that

$$p(x)q(x) = I_n(x) = \sum_{i=0}^{n} x^n, \tag{1}$$

where n is the sum of degrees of $p(x)$ and $q(x)$. In this case, the length of the canon is $n + 1$ beats.

PROPOSITION 1 (Existence and Uniqueness of a Rhythmic Canon). *A rhythmic canon generated by pattern* $\mathrm{P} \leftrightarrow \mathrm{p(x)}$ *can be* $\mathrm{n} + 1$ *beats long if and only if there exists a polynomial* $\mathrm{q(x)}$ *with coefficients* 0 *or* 1, *satisfying formula* (1). *If such a canon exists, it is unique to within permutation and union of voices.*

Proof. Indeed, if such a canon exists, the polynomial $I_n(x)$ from formula (1) is divisible by $p(x)$, and the result of the division, that is, some polynomial $q(x)$, is unique (Van der Waerden 1931). It means that the beats of entries of pattern P are uniquely determined, and the only freedom left is how to assign the patterns to voices. Q.E.D.

The reservation "unique to within permutation and union of voices" in Proposition 1 means that canons are considered equivalent if we (a) renumber the voices, or (b) reduce the number of voices by putting disjoint rhythmic patterns into the same voice. For instance, five voices in Example 2 can be reduced to three voices by uniting the voices 1 with 3 and 2 with 5.

Now note that the j^{th} augmentation $P^{(j)}$ of pattern P corresponds to the polynomial

$$P^{(j)} \longleftrightarrow p(x^{2^j}).$$

For instance, the augmentations from Example 1 correspond to the polynomials

$$\text{First augmentation} \longleftrightarrow p(x^2) = 1 + x^2 + x^8$$
$$\text{Second augmentation} \longleftrightarrow p(x^4) = 1 + x^4 + x^{16}.$$

Consequently, a rhythmic canon built from the rhythmic "theme" P and its two successive augmentations must satisfy the polynomial equation

$$p(x)q(x) + p(x^2)q_1(x) + p(x^4)q_2(x) = I_n(x), \qquad (2)$$

where polynomial $q_j(x)$ is associated with entry delays of the j^{th} augmentation. For example, the canon in Example 2 satisfies equation (2) for the following polynomials:

$$q(x) = 1 + x^2 + x^8 + x^{10}$$

$$q_1(x) = x^5$$
$$q_2(x) = 0$$
$$I_n(x) = 1 + x + \ldots + x^{14}.$$

Unlike (1), where the uniqueness of polynomial factorization implies the uniqueness of solution $q(x)$ (if it exists), we expect no uniqueness of a solution $q(x)$, $q_1(x)$, $q_2(x)$ to (2).

The isomorphism between rhythms and 0–1 polynomials is useful in analyzing properties of rhythmic canons. In particular, it enables to estimate the difficulties in finding a general analytical solution of the problem considered.

Note that polynomial classes inherit some properties of the number classes used for their coefficients (one can consider polynomials with integer coefficients, or rational coefficients, or real coefficients, etc.):

- Polynomials include numbers as polynomials of degree 0.
- Addition, subtraction, multiplication, and division are defined for polynomials.
- The division properties of polynomials are similar to those of real numbers, with the unique factorization into irreducible polynomials, which are analogous to primes.

From this standpoint, equation (2) is a polynomial version of the Diophantine equation

$$pq + p_1q_1 + p_2q_2 = I$$

with positive integer coefficients p, p_1, p_2, I to be solved in positive integers q, q_1, q_2. For instance, the Diophantine equation

$$5q + 7q_1 = 100 \tag{3}$$

has two solutions, (6, 10), and (13, 5).

The existence of a general analytical solution (with a formula) to (2) would mean the existence of an analytical solution to much more simple Diophantine equations in integers. Since no general solution to Diophantine equations is known, there is little hope to solve more general Diophantine equations for polynomials.[3]

4. Algorithm for Constructing Rhythmic Canons

An appropriate coding convention is often a ladder to success in combinations. Such a coding convention can imply an enumeration algorithm with the fewest parameters.

Proposition 2 (Coding Convention). *Under Assumptions* 1 *and* 2, *a rhythmic canon coded by a succession of entering rhythmic patterns is unique to within permutation and union of voices.*

Proof. The succession of rhythmic patterns, for instance, {11211} in Example 2, uniquely determines the instances of pattern entries, namely, each next pattern enters at the first common rest of previous patterns. Otherwise there would be a gap, against Assumption 1, or a double beat, against Assumption 2. Q.E.D.

Proposition 2 implies that a canon C can be unambiguously coded by a succession of entering patterns

$$C = \{\pi_1\pi_2...\pi_i\}, \text{ where } \pi_i = 1,2,3,$$

where 1 stands for the pattern P, 2 for its augmentation, and 3 for its double augmentation. Now rhythmic canons can be constructed by enumerating successions of numbers 1,2,3 as candidates for canons and sorting out inappropriate ones. More specifically, do the following:

0. Initialize the list C of *C*andidates for canon with $C[1] = \{1\}$ (a trivial sequence of entering patterns which consists of the single pattern P). Initialize the list S of *S*elected canons to be the empty list.
1. Append π = 1,2, or 3 to $C[1]$. This means that the pattern P (respectively, its augmentation, or double augmentation) enters at the first gap, i.e., at the first 0 of pulse train of $C[1]$.

   There are three possibilities:

   (a) The new succession $\{C[1],\pi\}$ is a rhythmic canon (= no gaps and no double beats). In such a case the new succession is appended to the list S of selected canons. This implies removing the whole branch of its descendants from further considerations. In our case, the first selected canon is $S[1] = \{11211\}$.

   (b) The new succession $\{C[1],\pi\}$ is not a candidate for canon, because the new pattern π entering at the first gap results in a double beat. By this reason the new succession is left out. Thereby all its descendants, containing the double beat, are removed from further considerations.

(c) The new succession $\{C[1],\pi\}$ is a candidate for canon, because the new pattern entering at the first gap results in no double beats. Then the new succession is appended to the bottom of the list C of candidates for canon.

2. After having performed all three trials with $\pi = 1,2,3$, delete the currently considered (first) candidate $C[1]$ from list C as unnecessary. Return to Item 1, considering the first remaining candidate in the list C.

Thus C is destroyed from the top, appended to the bottom, and some selected elements of C are moved to S.

This sorting algorithm resembles the famous sieve of Eratosthene (284–192 BC) for finding primes:

- If we remove an element (in our case, a candidate for canon) then we delete the branch with all its descendants that stems from this element.
- We always start with the first remaining element (in our case, a candidate for canon).

The list of selected canons has no repeats in the sense that no smaller canon is a part of a larger canon. Indeed, if a canon is accomplished then it is moved from the list of candidates to the selected list, leaving no descendants in list C. Thus, each selected canon is continuous, with the end of a rhythmic pattern in one voice occurring in the middle of a rhythmic pattern of some other voice.

The algorithm does not miss any canon, because it is based on enumerating all successions of numbers 1, 2, 3. Due to restrictions imposed by Assumptions 1–2, the number of branches retained remains within operational limits, enabling us to perform computations in reasonable time.

The implementation of the algorithm includes several important devices. First of all, the list C of candidates for canon should be stored and processed by portions to avoid a long processing time and running out of memory. In my implementation, the list C is stored in a series of temporary files, while keeping in memory only the first file (to be destroyed from the top) and the last file (to be appended from the bottom up to a certain size, after which a new file should be opened).

Moreover, for each candidate for canon, its pulse train after the first gap should be saved. It prevents from reconstructing the pulse train while appending a rhythmic pattern to the current candidate for canon.

The program has been written in the MATLAB (= MATrix LABoratory) C++-based computer programming environment for matrix and vector operations. The program output is a LaTeX text file with rhythmic scores of canons as in Example 2. A typical processing report for a program run on a PC with a Pentium II 300 MHz processor (note that MATLAB) is not a compiler but an interpreter) is given in Example 3.

These reports are helpful for composition. One can see that the number of canons found is large, so it is not convenient to examine all of them. For compositional tasks, some additional selection requirements

Totally tested combinations (candidates for canon)	1260234		
Maximal number of voices in preselection/selection		6	6
Maximal mean pattern number in preselection/selection		1.8	1.7
Periodicity in the preselection/selection		No	Yes
Found/preselected/selected canons of length 5	1	1	0
Found/preselected/selected canons of length 10	6	3	0
Found/preselected/selected canons of length 15	20	0	0
Found/preselected/selected canons of length 20	93	21	1
Found/preselected/selected canons of length 25	348	0	0
Found/preselected/selected canons of length 30	1460	0	0
Found/preselected/selected canons of length 35	5759	0	0
Found/preselected/selected canons of length 40	23502	961	15
Totally found/preselected/selected canons	31189	986	16
Maximal number of files on disk	120		
Maximum/average number of candidates for canon in memory	1000	296	
Time for computing/selection/making LaTeX file, in seconds	1856	10	7

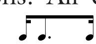
EXAMPLE 3: PROCESSING REPORT ON COMPUTING RHYTHMIC CANONS

may be formulated. Besides finding canons the program also classifies and selects canons according to several useful criteria:

- length
- maximal number of simultaneous voices, which indicates the number of instruments with which the canon can be performed
- prevailing pattern (the pattern, its augmentation, or double augmentation) which characterizes the relative rhythmic density
- uniformity of using all the patterns, which characterizes the variety of rhythms used, and
- periodicity in the canon structure, which is practical for making harmonic sequences

5. Example of Application

Example 4 contains the opening of *Eine kleine Mathmusik*.[4] The piece has been written partially algorithmically, partially heuristically. It was performed at IRCAM, Paris, on February 9, 2002. The Appendix contains the scores of rhythmic canons used in the piece.

Eine kleine Mathmusik is a piece centered on G for a woodwind sextet which uses a selection of eight computed rhythmic canons. All eight canons are built from the *basic rhythmic pattern* 11001 =  , its augmentation, and its double augmentation. In keeping with the length of the basic pattern, the basic *time signature* is $\frac{5}{16}$.

The reverse of the basic code 11001—that is, 10011—determines *basic melodic intervals* in the theme which are thirds and seconds. Thirds and seconds can be either minor or major. The principal *theme* motive is g_1, b_1, c_2.

In order to reduce the number of performers, non-overlapping rhythmic patterns are grouped into a few *composite voices* which are performed by the same instrument. For instance, the canon 11211 shown in Example 2 has five entering patterns which can be reduced to three composite voices. This is done heuristically in order to construct more developed *melodies from successive basic motives.*

Since the piece consists of a series of canons, they are separated by additional 1/16–3/16 rests inserted manually, which are rhythmically to be perceived as stops and harmonically emphasized as *cadences.* The basic $\frac{5}{16}$ measures are therefore extended, which is the cause of variable meter in the piece.

The piece suggests a loose analogy with formal procedures in tonal music. It was intended to be *neo-baroque* in a *sonata form* with two canons as contrasting "themes" implemented by "major-minor harmonies" and "tonal" development. The "harmony" is articulated in arpeggiations, not as score verticals, but rather operating in a time slot of several beats.

EXAMPLE 4: THE OPENING OF *EINE KLEINE MATHMUSIK*

The development is based on a certain *variation* principle. A canon is assumed to be a variation of some other canon if it has the same beginning but a new ending, e.g.

$$1121\boxed{1} \rightarrow 1121\boxed{331121}$$

Due to particularities of the algorithm, the list of canons selected is ordered with respect to their size, from shorter to longer, and within every size canons are ordered lexicographically, e.g., canons beginning with 112... come before the canons beginning with 113... . That means the closest variations of a given canon are its neighbors in the list.

The musical form of the piece is displayed in Example 5. As one can see, the harmonic plan of the piece is an analogy to Western tonal music. The first entry of the second theme is at the fifth (in analogy to the dominant), the development begins with the first theme in the "dominant," and the return to the main tonality passes through the "subdominant."

The selection of a particular canon for a particular purpose is motivated by several reasons:

1. For Theme 1, the shortest available canon (No. 1) is selected and used twice with harmonic modification, so that the rhythmic structure of Theme 1 is 1 + 1.

2. The closest variations of Theme 1, Canons Nos. 2–4 (the latter taken twice), are used to build a transition to Theme 2. The resulting rhythmic structure of the transition is 2 + 2 + 2 + 2.

3. Theme 2 (Canon No. 29, the first of relative length 4, i.e., four times longer than the theme, with fewer than six physical voices) is "slower" due to prevailing patterns of augmentation and second augmentation.

4. The "Variation of Theme 2" is a quite distant variation (Canon No. 55), but it is the only canon of the same length as Canon No. 29 with only four physical voices. The economy of physical voices is quite important to preserve harmonic transparency.

5. The Development contains the longest canon in the piece, with forty entries of the basic motive, which gives 120 beats. It has been selected due to its periodicity (which enables making harmonic sequences that are the norm in a classical development section) and economy of physical voices (six).

   Another selection criterion is the *mean pattern number* of the patterns used in the canon. For instance, the first canon 11211 has mean pattern number (1 + 1 + 2 + 1 + 1)/5 = 6/5 = 1.2, indicating that the basic rhythmic pattern with number 1 prevails over the augmentations coded by 2 or 3 (double augmentation). A low mean pattern number implies shorter durations, an easier recognizability of the theme, and a more vivid melodic development. Conversely, for slower sections a high mean pattern number can be desirable. But in the given piece (with a relatively high rhythmic density), a low mean was always preferred.

Section	Material	Measures	Description
Exposition	Theme 1	1–6	Canon No. 1, twice 11211 + 11211 $G \rightarrow D$ $C \rightarrow G$
	Transition 1	7–18	Canons No. 2 and 3 1121 [331121] + 1121 [332222] $C \rightarrow C^7$ $F \rightarrow F^{6/9}$
	Transition 2	19–30	Canon no. 4, twice 11 [31211211] + 11 [31211211] $Dm \rightarrow A^7$ $Dm \rightarrow F^6$
	Theme 2	31–42	Canon No. 29 112 [2223321113121121 1] $D \rightarrow F\sharp^7$
	Var. Theme 2	43–54	Canon No. 55 11 [3121331123321112 11] $E \rightarrow A^+$
Development	Theme 1 Var. Trans. 1	55–60 61–84	Canon No. 1, twice, D A, G D Canon No. 8005 with 3 periods 1 1222233211 1222233211 1222233211 121332222 G D, B^7, Am^6, E^7 G, E^7, Dm^6, A^7 C, A^7, Gm^6, D^7 $F \rightarrow G^7$
	Var. Trans. 2	85–96	Canon No. 49 1131211 [312113121121 1] $Cm \rightarrow A\flat$
	Theme 2 Var. Theme 2	97–108 109–20	Canon No. 29, $C \rightarrow E^7$ Canon no. 55, $D \rightarrow G^+$
Recapitulation	Theme 1 Trans. 2	121–6 127–38	Canon No. 1, twice, $G \rightarrow D$ $C \rightarrow G$ Canon No. 4, twice, $Gm \rightarrow D^7$, $Gm \rightarrow E\flat$
Coda	Theme 1	151–62	Canon No. 1, four times $Gm \rightarrow D$, $B\flat \rightarrow F$, $Fm \rightarrow Cm$, $D^{-9} \rightarrow G$

EXAMPLE 5: THE FORM OF *EINE KLEINE MATHMUSIK*

6. Generalizations

6.1 using several thematic elements

Instead of augmentations of the theme, the model can operate with some other arbitrary rhythmic patterns.

In fact, the algorithm fits several rhythmic patterns to a given pulse train. In our specific model these rhythmic patterns are restricted to the "theme," its augmentation, and double augmentation. However nothing prevents the model from using some other building blocks, e.g., two themes and/or some of their derivatives. The sorting algorithm will just operate on some other basic elements.

Thus besides rhythmic canons restricted to a single theme, one can construct, for instance, "rhythmic fugues" with several themes and counter-subjects.

6.2 producing an irregular pulse train

Assumption 1 (no gaps) is not obligatory for the model, and it can be replaced by a more general one. Instead of a regular pulse train which must be covered (tiled) by a restricted set of basic rhythmic patterns, an arbitrary pulse train can be considered:

For instance, consider the task of tiling the pulse train

110110...

with our three patterns from Example 1. A candidate for such a canon solution is shown in Example 6.

Voice number	Pattern number	Beat number											
		1	2	3	4	5	6	7	8	9	10	11	12
1	3	1	0	0	0	1	0	0	0	0	0	0	0
2	2	.	1	0	1	0	0	0	0	0	1	.	.
3	1	.	.	.	.	.	.	1	1	0	0	1	.
Simultaneous onsets		1	1	0	1	1	0	1	1	0	1	1	0

example 6: tiling the pulse train 110110...

The difference is that the sorting algorithm must fit given patterns not to the regular rhythmic grid but rather to a user-defined one. Then the resulting polynomial in equation (2) is

$$I_n(x) = 1 + x + 0x^2 + x^3 + x^4 + 0x^5 + \ldots .$$

6.3 Allowing Simultaneous Tone Onsets

Assumption 2 (no double beats) is not obligatory either. Double beats can be prohibited at certain beats, or allowed; triple beats can be required for certain accents, etc. Again, the performance of the model is determined by the resulting pulse train which can look, for instance, like

310121013101...

which means in particular that every fourth beat must be accentuated either with triple or double tone onset. Then the resulting polynomial in equation (2) is

$$I_n(x) = 3 + x + 0x^2 + x^3 + 2x^4 + x^5 + 0x^6 + 3x^7 + x^8 + 0x^9 + x^{10} + \ldots .$$

Then the model will fit the basic elements to such a grid.

It should be mentioned that the increasing degree of freedom will require more memory and will reduce the search speed.

6.4 Constructing Canons from Aggregate Blocks

Consider the original problem of constructing rhythmic canons, that is, with no double beats and no rhythmic gaps. Note that each candidate for canon has a regular pulse train until the first gap. Let us call the irregular pulse train after the first gap a *junction*. For instance, the candidate for canon 1121 has the following pulse train (see Example 2):

$$11111111110 \quad \underbrace{011}_{\textit{junction}}$$

The candidate for canon 1121 can be completed to a canon either by pattern 1 (as in Canon No. 1), or by patterns 331121 (as in Canon No. 2), or by patterns 332222 (as in Canon No. 3), or by some other sequence, all meeting the same junction.

Note that the length of a junction cannot be longer than the longest pattern considered minus two beats, in our case 15 beats. Not all junctions are possible. There is a finite number of junctions, and for every two junctions there must be a bridge (= a sequence of patterns providing a transition from one junction to another). Knowing all the bridges enables the constructing of rhythmic canons of arbitrary length. Then constructing canons can be reduced to manipulating a finite number of building blocks (like in puzzle games).

6.5 MOTIVES AS VECTOR NOTES

Motives can be regarded as vector notes. Similarly to the use of single notes restricted by certain rules of harmony in the Western tonal syntax, the use of vector notes in our model is restricted by the "no gap" and "no double beat" assumptions. The compatibility of vector notes can be developed into a theory similar to harmony for single notes. The difference is that this theory is rhythm-based, and therefore contributes to a theory of rhythm as well.

7. SUMMARY

Let us recapitulate the main results of the paper. We suggested an algorithmic solution to the problem of finding finite rhythmic canons with augmentations. The application of the model for practical composition was illustrated with an example of the piece *Eine kleine Mathmusik*. The model can be adapted for more general tasks which are outlined briefly: using several basic patterns to make rhythmic fugues, producing a user-defined pulse train, and/or allowing simultaneous notes for making special accents.

Appendix: Scores of Rhythmic Canons Used in *Eine Kleine Mathmusik*

Voice	Pattern	Score														
1	1	1	1	0	0	1	.	.	.	.	.	.	.	.	.	.
2	1	.	.	1	1	0	0	1	.	.	.	.	.	.	.	.
3	2	.	.	.	.	.	1	0	1	0	0	0	0	0	1	.
4	1	.	.	.	.	.	.	.	.	1	1	0	0	1	.	.
5	1	.	.	.	.	.	.	.	.	.	.	1	1	0	0	1

Canon no. 1 of length 15 beats with 3 simultaneous voices and mean pattern no. 1.2

Voice	Pattern	Score																													
1	1	1	1	0	0	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	1	.	.	1	1	0	0	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	2	.	.	.	.	.	1	0	1	0	0	0	0	0	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	1	.	.	.	.	.	.	.	.	1	1	0	0	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5	3	.	.	.	.	.	.	.	.	.	.	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	.	.	.
6	3	.	.	.	.	.	.	.	.	.	.	.	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	.	.
7	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	0	1	.	.	.	.	.	.	.	.	.
8	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	0	1	.	.	.	.	.	.	.
9	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	0	0	0	0	0	1
10	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	0	1	.

CANON NO. 2 OF LENGTH 30 BEATS WITH 4 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.6

Voice	Pattern	Score																													
1	1	1	1	0	0	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
2	1	·	·	1	1	0	0	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
3	2	·	·	·	·	·	1	0	1	0	0	0	0	0	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
4	1	·	·	·	·	·	·	·	·	1	1	0	0	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
5	3	·	·	·	·	·	·	·	·	·	·	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	·	·	·
6	3	·	·	·	·	·	·	·	·	·	·	·	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	·	·
7	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	0	1	0	0	0	0	0	1	·	·	·	·	·
8	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	0	1	0	0	0	0	0	1	·	·	·	·
9	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	0	1	0	0	0	0	0	1	·
10	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	0	1	0	0	0	0	0	1

CANON NO. 3 OF LENGTH 30 BEATS WITH 6 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.9

Voice	Pattern	Score					
1	1	1 1 0 0 1	. . . . .	. . . . .	. . . . .	. . . . .	. . . . .
2	1	. . 1 1 0	0 1 . . .	. . . . .	. . . . .	. . . . .	. . . . .
3	3	. . . . .	1 0 0 0 1	0 0 0 0 0	0 1 . . .	. . . . .	. . . . .
4	1	. . . . .	. . 1 1 0	0 1 . . .	. . . . .	. . . . .	. . . . .
5	2	. . . . .	. . . . .	1 0 1 0 0	0 0 0 1 .	. . . . .	. . . . .
6	1	. . . . .	. . . . .	. . . 1 1	0 0 1 . .	. . . . .	. . . . .
7	1	. . . . .	. . . . .	. . . . .	1 1 0 0 1	. . . . .	. . . . .
8	2	. . . . .	. . . . .	. . . . .	. . . . .	1 0 1 0 0	0 0 0 1 .
9	1	. . . . .	. . . . .	. . . . .	. . . . .	. . . 1 1	0 0 1 . .
10	1	. . . . .	. . . . .	. . . . .	. . . . .	. . . . .	1 1 0 0 1

CANON NO. 4 OF LENGTH 30 BEATS WITH 4 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.4

Voice	Pattern	Score											
1	1	11001	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
2	1	..110	01...	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
3	2	.....	10100	0001.	.....	.....	.....	.....	.....	.....	.....	.....	.....
4	2	.....	...10	10000	01...	.....	.....	.....	.....	.....	.....	.....	.....
5	2	.....	1	01000	001..	.....	.....	.....	.....	.....	.....	.....	.....
6	2	.....	.....	..101	00000	1....	.....	.....	.....	.....	.....	.....	.....
7	3	.....	.....	.....	10001	00000	00000	01...	.....	.....	.....	.....	.....
8	3	.....	.....	.....	...10	00100	00000	00001	.....	.....	.....	.....	.....
9	2	.....	.....	.....	.....	.1010	00001	.....	.....	.....	.....	.....	.....
10	1	.....	.....	.....	.....	1	1001.	.....	.....	.....	.....	.....	.....
11	1	.....	.....	.....	.....	.....	.1100	1....	.....	.....	.....	.....	.....
12	1	.....	.....	.....	.....	.....	.....	..110	01...	.....	.....	.....	.....
13	2	.....	.....	.....	.....	.....	.....	.....	10100	0001.	.....	.....	.....
14	1	.....	.....	.....	.....	.....	.....	.....	...11	001..	.....	.....	.....
15	3	.....	.....	.....	.....	.....	.....	.....	.....	10001	00000	00000	01...
16	3	.....	.....	.....	.....	.....	.....	.....	.....	.1000	10000	00000	001..
17	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.1100	1....	.....
18	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	...11	001..	.....
19	2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.1010	00001
20	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	1	1001.

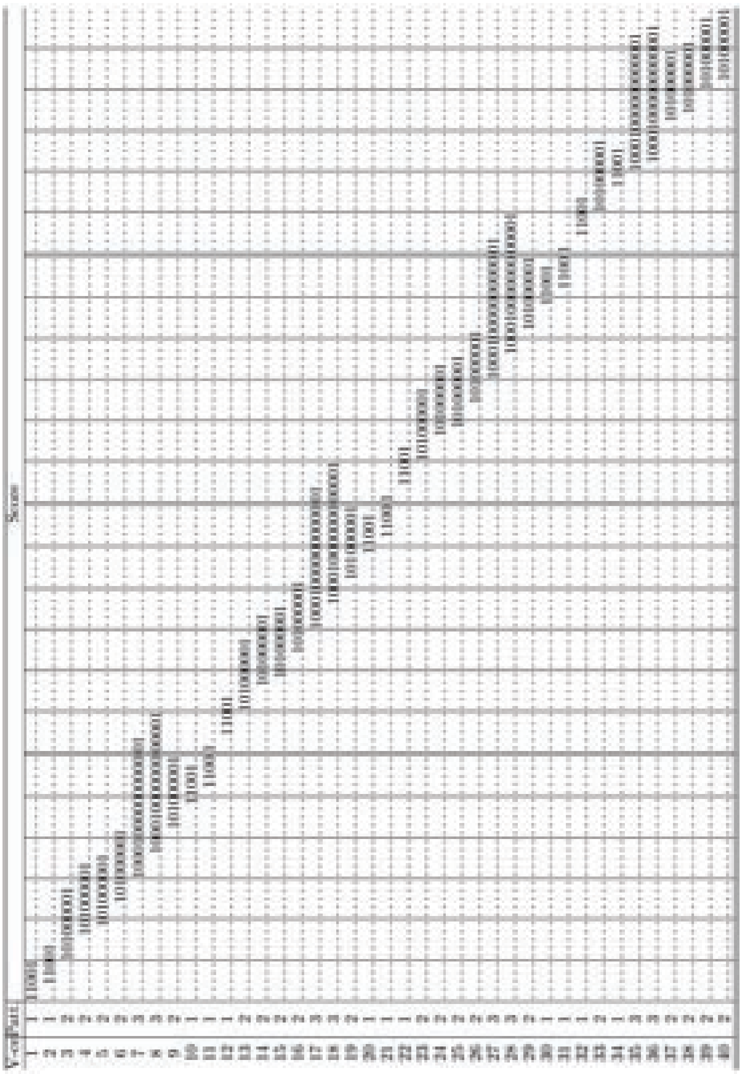
CANON NO. 29 OF LENGTH 60 BEATS WITH 5 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.8

Voice	Pattern	Score											
1	1	11001	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
2	1	..110	01...	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
3	3	.....	10001	00000	00000	01...	.....	.....	.....	.....	.....	.....	.....
4	1	.....	..110	01...	.....	.....	.....	.....	.....	.....	.....	.....	.....
5	2	.....	.....	10100	0001.	.....	.....	.....	.....	.....	.....	.....	.....
6	1	.....	.....	...11	001..	.....	.....	.....	.....	.....	.....	.....	.....
7	1	.....	.....	.....	11001	.....	.....	.....	.....	.....	.....	.....	.....
8	3	.....	.....	.....	.....	10001	00000	00000	01...	.....	.....	.....	.....
9	1	.....	.....	.....	.....	..110	01...	.....	.....	.....	.....	.....	.....
10	2	.....	.....	.....	.....	.....	10100	0001.	.....	.....	.....	.....	.....
11	1	.....	.....	.....	.....	.....	...11	001..	.....	.....	.....	.....	.....
12	1	.....	.....	.....	.....	.....	.....	11001	.....	.....	.....	.....	.....
13	3	.....	.....	.....	.....	.....	.....	.....	10001	00000	00000	01...	.....
14	1	.....	.....	.....	.....	.....	.....	.....	..110	01...	.....	.....	.....
15	2	.....	.....	.....	.....	.....	.....	.....	.....	10100	0001.	.....	.....
16	1	.....	.....	.....	.....	.....	.....	.....	.....	...11	001..	.....	.....
17	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	11001	.....	.....
18	2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	10100	0001.
19	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	...11	001..
20	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	11001

CANON NO. 49 OF LENGTH 60 BEATS WITH 4 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.5

Voice	Pattern	Score											
1	1	11001	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
2	1	..110	01...	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
3	3	.....	10001	00000	00000	01...	.....	.....	.....	.....	.....	.....	.....
4	1	.....	..110	01...	.....	.....	.....	.....	.....	.....	.....	.....	.....
5	2	.....	.....	10100	0001.	.....	.....	.....	.....	.....	.....	.....	.....
6	1	.....	.....	...11	001..	.....	.....	.....	.....	.....	.....	.....	.....
7	3	.....	.....	.....	10001	00000	00000	01...	.....	.....	.....	.....	.....
8	3	.....	.....	.....	.1000	10000	00000	001..		.....	.....	.....	.....
9	1	.....	.....	.....	.....	..110	01...	.....	.....	.....	.....	.....	.....
10	1	.....	.....	.....	.....	1	1001.	.....	.....	.....	.....	.....	.....
11	2	.....	.....	.....	.....	.....	..101	00000	1....	.....	.....	.....	.....
12	3	.....	.....	.....	.....	.....	.....	10001	00000	00000	01...	.....	.....
13	3	.....	.....	.....	.....	.....	.....	...10	00100	00000	00001	.....	.....
14	2	.....	.....	.....	.....	.....	.....	.....	.1010	00001	.....	.....	.....
15	1	.....	.....	.....	.....	.....	.....	.....	1	1001.	.....	.....	.....
16	1	.....	.....	.....	.....	.....	.....	.....	.....	.1100	1....	.....	.....
17	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	..110	01...	.....
18	2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	10100	0001.
19	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	...11	001..
20	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	11001

CANON NO. 55 OF LENGTH 60 BEATS WITH 5 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.7



CANON NO. 8005 OF LENGTH 120 BEATS WITH PERIOD 10 WITH 6 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.9

Notes

The author thanks Tom Johnson for fruitful discussions and thoughtful reading of the draft of the paper and Professor Robert Morris for numerous suggestions which improved both the content and the style.

1. As noted by Robert Morris, Messiaen's "non-retrogradable rhythms" are already a transference of pitch to rhythm, except that the "modes" are invariant under shift of pitch (transposition) and the non-retrogradable rhythms are invariant under (retrograde) inversion.
2. The English translation is given by Xenakis (1971, 8).
3. Recall that Fermat (1601–1665) stated his Last Theorem as a marginal note in Diophante's Arithmetic as a step towards the unsolvable general case.
4. The full score is available from the author of the paper.

References

Amiot, Emmanuel. 2002A. "From Vuza-Canons to Economical Rhythmic Canons." <http://www.ircam.fr/equipes/repmus/mamux/documents/MaMuXTiling.html>.

———. 2002B. "A Solution to Johnson-Tangian Conjecture." <http://www.ircam.fr/equipes/repmus/mamux/documents/johnsonConjecture.pdf>.

———. 2003. "Outil théoriques et pratiques pour les canons rythmiques." <http://perso.wanadoo.fr/chucky/mamux2003/>.

Andreatta, Moreno, Thomas Noll, C. Agon, and Gerard Assayag, 2001. "The Geometrical Groove: Rhythmic Canons Between Theory, Implementation, and Musical Experiment." In *Les Actes des 8e Journées d'Informatique Musicale, Bourges, 7–9 Juin 2001*, 93–8. Bourges: Inst. Inetrn. de Musique Electroacoustique de Bourges (IMEB), École Nationale Supérieure d'Ingénieurs de Bourges.

Fripertinger, Harald. 2002. "Enumeration of Non-Isomorphic Canons." *Tatra Mountains Mathematical Publications* 23: 47–57.

Johnson, Tom. 2001. "Tiling the Line (Pavage de la ligne): Self-Replicating Melodies, Rhythmic Canons, and an Open Problem." In *Les Actes des 8e Journées d'Informatique Musicale, Bourges, 7–9 Juin 2001*, 147–52. Bourges: Inst. Inetrn. de Musique Electroacoustique de Bourges (IMEB), École Nationale Supérieure d'Ingénieurs de Bourges.

Mazzola, Guerino. 2002. *The Topos of Music: Geometric Logic of Concepts, Theory, and Performance*. Basel: Birkhäuser.

Messiaen, Olivier. 1944. *Technique de mon langage musical*. Paris: Leduc.

Tanguiane, Andranick S. 1993. *Artificial Perception and Music Recognition*. Berlin: Springer.

———. 1995. "Towards Axiomatization of Music Perception." *Journal of New Music Research* 24, no. 3 (September): 247–81.

Tangian, Andranik S. 2001. "How Do We Think: Modeling Interactions of Memory and Thinking." *Cognitive Processing* 2: 117–51.

Van der Waerden, Bartel Leendert. 1931. *Moderne Algebra*. Berlin: Springer.

Vieru, Anatol. 1993. *The Book of Modes.* Bucharest: Editura Muzicala.

Vuza, Dan Tudor. 1991. "Supplementary Sets and Regular Complementary Unending Canons, Part 1." *Perspectives of New Music* 29, no. 2 (Summer): 22–49.

———. 1992a. "Supplementary Sets and Regular Complementary Unending Canons, Part 2." *Perspectives of New Music* 30, no. 1 (Winter): 184–207.

———. 1992b. "Supplementary Sets and Regular Complementary Unending Canons, Part 3." *Perspectives of New Music* 30, no. 2 (Summer): 102–25.

———. 1993. "Supplementary Sets and Regular Complementary Unending Canons, Part 4." *Perspectives of New Music* 31, no. 1 (Winter): 270–305.

———. 1995. "Supplementary Sets: Theory and Algorithms." *Muzica* 1: 75–99.

Xenakis, Iannis. 1963. *Musiques Formelles.* Paris: Edition Richard-Masse. English translation: *Formalized Music*. Bloomington: Indiana University Press, 1971.

Constructing rhythmic fugues (unpublished addendum to *Constructing rhythmic canons*)

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Abstract

A fugue is a polyphonic piece whose voices lead a few melodies (subjects and counter-subjects) with different delays. A rhythmic fugue is one whose tone onsets result in a regular pulse train with no simultaneous tone onsets at a time.

Tiling in geometry is covering an area by disjoint equal figures, e.g., a rectangle by triangles. In that sense, a rhythmic fugue tiles the time, providing a covering of a regular pulse train by a few disjoint rhythmic patterns.

The paper suggests a general computational method for constructing rhythmic fugues. The latter generalize the notion of a rhythmic canon considered by Vuza (1991–1995) and other authors in four ways: (1) by inclusion of augmentation and diminution of the subjects and/or counterpoints, as well as of their inverse, retrograde, and inverse retrograde transformations, (2) by fitting the tone onsets to an irregular pulse train, (3) by applying both linear and circular time (to construct both ending and unending fugues), and (4) by allowing multiple voices to sound on certain beats.

Shortly, the model allows to tile an arbitrary rhythm, linear or circular, by means of an arbitrary set of rhythmic patterns, emphasizing certain time events by multiple tone onsets. Of course, there can be no solution, but if there is any then the model finds it. The algorithm follows the previous approach of the author based on an isomorphism of rhythms and special polynomials. The method is used to make a composition *Eine kleine Mathmusik 2*.

1 Introduction

The time-tiling problem[1] has profound roots in music theory. The inspirative prototypes were Messiaen's (1944) *modes of limited transposition*, that is, pitch scales with specific interval relations which transpositions disjointly covered the 12-tone tempered scale. For

[1] *Tiling* in geometry is covering an area by disjoint equal figures, e.g., a rectangle by triangles.

instance, this is the case of the mode with pitches $\{c, e\flat, f\sharp, a\}$ and its two transpositions, by one and by two semitones.

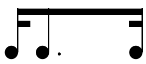
Vuza (1991–1993, 1995) and Vieru (1993) transferred Messian's ideas from the domain of pitch to the domain of rhythm. By analogy with covering the 12-tone tempered scale by a mode and its transpositions, the regular pulse train was covered by a rhythmic pattern with shifts. The disjointedness of pitch classes implied the prohibition of beat overlaps, and the circularity of pitch (= octave periodicity) corresponded to circular time (= periodicity of rhythmic structure). Vieru and Vuza intended such "rhythms of limited transposition", or, better, "rhythms of limited delay", for constructing unending (= infinite, periodic) canons, where the generative rhythm was the "theme" performed by "polyphonic voices" with different delays.

Making polyphonic pieces from a single rhythmic/melodic pattern meets the principle of economy in both classical and 20th century music: recall long phrases built from the opening four-note motive in Beethoven's Fifth Symphony, 12-tone composition, etc. On the other hand, in rhythmic canons the independence of voices is maximal, because no two tones occur simultaneously, and there are no simultaneous cadences in different voices. All of these imply continuous music development without melodic stops, which is much appreciated in polyphony.

After contributions of Vieru and Vuza, time-tiling attracted attention of several music theorists (Amiot 2002a, Andreatta et al. 2001 and 2002, Fripertinger 2002, 2003). However, solutions to the time-tiling problem appeared to be trivial and musically not interesting. A typical solution was a metronome rhythm entering with equal delays, e.g., a sequence of every fourth beat, entering at the first, at the second, and at the third beat. Non-trivial solutions were found by Vuza for a circular time with periods $72, 108, 120, \ldots$

These long complex solutions were difficult for music perception, because the canon structure could be hardly captured by hearing. To find simple rhythmic canons, Johnson (2001) relaxed the time-tiling constraints in two directions. Additionally to the theme, he authorized the use of its augmentation and double augmentation, like in Bach's *The Art of the Fugue*. Additionally to unending canons with circular time, he also considered finite canons with linear time. The Vuza's method was however adaptable neither to using augmentations of the theme, nor to linear time. Having no general method, Johnson nevertheless heuristically constructed a non-trivial finite rhythmic canon, which he discussed at the conference *Journées de l'informatique musicale, Bourges, 7–9 June 2001.*

The theme (or subject) was a rhythm whose beats were coded by 1's at the scale of sixteenths:



$$1\ 1\ 0\ 0\ 1 \qquad (1)$$

The complete structure — the rhythmic canon — was built of five instances of the theme, with one instance in augmentation (in twice slower tempo). The sum of all beats constituted a regular pulse train with no two onsets at a time, see Table 1. Thereby the regular pulse train was tiled (covered) by one rhythmic pattern and its augmentation with no beat overlaps. From the musical viewpoint the result was a canon, since it was generated by one theme entering with different delays.

The question posed by Johnson was the existence of other rhythmic canons generated

Table 1: Johnson's rhythmic canon

Voice	Pattern	Pattern type	
1	1	Theme	1 1 0 0 1 . . . .
2	1	Theme	. . 1 1 0 0 1 . . . .
3	2	Theme in augmentation	. . . . . 1 0 1 0 0 0 0 0 1 .
4	1	Theme	. . . . 1 1 0 0 1 . .
5	1	Theme	. . . . 1 1 0 0 1
		Pulse train (sum of onsets at a time)	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

by the given rhythmic pattern. A computational approach to find such canons has been developed by Tangian (2001b and 2003). In particular, a large number of rhythmic canons of different length were generated with the Johnson's rhythm (1) and used in the author's composition *Eine kleine Mathmusik* for woodwind sextet. Both the computational model and the musical piece were presented at the MaMuX seminar, IRCAM, Paris (Tangian 2002). Johnson (2005) also used these rhythmic structures in his piece *Tilework for Log Drums*.

The computational approach to finding finite rhythmic canons is based on an isomorphism between binary strings (coded rhythms) and polynomials with binary coefficients introduced in some earlier publications by Tangian (1993–1995, 2001a). This isomorphism reduces constructing rhythmic canons to factorizing such polynomials, with one factor representing the generative rhythm and another factor — the 'rhythm of its entries', that is, the set of time points where the generative rhythm starts. The techniques for constructing rhythmic canons resembles the enumeration of number products in the sieve of Eratosthene (284–192 BC) for finding primes; for details see Tangian (2001b, 2003a).

Although the isomorphism between rhythms and polynomials was originally intended for computational purposes, it was used by other researches for theoretical purposes as well; see Amiot (2002–2005), Gilbert (2007) and some other authors cited at the dedicated webpage of Seminar MaMuX (2002–2008); for a recent surveys see Amiot, Andreatta, and Agon (2005) and Amiot (2008).

The computational approach differs from the theoretical approach to rhythmic tiling. The former finds rhythmic canons generated by a *given* rhythmic pattern, whereas the latter focuses on properties of rhythmic canons and tiling conditions. Drawing analogy to numbers, the computational approach constructs multiples of a given number, whereas the theoretical approach aims at formulating characteristic properties of products, that is, non-primes, and finding their factorizations.

In a sense, the computational approach is synthetic, operating from bottom to top by starting from a low-level rhythmic pattern and constructing a high-level rhythmic structure. The theoretical approach is analytically top-to-bottom oriented, studying rhythmic sequences coverable by low-level tiles. For example, finding *all* tiling possibilities of the same pulse train is beyond the scope of the computational approach but is a subject for theoretical considerations.

On the other hand, the theoretical approach fails, at least at the current stage, to deal with several tiling patterns. The restriction to a single tiling pattern was essential in

Vuza's tiling solution, but it was not extendable to the Johnson's tiling problem with rhythm augmentations. On the contrary, the computational approach operates on any set of tile patterns, particularly including rhythm augmentations.

This flexibility of the computational approach was used by Wild (2003) for generalization of Messian's modes of limited transposition in the domain of pitch. Wild has developed a flexible algorithm with a computer program to construct special modes on scales with or without gaps (missed notes), inversions, and circular loops (pitch classes), which have a clear parallel to rhythmic tilings, both for linear and circular time.

The given paper describes the author's second computational model illustrated by the piece *Eine kleine Mathmusik 2*, also for the woodwind sextet, presented at the MaMuX seminar, IRCAM, Paris (Tangian 2003b). The second version contains a number of generalizations mentioned by Tangian (2003a, pp. 81–82), which were also considered by other contributors to the topic.

First, to avoid toccata-like rhythmic homogeneity, patterns can be fitted to a pulse train with gaps. An example of such a structure is shown in Table 2. Drawing analogy to geometry, the area to be covered can have holes, and their location and size can be irregular. It follows Vuza's (1992b, p. 109–112) examples of rhythms with silent points and Johnson's (2005) *Tiling with holes*.

Table 2: Rhythmic canon with gaps

Voice	Pattern	Pattern type	
1	2	Theme in augmentation	1 0 1 0 0 0 0 0 1 . . .
2	2	Theme in augmentation	. . . 1 0 1 0 0 0 0 0 1
3	1	Theme	. . . . . . 1 1 0 0 1 .
Pulse train with gaps			1 0 1 1 0 1 1 1 1 0 1 1

Second, a special application of using pulse trains with gaps is constructing unending rhythmic canons by the technique for finite canons. For instance, a linear pulse train must have "matching ends" as shown in 3. If the "matching ends" are connected (in the given example by fitting the 0 1 at the end to the 1 0 at the beginning), the linear canon turns into a loop with a regular pulse train. Circular canons are considered in works of Vieru and Vuza already cited, but their methods are restricted to using a single rhythmic pattern.

Table 3: Unending rhythmic canon

Voice	Pattern	Pattern type	
1	3	Theme in double augmentation	1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1
2	1	Theme	. . 1 1 0 0 1 . . . .
3	2	Theme in augmentation	. . . . . 1 0 1 0 0 0 0 0 1 . . . .
4	1	Theme	. . . . 1 1 0 0 1 . . . . .
5	1	Theme	. . . . 1 1 0 0 1 . . .
Pulse train with matching ends (loop)			1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1

Third, the irregular pulse train can be made even "more irregular" by allowing beat overlaps to produce accents of variable strength as shown in Table 4. This type of pulse trains was also considered by Vuza (1992a, p. 195) but exclusively for using a single rhythmic pattern and for circular time.

Table 4: Rhythmic canon with an irregular pulse train (gaps and double beats)

Voice	Pattern	Pattern type	
1	1	Theme	1 1 0 0 1 . . . .
2	2	Theme in augmentation	. . . . 1 0 1 0 0 0 0 0 1 .
3	2	Theme in augmentation	. . . . . 1 0 1 0 0 0 0 0 1
4	3	Retrograde theme	. . . . . . 1 0 0 1 1 . . .
5	3	Retrograde theme	. . . . 1 0 0 1 1 .
Irregular pulse train with multiple beats			1 1 0 0 2 1 2 1 1 1 1 1 2 1

Finally, tiling is performed with a few rhythmic patterns. For instance, one can use two different patterns, ”theme” and ”counterpoint”, as in fugues. Drawing analogy to tiling in geometry, one can cover an area with figures of two different shapes, say, with squares and triangles. In particular, the theme can be in diminution and augmentation, inverse, or retrograde, or both, as traditional in polyphony. The importance of these transformations in rhythmic tiling is emphasized by Amiot (2007). Wild (2003) refers to a medieval ’trichord theorem’ — that any three-note rhythm and its retrograde can tile a regular pulse train; see also Johnson (2004 and 2005), and *Rhythmic canons* (2010). An example of such a structure is shown in Table 4, where the retrograde theme can be regarded as ’counterpoint’.

The goal of the paper is developing a practical instrument (algorithm and a computer program) for constructing *rhythmic fugues,* that is, rhythmic structures generated by one or few patterns fitted to a given regular or irregular, and linear or circular pulse train, in other words, tiling an arbitrary sequence of time events with a few given patterns. Of course, there can be no solution, but if there is any, then the model finds it. All of these is implemented in the piece *Eine kleine Mathmusik 2*, which unites a number of rhythmic fugues, similarly to *Eine kleine Mathmusik*, which unites a number of rhythmic canons.

2 Isomorphism between rhythms and polynomials

Associate rhythmic patterns P with 0–1 polynomials $P(x)$, that is, with coefficients 0, 1:

$$P = \delta_1 \ \dots \ \delta_n \quad \text{where} \quad \delta_i = 0, 1 \quad \longleftrightarrow \quad P(x) = \sum_{i=0}^{n} \delta_i x^i$$

For example, Johnson’s rhythmic pattern (1) is represented by a polynomial as follows:

$$J = 1\ 1\ 0\ 0\ 1 \quad \longleftrightarrow \quad J(x) = 1 + 1x + 0x^2 + 0x^3 + 1x^4 \ .$$

Delay. A delay of a rhythmic pattern by k beats corresponds to multiplying the associated polynomial by x^k. For instance, the delay of J by two beats implies

$$J_2 = \ 0\ 0\ 1\ 1\ 0\ 0\ 1 \quad \longleftrightarrow \quad J(x)x^2 = 0 + 0x + 1x^2 + 1x^3 + 0x^4 + 0x^5 + 1x^6 \ .$$

Augmentation. The augmentation of a rhythmic pattern corresponds to taking the associated polynomial with the argument x^2. For instance,

$$\begin{array}{rcl} \text{Augmentation of } J & \longleftrightarrow & J(x^2) = 1 + x^2 + x^8 \\ \text{Double augmentation of } J & \longleftrightarrow & J\left((x^2)^2\right) = J(x^4) = 1 + x^4 + x^{16} \ . \end{array}$$

Superposition. A superposition of rhythmic patterns corresponds to the sum of the associated polynomials. For instance, the superposition of Johnson's pattern with itself delayed by three beats implies

$$J + J_3 = \begin{array}{cccccccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 2 & 0 & 0 & 1 \end{array} \quad \longleftrightarrow \quad J(x) + J(x)x^3 = J(x)(1 + x^3) \ .$$

Pulse train. The pulse train constituted by the sum of onsets at a time is associated with the polynomial

$$S(x) = \sum_{i=0}^{n} s_i x^i \quad \text{where} \quad s_i = \text{the number of overlaps at the } i\text{th beat.}$$

The regular pulse train corresponds to the case when all the coefficients $s_i = 1$.

Rhythmic fugues as polynomial equations. Rhythmic fugues are associated with polynomial equations. For instance, fugue in Table 4 is associated with the equation

$$J(x)U(x) + J(x^2)V(x) + J^R(x)W(x) = S(x) \ , \tag{2}$$

where

$J(x)$ is the 0–1 polynomial of pattern J

$J(x^2)$ is the 0–1 polynomial of augmented pattern J

$J^R(x)$ is the 0–1 polynomial of retrograde pattern J

$S(x)$ is the polynomial with non-negative integer coefficients of the pulse train in (4), and

$U(x)$, $V(x)$, and $W(x)$ are unknown 0–1 polynomials which determine the entries of pattern J, of augmented J, and of retrograde J, respectively.

Note that equations like (2) can have numerous solutions, a unique solution, or none.

3 Algorithm for constructing rhythmic fugues

We shall describe the algorithm for solving equations like (2) in terms of rhythmic patterns. First of all define and enumerate the patterns to be used, for example

1	Theme	1 1 0 0 1
2	Theme in augmentation	1 0 1 0 0 0 0 0 1
3	Retrograde theme	1 0 0 1 1
4	Retrograde theme in augmentation	1 0 0 0 0 0 1 0 1
	Pulse train	1 1 0 0 ‖:2 1 2 1 1 1 1 1 2 1:‖

where ‖:   :‖ are repeat signs.

It should be emphasized that we are not restricted to use both augmentation and diminutions. In fact, we are allowed to take any rhythmic patterns we want. The four patterns above could have no similarity, and it is our choice that they are four and have some particular structural relation to each other.

Note that a fugue is completely determined by given pulse train and succession of entering patterns which enter at the first available slot — not covered or 'insufficiently covered' beat. For example, consider the fugue in Table 4. If we start with Pattern 1 then the first slot is the 5th beat, because Pattern 1 covers all the first beats of the Pulse train except beat 5, which should be covered twice, — and this is the first available slot for entering the next pattern. Proceeding this way, one can successively fit patterns 1, 2, 2, 3, 3 to beats 1, 5, 6, 7, 9, respectively. Thus, given a pulse train, a rhythmic fugue can be labelled with the succession of entering patterns. For instance, fugue (4) is labelled "12233".

This notation enables to construct rhythmic fugues by building labels. Each such a sequences of pattern numbers is regarded as a *candidates for fugue*. A new number is appended to the label after the matching test, that is, if the tail of the resulting succession of patterns is compatible with the given pulse train. The mismatching candidates are deleted from further consideration.

More specifically, do the following.

1. Initialize the list C of candidates for fugue with label "1" (a sequence with a single pattern — the theme). Initialize the list F of fugues to be the empty list.

2. Append number $\pi = 1, 2, 3$, or 4 to the label of the first candidate c in list C, making four new candidates from one root $[c, 1], \ldots, [c, 4]$. As for the resulting score, the pattern with number π must enter at the very first slot, where the sum of onsets of the root candidate c is less than in the pulse train.

   For every new candidate $[c, \pi]$, perform the matching test with three outcomes:

   (a) The matching test fails, that is, the sum of onsets of the new candidate $[c, \pi]$ surpasses the given pulse train at some beat. Then the new candidate is left out.

   (b) The matching test reveals a complete fugue, that is, a perfect fit of the sum of onsets of the new candidate $[c, \pi]$ along its full length to the given pulse train. Then append the new candidate to the list F of fugues: $F = \{F, [c, \pi]\}$.

   (c) The matching test does not fail and reveals no complete fugue, "leaving a chance" for future accomplishments. Then append it to the end of list of candidates: $C = \{C, [c, \pi]\}$.

3. After having tested four new candidates delete the root candidate c (the first in C) and return to Item 2. This root candidate is no longer needed, because it is already built up in all possible ways, and the 'perspective built-ups' are appended to the list C.

Thus the list of candidates C is destroyed from the top and appended from the bottom with a "new generation" of candidates. The "successful candidates", that is, complete fugues, are moved from C to F. This sorting algorithm resembles the sieve of Eratosthene (284–192 BC) for finding primes:

- If we remove an element (in our case, a candidate for fugue) then we delete the whole branch with all its descendants which stem from this element.
- We always start with the first retained element (in our case, a candidate for fugue).

The list of selected fugues has no repeats in the sense that no smaller fugue is a part of a larger fugue. Indeed, if a fugue is accomplished then it is moved from the list of candidates to the selected list, leaving no descendants in list C. Thus, each selected fugue is "continuous", with the end of a rhythmic pattern in one voice occurring in the middle of a rhythmic pattern of some other voice. The algorithm does not miss any fugue, because it is based on enumerating all successions of numerals 1,2,3,4.

The implementation of the algorithm includes several technicalities. First of all, the list C of candidates is stored and processed by portions to avoid runs out of memory and long disk exchanges. In my implementation the list C is stored in a series of temporary files, while keeping in memory only the first and the last one (the list C is destroyed from the top and appended from the bottom). If the last file surpasses a certain size, a new file is opened.

Second, to simplify the matching test, each candidate for fugue is saved together with the "tail of its score", that is, with the sum of onsets starting at the first mismatch with the pulse train.

The computer program has been written in MATLAB. It outputs a LaTeX text file, containing rhythmic scores like Table 4 together with specifications of each fugue:

- first pattern, e.g., the theme,
- length of the fugue in beats,
- number of entering patterns,
- maximal number of simultaneous voices, i.e., sufficient size of performing ensemble,
- prevailing patterns (the pattern, its augmentation, or double augmentation) to characterize the relative rhythmic density,
- uniformity of using the patterns to characterizes the variety of rhythms used, and
- periodicity in the fugue structure which is practical for making harmonic sequences.

The program can also preselect fugues with respect to their particular specifications.

4 Assembled and unending rhythmic fugues

Assemble one pulse train from two pulse trains with "matching ends":

Pulse train I	1	1	1	1	1	0	1	.	.	.	.	.	.	.	.
Pulse train II	.	.	.	.	.	1	0	1	1	1	1	1	1	1	1
Total pulse train	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

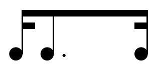
Fitting rhythmic pattern (1) and its augmentation to Pulse trains I and II results in two parts of the rhythmic canon in Table 1: Voices 1–2 for Pulse train I, and Voices 3–5 for Pulse train II. This way the rhythmic cannon can be assembled of two sections.

Of course, "matching ends" can be longer and with different "profiles". However, their variety is limited, in particular, by the length of the rhythmic patterns used in the construction. Therefore, short blocks with different ends on both sides exhaust all elements with which rhythmic fugues of arbitrary length can be assembled.

A pulse train with both ends matching to each other can be transformed into a pulse train loop. This device is practical for constructing unending rhythmic fugues and canons. An example is shown in Table 3.

5 Application to composition

Appendix 1 contains the score of *Eine kleine Mathmusik 2*, a piece in C for woodwind sextet. It is based on seven rhythmic fugues computed with the model which scores are given in Appendix 2. The pitches are "manually" assigned to time events, and the tonal development is performed as a walk on the tonal map described in the next section. The first electroacoustic performance took place at the MaMuX seminar, IRCAM, Paris, on January 25, 2003.

All the seven fugues are built from the basic rhythmic pattern 11001 =  , its augmentation, its retrograde version, and the retrograde version in augmentation.

The retrograde basic code 11001, that is, 10011 determines melodic intervals of the theme, third and second. Thirds and seconds can be either minor or major. The principal thematic motive is c_1, a, g.

The non-overlapping rhythmic patterns are grouped into a few physical voices, each being played by one instrument. For instance, five patterns of the first fugue are assigned to four physical voices. This is done not only to reduce the number of performers, but also to construct musically more interesting longer motives.

To embed the fugues into the metric structure of the piece, some fugues are separated by additional rests. They are perceived as stops and are harmonically emphasized as cadences. The harmony is articulated in arpeggiations, not as score verticals but rather as operating in a time-slot of several beats.

The development is based on a certain variation principle. A fugue is assumed to be a variation of some other fugue if it has the same beginning but a new ending. For instance,

Table 5: The form of *Eine kleine Mathmusik 2*

Section	Material	Bars	Description
Exposition (repeated)	Theme	0–4	Fugues 1 & 2 $\underbrace{12\boxed{233}}_{C\to G^7}+\underbrace{12\boxed{411}}_{g_m^6\to d_m}$
	Variation 1	4–8	Fugue 3 $\underbrace{1223\boxed{4}12\boxed{34331}}_{F\to C}$
Development (repeated)	Variation 2	9–13	Fugues 4 & 5 $\underbrace{12\boxed{3441}+1\boxed{31431}}_{g_m\to A\flat}$
	Variation 3	13–17	Fugue 6 $\underbrace{12\boxed{3431}1\boxed{43411}}_{A\flat\to f_m}$
Slow trio (repeated)	Variation 4	18–34	Fugue 7 $\underbrace{12\boxed{31313411343114331143411}}_{F\to A\flat}$
Recapitulation	Theme	35–39	Fugues 1 & 2 $\underbrace{12\boxed{233}}_{C\to G^7}+\underbrace{12\boxed{411}}_{g_m^6\to d_m}$
	Variation 1m (minor subdominant)	39–43	Fugue 3 $\underbrace{1223\boxed{4}12\boxed{34331}}_{F\to C}$

the second and the third fugues are variations of the first one. It can be seen in their labels which indicate entering patterns:

$$\begin{array}{lcl} \text{Fugue 1: } 12\boxed{233} & \to & \text{Fugue 2: } 12\ \boxed{411} \\ \text{Fugue 1: } 1223\boxed{3} & \to & \text{Fugue 3: } 1223\boxed{41234331} \end{array}$$

The musical form of the piece is summarized in Table 5. As one can see, the harmonic plan of the piece is in analogy to Western tonal music. The development begins with the theme at the fifth (”dominant”), and the return to the main tonality passes through the ”subdominant”.

The selection of a particular fugue for a particular purpose is motivated by several reasons:

1. For the theme, the two shortest fugues of equal length are selected, so that the form of the theme is $1 + 1$.

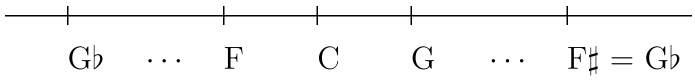
2. Variation 1 is twice longer than the theme. Thereby the exposition (Theme and Variation 1) has the form $1 + 1 + 2$.

3. The development has the same form as the exposition. For continuity, the fugues selected are somewhat longer, so that there are no gaps between them.

4. Trio is a 60-beats long rhythmic fugue with at most three simultaneous voices. To make it sound even longer, its tempo is made twice slower, so that it actually takes 120 beats. Thereby trio provides a counterbalance to the exposition and development. As usual, trio with its "thin harmony" due to few voices is put before the recapitulation.

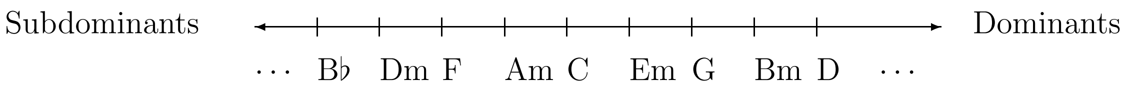
6 Tonal map in *Eine kleine Mathmusic 2*

There exist a number of maps for visualizing relationships between tonalities and chords; see Krumhansl (2002) for a survey. The best known is the line of fifths often rolled into the enharmonic circle



Here, the distance between two chords/tonalities is the number of fifths between their roots. A similar representation exists for minor chords.

Subdominant–dominant axis It is however possible to visualize the tonal proximity of both major and minor chords by putting them on one axis



and defining the distance between the neighboring chords to be 0.5. (At the line of fifths, each chord differs from its neighbor in two notes, whereas here in one.)

The left neighbors of a chord are its subdominants of different degree. For instance, if C is tonic then Am is the half-subdominant, F is the 1st subdominant, Dm is the 1.5-subdominant, $B\flat$ is the 2nd subdominant, and so on. The common subdominant nature of all left-hand neighbors of a chord is illustrated by perceptual similarity of the three chord progressions

C	Am	F	G
C	Am	Dm	G
C	F	Dm	G

Here, the subdominant-dominant direction of change is more important than the leaps between the chord roots. Similarly, in melodic variations, the melody remains recognizable if the ascending/descending direction of melodic intervals at metrical accents is preserved, while the interval values being less important (Zaripov 1983).

The right neighbors of a chord are its dominants ranked in the same way. For instance, if Am is tonic, C is the half-dominant. Recall that the parallel major is often used in sonatas in minor for the second theme or for the second movement, thereby playing the same role as dominant in sonatas in major.

Tonal function of mediant chords Recall that mediant chords (e.g. Am, Em and Dm in C major) have been qualified by Weber (1817–1821), Tchaikovski (1872), and Rimski-Korsakov (1886) as auxiliary to three main harmonic functions of tonic (C), dominant (G) and subdominant (F). In our arrangement all the chords are considered together and distinguished by numerical grades.

Riemann (1893) has qualified mediants as parallel tonic, parallel dominant and parallel subdominant. In our case, the chord system is not split into parallel classes but is located on one axis with a unique tonic.

Catuar (1924), Schenker (1906–1935) and Tulin-Privano (1965) have proposed a context-dependent functional interpretation of mediant chords. For instance, Am in C major can be regarded either as tonic, or subdominant. Such an ambiguity is surmounted by considering half-steps in the discrimination of tonal functions. For instance, Am is separable both from tonic C and from subdominant F.

Visualizing the proximity of major and minor chords of the same root The proximity of major and minor chords of the same root can be reflected by the two-dimensional map in Figure 1.

						Dominants
	⋮	⋮	⋮	⋮	⋮	↗
⋯	F♯m	A	C♯m	E	G♯m	⋯
⋯	F	Am	C	Em	G	⋯
⋯	Fm	A♭	Cm	E♭	Gm	⋯
↙	⋮	⋮	⋮	⋮	⋮	
Subdominants						

Figure 1: Map of major and minor tonalities/chords used in *Eine kleine Mathmusic 2*

The difference between horizontally neighboring chords is one note, and between vertically neighboring chords — one or two notes (respectively between C and Cm, or between C and C♯m). Therefore, the distance between vertically neighboring chords can be considered, depending on the case, as 0.5, or 1. rThe proximity of the chords with the same root can be also visualized by rolling the subdominant–dominant axis into a coil shown in Figure 2.

The chord map with enharmonic equivalence Recall that the enharmonic circle is obtained from identifying pitch classes which are 12 fifths apart. Then the cylindrical coil is rolled into a toroidal coil. It should be however emphasized that the toroidal model should not be used for finding modulation paths. For example, the enharmonic tonic, appearing in distant modulations, sounds different from the true tonic. Therefore, the return to the tonic should be done through successive back-steps on the plane map in Figure 1 rather than by enharmonic shortcuts on the toroid.

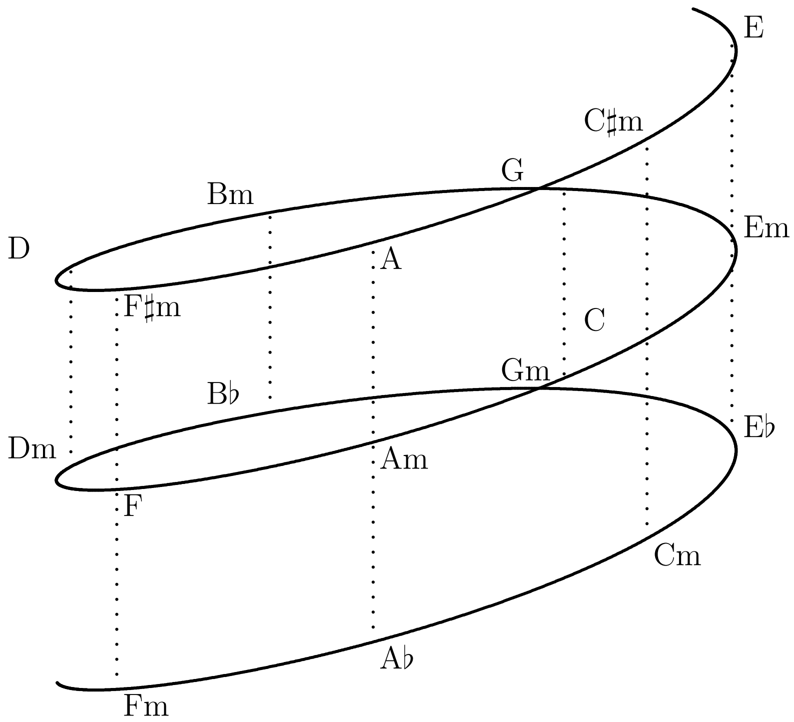


Figure 2: Subdominant-dominant coil

7 Summary

Let us recapitulate the main results.

The paper describes a general computational method for tiling musical events with a few rhythmic patterns. It enables constructing finite and infinite rhythmic canons and rhythmic fugues.

The method is based on the isomorphism of rhythmic structures with polynomial equations. It is implemented in a computer program which outputs rhythmic scores.

The model applications to practical composition is illustrated with a piece *Eine kleine Mathmusik 2*. It is based on rhythmic fugues computed, and the harmonic development is designed with a special tonal map.

8 References

Amiot, E. (2002a) "From Vuza-canons to economical rhythmic canons," `www.ircam.fr/equipes/repmus/mamux/documents/MaMuXTiling.html`

Amiot, E. (2002b) "A solution to Johnson-Tangian conjecture," `www.ircam.fr/equipes/repmus/mamux/documents/MaMuXTiling.html`

Amiot, E. (2003) "Un outil d'exploration des canons rythmiques — ce qu'il peut apporter (dans un proche avenir?)" `www.ircam.fr/equipes/repmus/mamux/documents2002-2003/ResumesJanv.html`

Amiot, E. (2004) "Why rhythmic canons are interesting." In: G. Mazzola, Th. Noll, and E. Lluis-Puebla (Eds.) *Perspectives of Mathematical and Computational Music Theory.* Osnabrück, epOs Music, 194–213.
`http://canonsrythmiques.free.fr/zurichepos.pdf`

Amiot, E. (2005a) "À propos des canons rythmiques", *Gazette des mathmatiques*, 106, Octobre 2005.
`http://canonsrythmiques.free.fr/zurichepos.pdf`

Amiot, E. (2008) From Vuza canons and their mathematical models to the Spectral Conjecture. Paper at the MaMuX Seminar on 02.02.2008. Paris, IRCAM
`http://recherche.ircam.fr/equipes/repmus/mamux/`

Amiot, E. (2005b) "Rhythmic canons and galois theory", In: H. Fripertinger and L. Reich (eds.), *Proceedings of the Colloquium on Mathematical Music Theory, Grazer Mathematische Berichte*, vol. 347, Graz, Austria, 1–25.
`http://canonsrythmiques.free.fr/zurichepos.pdf`

Amiot, E. (2007) *Canons rythmiques. I–Une notion musicale qui vaut son pesant dalgèbre.* http://canonsrythmiques.free.fr/Quadrature/canonsQuadra1.pdf

Amiot, E., Andreatta, M., and Agon, C. (2005) "Tiling the (musical) line with polynomials : Some theoretical and implementational aspects", *International Computer Music Conference 2005*, Barcelona, Espagne, 227–230.
`http://canonsrythmiques.free.fr/zurichepos.pdf`

Andreatta, M., Agon, C., and Amiot, E. (2002) *Tiling problems in music composition: Theory and Implementation.* Paper at the MaMuX Seminar on 09.02.2002. Paris, IRCAM
`http://recherche.ircam.fr/equipes/repmus/moreno/AndreattaAgonAmioticmc2002.pdf`

Andreatta, M., Noll, T., Agon, C., and Assayag, G. (2001) "The geometrical groove: rhythmic canons between theory, implementation and musical experiment," *Les Actes des 8e Journées d'Informatique Musicale, Bourges, 7–9 Juin 2001,* Bourges, Inst.Inetrn. de Musique Electroacoustique de Bourges (IMEB), École Nationale Supérieure d'Ingénieurs de Bourges, 93–98.

Catuar, G.L. (1924) *Prakticheskij kurs garmonii* (=Practical Course of Harmony). Part I. Moscow (Russian).

Fripertinger, H. (2002) "Enumeration of non-isomorphic canons," *Tatra Mountains Mathematical Publications,* (Bratislava), 23, 47–57.

Fripertinger, H (2003) "Tiling problems in music theory" In: G. Mazzola, Th. Noll, and E. Lluis-Puebla (Eds.) *Perspectives of Mathematical and Computational Music Theory.* Osnabrück, epOs Music.

Gilbert, E. (2007) Polynômes cyclotomiques, canons mosaïques et rythmes k-asymétriques. Mémoire de Master ATIAM.
`http://recherche.ircam.fr/equipes/repmus/mamux/`

Johnson, T. (2001) "Tiling the line (pavage de la ligne), self-replicating melodies, rhythmic canons, and an open problem," *Les Actes des 8e Journées d'Informatique Musicale, Bourges, 7–9 Juin 2001,* Bourges, Inst.Inetrn. de Musique Electroacoustique de Bourges (IMEB), École Nationale Supérieure d'Ingénieurs de Bourges, *Les Actes des 8èmes Journées d'Informatique Musicale, Bourges 7–9 Juin 2001*, 147–152.

Johnson, T. (2004) *Perfect Rhythmic Tilings.* Lecture delivered at MaMuX meeting, IRCAM, January 24, 2004.
http://recherche.ircam.fr/equipes/repmus/mamux/documents/Perfectrhythmictilings.html

Johnson, T. (2005) *Tiling in my music.* Paris, Editions 75.
http://www.editions75.com/Articles/Tiling%20in%20my%20music.pdf

Krumhansl, C. (2002) Music as cognition: Mental maps and models. In: C. Stevens, D. Burnham, G. McPherson, E. Schubert, and J. Renwick (Eds.) *Proceedings of the 7th International Conference on Music Perception and Cognition, 17-21 July 2002, Sydney, Australia.* Sydney: The University of New South Wales, p. 2.

Messiaen, O. (1944) *Technique de mon langage musical, Vol. 1.* Paris, Leduc.

Riemann, H. (1893) *Vereinfachte Harmonielehre oder die Lehre von der tonalen Funktionene der Akkorde.* London- New York.

Rimsky-Korsakov, N.A. (1886) *Prakticheskij Uchebnik Garmonii (= Practical Manual of Harmony).* Sankt-Peterburg: Bitner (Russian).

Rhythmic Canons (2010) http://canonsrythmiques.free.fr/menuEnglish.html

Séminaire MaMuX (2002–2008). *Mosaïques et pavages en théorie et composition musicales* (8 sessions).
http://recherche.ircam.fr/equipes/repmus/mamux/

Shenker, H. (1906-1935) *Neue musikalische Theorien und Phantasien, Bd. 1-3.* Stuttgart-Berlin-Wien.

Tangian, A. (Tanguiane, A.) (1993) *Artificial Perception and Music Recognition,* Berlin, Springer.

Tangian, A. (Tanguiane, A.) (1994) "A Principle of Correlativity of Perception and Its Applications to Music Recognition," *Music Perception,* 11 (4), 465–502.

Tangian, A. (Tanguiane, A.) (1995) "Towards axiomatization of music perception," *Journal of New Music Research*, 24 (3), 247–281.

Tangian, A. (2001a) "How do we think: Modeling interactions of memory and thinking," *Cognitive Processing*, 2, 117–151.

Tangian, A. (2001b) *The sieve of Eratosthene for Diophantine equations in integer polynomials and Johnson's problem.* FernUniversität Hagen: Discussion Paper No. 309.

Tangian, A. (2002a) “Enumeration of Rhythmic Canons and Fugues”. Paper at the MaMuX Seminar on 09.02.2002. Paris, IRCAM
`http://recherche.ircam.fr/equipes/repmus/mamux/`

Tangian, A. (2002b) “A compositional application”. Music performance at the MaMuX Seminar on 09.02.2002. Paris, IRCAM
`http://recherche.ircam.fr/equipes/repmus/mamux/`

Tangian, A. (2003a) “Constructing rhythmic canons,” *Perspectives of New Music*, 41(2), 66–94.

Tangian, A. (2003b) “Making rhythmic fugues (with a music piece)” Paper and music performance at the MaMuX Seminar on 25.01.2003. Paris, IRCAM
`http://recherche.ircam.fr/equipes/repmus/mamux/`

Tchaikovski, P.I. (1872) *Rukovodstvo k prakticheskomu izucheniju garmonii* (=Instruction to Practically Learning Harmony). Moscow (Russian).

Tulin, Yu., and N. Privano (1965) *Teoreticheskije osnovy garmonii* (=Theoretical Foundations of Harmony). Moscow (Russian).

Vieru, A. (1993) *The Book of Modes*, Bucarest, Editura Mizicala.

Vuza, D.T. (1991) “Supplementary sets and regular complementary uneding canons, part 1”, *Perspectives of New Music,* 29 (2), 22–49.
`http://recherche.ircam.fr/equipes/repmus/mamux/`

Vuza, D.T. (1992a) “Supplementary sets and regular complementary uneding canons, part 2”, *Perspectives of New Music,* 30 (1), 184–207.
`http://recherche.ircam.fr/equipes/repmus/mamux/`

Vuza, D.T. (1992b) “Supplementary sets and regular complementary uneding canons, part 3”, *Perspectives of New Music,* 30 (2), 102–125.
`http://recherche.ircam.fr/equipes/repmus/mamux/`

Vuza, D.T. (1993) “Supplementary sets and regular complementary uneding canons, part 4”, *Perspectives of New Music,* 31 (1), 270–305.
`http://recherche.ircam.fr/equipes/repmus/mamux/`

Vuza, D.T. (1995) “Supplementary sets — theory and algorithms”, *Muzica* (Bucarest) No. 1, 75–99.

Weber, J.G. (1817-1821) *Versuch einer geordneten Theorie der Tonsatzkunst. Bd. 1-3.* Mainz.

Wild, J. (2003) Tiling Gapped and Warped Harmonic Spaces. Paper at the MaMuX Seminar on 13.12.2003. Paris, IRCAM.
`http://recherche.ircam.fr/equipes/repmus/mamux/documents/ResumesDec03.html`

Zaripov, R.H. (1983) *Mashinnyi poisk variantov pri modelirovanii tvorcheskogo prozessa* (= Computer Generation of Variants for Modeling Human Creativity). Moscow: Nauka (Russian).

9 Appendix 1: Score of *Eine kleine Mathmusik 2*

Eine kleine Mathmusik 2 (Romance)

Andranik Tangian
1952*

Herdecke, 23 January 2003


18
28
37

10 Appendix 2: Scores of rhythmic fugues used in *Eine kleine Mathmusik 2*

Table 6: *Eine kleine Mathmusik 2.* Fugue No. 1 with 4 voices, 14 beats long, and mean pattern No. 2.2

		Measures													
Voice	Pattern	0					1								2
1	1	1	1	0	0	1	.	.	.	.	.	.	.	.	.
2	2	.	.	.	.	1	0	1	0	0	0	0	0	1	.
3	2	.	.	.	.	.	1	0	1	0	0	0	0	0	1
4	3	.	.	.	.	.	.	1	0	0	1	1	.	.	.
5	3	.	.	.	.	.	.	.	.	1	0	0	1	1	.
Simultaneous voices		1	1	1	1	2	2	3	3	4	4	4	3	3	1
Pulse train		1	1	0	0	2	1	2	1	1	1	1	1	2	1

Table 7: *Eine kleine Mathmusik 2.* Fugue No. 2 with 4 voices, 14 beats long, and mean pattern No. 1.8

		Measures													
Voice	Pattern	0					1								2
1	1	1	1	0	0	1	.	.	.	.	.	.	.	.	.
2	2	.	.	.	.	1	0	1	0	0	0	0	0	1	.
3	4	.	.	.	.	.	1	0	0	0	0	0	1	0	1
4	1	.	.	.	.	.	.	1	1	0	0	1	.	.	.
5	1	.	.	.	.	.	.	.	.	1	1	0	0	1	.
Simultaneous voices		1	1	1	1	2	2	3	3	4	4	4	3	3	1
Pulse train		1	1	0	0	2	1	2	1	1	1	1	1	2	1

Table 8: *Eine kleine Mathmusik 2.* Fugue No. 3 with 4 voices, 30 beats long, and mean pattern No. 2.42

Voice	Pattern	Measures 0	1	2	3	4
1	1	1 1 0 0 1	. . . .	. . . .	. . . .	.
2	2	 1	0 1 0 0 0 0 0 1	. . . .	. . . .	.
3	2	. . . . .	1 0 1 0 0 0 0 0	1	. . . .	.
4	3	. . . . .	. 1 0 0 1 1 . .	. . . .	. . . .	.
5	4	. . . . .	. . . 1 0 0 0 0	0 1 0 1	. . . .	.
6	1	. . . . .	. . . . . . 1 1	0 0 1	. . . .	.
7	2	. . . . .	. . . .	. 1 0 1 0 0 0 0	0 1	.
8	3	. . . . .	. . . .	. . . . 1 0 0 1	1	.
9	4	. . . . .	. . . .	. . . . . 1 0 0	0 0 0 1 0 1 . .	.
10	3	. . . . .	. . . .	. . . . . . 1 0	0 1 1	.
11	3	. . . . .	. . . .	. . . .	. . . 1 0 0 1 1	.
12	1	. . . . .	. . . .	. . . .	. . . . 1 1 0 0	1
Simultaneous voices		1 1 1 1 2	2 3 3 4 4 4 4 4	3 3 3 2 2 3 4 4	4 3 2 2 3 3 2 2	1
Pulse train		1 1 0 0 2	1 2 1 1 1 1 1 2	1 2 1 2 1 1 1 1	1 2 1 2 1 2 1 1	1

Table 9: *Eine kleine Mathmusik 2.* Fugue No. 4 with 4 voices, 16 beats long, and mean pattern No. 2.5

Voice	Pattern	Measures 0	1	2
1	1	1 1 0 0 1	. . . .	. . .
2	2	 1	0 1 0 0 0 0 0 1	. . .
3	3	. . . . .	1 0 0 1 1 . . .	. . .
4	4	. . . . .	. 1 0 0 0 0 0 1	0 1 .
5	4	. . . . .	. . 1 0 0 0 0 0	1 0 1
6	1	. . . . .	. . . . . 1 1 0	0 1 .
Simultaneous voices		1 1 1 1 2	2 3 4 4 4 4 4 4	3 3 1
Pulse train		1 1 0 0 2	1 2 1 1 1 1 1 2	1 2 1

Table 10: *Eine kleine Mathmusik 2.* Fugue No. 5 with 3 voices, 16 beats long, and mean pattern No. 2.17

Voice	Pattern	Measures 0	1	2
1	1	1 1 0 0 1	. . . .	. . .
2	3	 1	0 0 1 1	. . .
3	1	. . . . .	1 1 0 0 1 . . .	. . .
4	4	. . . . .	. 1 0 0 0 0 0 1	0 1 .
5	3	. . . . .	. . . . . 1 0 0	1 1 .
6	1	. . . . .	. . . . . . 1 1	0 0 1
Simultaneous voices		1 1 1 1 2	2 3 3 3 2 2 3 3	3 3 1
Pulse train		1 1 0 0 2	1 2 1 1 1 1 1 2	1 2 1

Table 11: *Eine kleine Mathmusik 2.* Fugue No. 6 with 4 voices, 30 beats long, and mean pattern No. 2.33

Voice	Pattern	Measures 0	1	2	3	4
1	1	1 1 0 0 1	. . . .	. . . .	. . . .	.
2	2	 1	0 1 0 0 0 0 0 1	. . . .	. . . .	.
3	3	. . . . .	1 0 0 1 1 . . .	. . . .	. . . .	.
4	4	. . . . .	. 1 0 0 0 0 0 1	0 1	. . . .	.
5	3	. . . . .	. . 1 0 0 1 1 .	. . . .	. . . .	.
6	1	. . . . .	. . . .	1 1 0 0 1 . . .	. . . .	.
7	1	. . . . .	. . . .	. . 1 1 0 0 1 .	. . . .	.
8	4	. . . . .	. . . .	. . . 1 0 0 0 0	0 1 0 1	.
9	3	. . . . .	. . . .	. . . . . 1 0 0	1 1	.
10	4	. . . . .	. . . .	 1	0 0 0 0 0 1 0 1	.
11	1	. . . . .	. . . .	. . . .	. . 1 1 0 0 1 .	.
12	1	. . . . .	. . . .	. . . .	. . . . 1 1 0 0	1
Simultaneous voices		1 1 1 1 2	2 3 4 4 4 3 3 2	2 2 2 3 3 3 3 3	3 3 3 3 3 3 3 2	1
Pulse train		1 1 0 0 2	1 2 1 1 1 1 1 2	1 2 1 2 1 1 1 1	1 2 1 2 1 2 1 1	1

Table 12: *Eine kleine Mathmusik 2.* Fugue No. 7 with 3 voices, 60 beats long, and mean pattern No. 2.28

Voice	Pattern	Measures 0	1	2	3	4	5	6	7	8
1	1	1	1 0 0 1	. . . .	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
2	2	.	. . . 1 0 1 0 0	0 0 0 1	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
3	3	.	. . . . 1 0 0 1	1	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
4	1	.	. . . . . 1 1 0	0 1	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
5	3	.	. . . .	. . 1 0 0 1 1 .	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
6	1	.	. . . .	. . . 1 1 0 0 1	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
7	3	.	. . . .	. . . . . 1 0 0	1 1	. . . .	. . . .	. . . .	. . . .	. . .
8	4	.	. . . .	 1	0 0 0 0 0 1 0 1	. . . .	. . . .	. . . .	. . . .	. . .
9	1	.	. . . .	. . . .	. . 1 1 0 0 1 .	. . . .	. . . .	. . . .	. . . .	. . .
10	1	.	. . . .	. . . .	. . . . 1 1 0 0	1	. . . .	. . . .	. . . .	. . .
11	3	.	. . . .	. . . .	 1	0 0 1 1	. . . .	. . . .	. . . .	. . .
12	4	.	. . . .	. . . .	. . . .	. 1 0 0 0 0 0 1	0 1	. . . .	. . . .	. . .
13	3	.	. . . .	. . . .	. . . .	. 1 0 0 1 1 . .	. . . .	. . . .	. . . .	. . .
14	1	.	. . . .	. . . .	. . . .	. . . . . . 1 1	0 0 1	. . . .	. . . .	. . .
15	1	.	. . . .	. . . .	. . . .	. . . .	1 1 0 0 1 . . .	. . . .	. . . .	. . .
16	4	.	. . . .	. . . .	. . . .	. . . .	. . . 1 0 0 0 0	0 1 0 1	. . . .	. . .
17	3	.	. . . .	. . . .	. . . .	. . . .	. . . 1 0 0 1 1	. . . .	. . . .	. . .
18	3	.	. . . .	. . . .	. . . .	. . . .	. . . . . 1 0 0	1 1	. . . .	. . .
19	1	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . 1 1 0 0 1 .	. . . .	. . .
20	1	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . . . 1 1 0 0	1	. . .
21	4	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . . . . 1 0 0	0 0 0 1 0 1 . .	. . .
22	3	.	. . . .	. . . .	. . . .	. . . .	. . . .	 1	0 0 1 1	. . .
23	4	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . . .	. 1 0 0 0 0 0 1	0 1 .
24	1	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . . .	. . . . 1 1 0 0	1 . .
25	1	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . . .	. . . . . . 1 1	0 0 1
Simult.voices		1	1 1 1 2 2 3 3 3	3 2 2 3 2 3 3 3	2 2 2 2 3 3 3 3	2 3 3 3 2 2 2 2	3 3 2 3 3 3 3 3	2 2 2 2 2 3 3 3	3 3 3 3 3 3 3 3	3 2 1
Pulse train		1	1 0 0 2 1 2 1 1	1 1 1 2 1 2 1 2	1 1 1 1 1 2 1 2	1 2 1 1 1 1 1 2	1 2 1 2 1 1 1 1	1 2 1 2 1 2 1 1	1 1 1 2 1 2 1 2	1 1 1