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THE INTERPRETATION OF INTERACTION IN CONTINGENCY TABLES

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SUMMARY

THE definition of second order interaction in a $(2 \times 2 \times 2)$ table given by Bartlett is accepted, but it is shown by an example that the vanishing of this second order interaction does not necessarily justify the mechanical procedure of forming the three component 2×2 tables and testing each of these for significance by standard methods.*

1. In a $2 \times 2 \times 2$ contingency table in which each entry is classified according to its possession or not of each of three attributes, there may exist not only associations or interactions of these attributes in pairs, but also a second order interaction of all three taken together. Bartlett (1935) has outlined a test for the presence of such a second order interaction, and Norton (1945) has discussed the numerical processes involved in carrying it out. The purpose of this note is to examine more fully the meaning of the test and its interpretation in practical examples.

2. Suppose a $2 \times 2 \times 2$ table is made up by classifying entries according to their possession of the attributes A or $\bar{A}$, B or $\bar{B}$, C or $\bar{C}$, where as usual $\bar{A}$ denotes "not-A" and so on, and let $a, b, \ldots h$ be the probabilities that an entry will fall in one of the eight classes so formed, thus:

TABLE 1

	CB	$C\bar{B}$	$\bar{C}B$	$\bar{C}\bar{B}$
A . . .	a	c	e	g
$\bar{A}$ . . .	b	d	f	h

Obviously $a + b + c + d + e + f + g + h = 1$.

The extension to this case of the hypothesis of independence which is commonly applied to the 2×2 table, namely that the probability of the class AB is the product of the probabilities of the classes A and B, gives us equations of the form

$$a = (a + c + e + g)(a + b + e + f)(a + b + c + d).$$

This is the statement of the hypothesis of complete independence of the three attributes, and any given experimental data could be tested on it by calculating the theoretical cell contents from the totals of A, of B and of C in the total sample and forming a χ^2 which would have 4 degrees of freedom.

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* This paper should be read in conjunction with the following paper by H. O. Lancaster. Bartlett's condition for a zero second order interaction is (in Simpson's notation)

$$\left(\frac{a}{b}\right)\Big/\left(\frac{c}{d}\right) = \left(\frac{e}{f}\right)\Big/\left(\frac{g}{h}\right).$$

Lancaster defines the second order interaction so as to make χ^2 additive. In general his χ^2 component for interaction is different from Bartlett's, but he shows that they are asymptotically the same. On Lancaster's definition the condition for zero second order interaction is

$$a(q\,q'\,q'') - b(p\,q'\,p'') - c(q\,p'\,q'') + d(p\,p'\,q'') - e(q\,q'\,p'') + f(p\,q'\,p'') + g(q\,p'\,p'') - h(p\,p'\,p'')$$

where

$$p = (a + c + e + g)/N, \qquad p' = (a + b + e + f)/N, \qquad p'' = (a + b + c + d)/N.$$

This satisfies the condition of symmetry mentioned by Simpson.—ED.

3. The four degrees of freedom of χ^2 suggest that there are four ways in which the cell probabilities can depart from complete independence, and it is reasonable to associate with them the three first order interactions, A with B, B with C, C with A, and something which we shall call the second order interaction. So far it is open to us to define as we like the boundary between first order and second order interaction, since no definition of the latter is forced upon us by what has gone before. Common sense suggests some statement such as this:

> "Granting that there is an association between the attributes A and B, we shall say that there is no second order interaction between A and B on the one hand and C on the other if the degree of association between A and B is the same in the class C as it is in the class $\bar{C}$."

This is a rather more general statement than that given by Norton (1939). Two difficulties immediately present themselves:

(i) We now require to measure the degree of association of two attributes in a 2×2 table. Various measures have been suggested (see for instance Kendall (1945), chapter 13), and it is reasonable to expect the use of different measures in the simple table to lead to different definitions of second order interaction in the more complex one.

(ii) The above statement is not symmetrical in all three attributes. We could similarly define the absence of second order interaction between B-and-C and A, and between C-and-A and B. Should the three sets of conditions be identical?

If on consideration of (ii) we decide that our definition of "no second order interaction" should be symmetrical with respect to the three attributes, and this is a logically attractive condition, then the choice of measures under (i) will be considerably restricted. For if in Table 1 we choose some function $\psi(a, b, c, d)$ to measure the association of A and B in the class C, the function must be such that the equation

$$\psi(a, b, c, d) = \psi(e, f, g, h)$$

implies and is implied by the equations

$$\psi(a, c, e, g) = \psi(b, d, f, h) \text{ and } \psi(a, b, e, f) = \psi(c, d, g, h).$$

This condition of symmetry is not satisfied, for instance, by the root mean square contingency defined by

$$\frac{ad - bc}{\{(a + b)(a + c)(b + d)(c + d)\}^{\frac{1}{2}}}.$$

But it is satisfied by $\psi = bc/ad$, which was used in rather similar circumstances by Fisher (1935), page 50, and by any simple function of bc/ad such as the coefficient of association

$$Q = (1 - bc/ad) \div (1 + bc/ad)$$

or the coefficient of colligation

$$Y = (1 - \sqrt{bc/ad}) \div (1 + \sqrt{bc/ad}).$$

If we use either bc/ad, Q or Y as our measure, the definition of absence of second order interaction becomes

$$adfg = bceh,$$

which is the form of the hypothesis stated by Bartlett (1935).

4. In order to obtain a clearer notion of the assumptions underlying this Bartlett test let us take an example in which, as is often the case, B and C are treatments applied to a patient of some kind, A is the classification "alive" and $\bar{A}$ the classification "dead". To describe the fatality in each of the treatment classes we shall use, not the conventional fatality rate, but the ratio of deaths to survivals. The effect of a treatment is to multiply this ratio by some factor which will be less than unity if the treatment is beneficial. Referring back to Table 1, the ratio in the untreated class $\overline{BC}$ is h/g, that in the class treated by B alone is f/e and that in the class treated by C alone

is d/c, and so the factors associated with B and C are fg/eh and dg/ch respectively. The combination of treatments BC has a factor bg/ah, and the condition for no second order interaction, $adfg = bceh$, is seen at once to be the same as

$$\frac{bg}{ah} = \frac{fg}{eh} \frac{dg}{ch},$$

that is, the factor for treatment BC is the product of the factors for B and C alone. In other words, the effect of treatment B is to multiply the dead/alive ratio by the same factor whatever other treatments may have been used; and similarly for treatment C.

5. Returning to our contingency table, it is of interest to notice the consequences of thinking in terms of fatality rates instead of the dead/alive ratio. In an example quoted by Norton (1939) the classes C and $\bar{C}$ are male and female animals, B is a treatment, and A is "alive". Norton then states that to test for second order interaction we find the difference in death rates between "controls" (untreated) and "experimentals" (treated) and compare the values of the difference in the male and female categories. In the notation of Table 1 this gives as the condition for the absence of second order interaction

$$\frac{b}{a+b} - \frac{d}{c+d} = \frac{f}{e+f} - \frac{h}{g+h}$$

or

$$\frac{bc - ad}{(a+b)(c+d)} = \frac{fg - eh}{(e+f)(g+h)}.$$

This is equivalent to taking as a measure of association in a 2×2 table

$$\frac{bc - ad}{(a+b)(c+d)},$$

which does not satisfy the symmetry condition.

6. Let us now consider the interpretation to be placed on the three first order interactions in the two cases when second order interaction does or does not exist.

7. If second order interaction does exist, the course of action is clear but dull. No two classifications can be said to be independent, for if, say, A and B are independent in C, then by hypothesis they cannot be independent in $\bar{C}$. In general it is impossible to summarize the relationship of any two classifications without reference to the third, and the only course is to set out the six possible 2×2 tables and enumerate the relationsips of A and B in C, of A and B in $\bar{C}$, and so on.

8. If, however, there is no second order interaction, there is considerable scope for paradox and error. The dangers of amalgamating 2×2 tables are well known and are referred to, for example, on page 317 of Kendall (1945), vol. I. Kendall's example illustrates that if A and B are associated positively in C and negatively in $\bar{C}$ they may appear as independent in the whole population; but a more curious case than this can be constructed. Consider the following artificial example.

9. An investigator wished to examine whether in a pack of cards the proportion of court cards (King, Queen, Knave) was associated with colour. It happened that the pack which he examined was one with which Baby had been playing, and some of the cards were dirty. He included the classification "dirty" in his scheme in case it was relevant, and obtained the following probabilities:

TABLE 2

	Dirty		*Clean*	
	Court	*Plain*	*Court*	*Plain*
Red . . .	4/52	8/52	2/52	12/52
Black . . .	3/52	5/52	3/52	15/52

It will be observed that Baby preferred red cards to black and court cards to plain, but showed no second order interaction on Bartlett's definition. The investigator deduced a positive association between redness and plainness both among the dirty cards and among the clean, yet it is the combined table

TABLE 3

	Court	*Plain*
Red	6/52	20/52
Black	6/52	20/52

which provides what we would call the sensible answer, namely, that there is no such association.

10. Suppose we now change the names of the classes in Table 2 thus:

TABLE 4

	Male		*Female*	
	Untreated	*Treated*	*Untreated*	*Treated*
Alive	4/52	8/52	2/52	12/52
Dead	3/52	5/52	3/52	15/52

The probabilities are exactly the same as in Table 2, and there is again the same degree of positive association in each of the 2×2 tables. This time we say that there is a positive association between treatment and survival both among males and among females; but if we combine the tables we again find that there is no association between treatment and survival in the combined population. What is the "sensible" interpretation here? The treatment can hardly be rejected as valueless to the race when it is beneficial when applied to males and to females.

11. It is sometimes said—for example in Norton (1939) and on page 203 of Snedecor (1946)—that provided there is no second order interaction it is permissible to add a $2 \times 2 \times 2$ table in each of the three possible ways and to test for independence in the resulting three 2×2 tables. The example just given shows that this is false. What can be said is that, assuming there is no second order interaction, the degree of association in the combined table (measured by ψ) will be the same as that in the separate tables C and $\bar{C}$ if and only if the classification C which is being submerged is independent either of A in both B and $\bar{B}$ or of B in both A and $\bar{A}$. For suppose that in Table 1 there is no second order interaction, so that $bc/ad = fg/eh$, and that the two 2×2 tables in C and $\bar{C}$ respectively are combined in the ratio $1 : \lambda$. If the resulting table is to show the same degree of association between A and B we must have

$$\frac{(b + \lambda f)(c + \lambda g)}{(a + \lambda e)(d + \lambda h)} = \frac{b\,c}{a\,d} = \frac{fg}{eh},$$

which reduces to

$$\lambda(af - be)(ag - ce)h/a = 0.$$

Thus (trivial cases apart) no linear combination of the tables will preserve the value of the association unless either $af = be$ or $ag = ce$, which represent respectively the conditions that C and A are independent in B (and consequently also in $\bar{B}$ though not necessarily in the whole population) and that C and B are independent in A. Kendall's statement that if A and B are independent in both C and $\bar{C}$ they are not independent in the population as a whole unless C is independent of A or B or both is a special case of this rule.

12. I am indebted to Professor M. S. Bartlett for having suggested, some years ago now, that this subject needed further consideration.

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