240AB Differential Geometry

Jeff A. Viaclovsky

Fall 2018 & Winter 2019

Contents

Introduction

References for basic material [?, Lee97, Spi79, War83].

More advanced references: [Bes87, Pet06, Poo81],

1 Lecture 1

1.1 Vectors, and one-forms

Let M be a smooth manifold. A vector field is a section of the tangent bundle, $X \in \Gamma(TM)$. In coordinates,

$$X = X^i \partial_i, \quad X^i \in C^\infty(M), \tag{1.1}$$

where

$$\partial_i = \frac{\partial}{\partial x^i}, \tag{1.2}$$

is the coordinate partial. We will use the Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise noted.

A 1-form is a section of the cotangent bundle, $X \in \Gamma(T^*M)$. In coordinates,

$$\omega = \omega_i dx^i, \quad \omega_i \in C^\infty(M). \tag{1.3}$$

Remark 1.1. Note that components of vector fields have upper indices, while components of 1-forms have lower indices. However, a collection of vector fields will be indexed by lower indices, $\{Y_1, \ldots, Y_p\}$, and a collection of 1-forms will be indexed by upper indices $\{dx^1, \ldots, dx^n\}$. This is one reason why we write the coordinates with upper indices.

Note that a smooth mapping $f : M \to N$ induces mappings

$$f_* : TM \to TN \tag{1.4}$$

$$f^* : T^*N \to T^*M. \tag{1.5}$$

The first mapping is defined as follows. If $X \in T_pM$, let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve satisfying $\gamma(0) = p$, $\gamma'(0) = X$. Then

$$f_*(X) = \frac{d}{dt}(f \circ \gamma)|_{t=0}. \tag{1.6}$$

Alternatively, since a tangent vector is equivalent to a linear derivation on germs of smooth functions around a point, we can define

$$(f_*X)_{f(p)}\phi = X(\phi \circ f), \tag{1.7}$$

where ϕ is a germ of a smooth function at $f(p)$.

The second mapping is then defined by

$$(f^*\omega)(v) \equiv \omega(f_* v). \tag{1.8}$$

Another way to say this is that under a mapping, we can *push forward* a vector, and *pull back* a one-form.

We always have mapping

$$f^* : \Gamma(T^*N) \to \Gamma(T^*M). \tag{1.9}$$

However, in general there is not a mapping

$$f_* : \Gamma(TM) \to \Gamma(TN), \tag{1.10}$$

but later we will be able to make sense of the following: if $X \in \Gamma(TM)$, then

$$f_* X \in \Gamma(f^*TN), \tag{1.11}$$

where f^*TN is called a *pull-back bundle*.

Note the following important proposition.

Proposition 1.2 (The chain rule)**.** *If $f : M \to N$, and $h : N \to M'$ are smooth maps, then*

$$(h \circ f)_* = h_* \circ f_* : TM \to TM' \tag{1.12}$$

$$(h \circ f)^* = f^* \circ h^* : TM' \to TM. \tag{1.13}$$

1.2 Exterior algebra and wedge product

For a real vector space V, a differential form is an element of $\Lambda^p(V^*)$. The wedge product of $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ is a form in $\Lambda^{p+q}(V^*)$ defined as follows. The exterior algebra $\Lambda(V^*)$ is the tensor algebra

$$\Lambda(V^*) = \Big\{ \bigoplus_{k\geq 0} (V^*)^{\otimes^k} \Big\} / \mathcal{I} = \bigoplus_{k \geq 0} \Lambda^k(V^*) \tag{1.14}$$

where $\mathcal{I}$ is the two-sided ideal generated by elements of the form $\alpha \otimes \alpha \in V^* \otimes V^*$. The wedge product of $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ is just the multiplication induced by the tensor product in this algebra.

The space $\Lambda^k(V^*)$ satisfies the universal mapping property as follows. Let W be any vector space, and $F : (V^*)^{\otimes^k} \to W$ an alternating multilinear mapping. That is, $F(\alpha^1, \ldots, \alpha^k) = 0$ if $\alpha^i = \alpha^j$ for some i, j. Then there is a unique linear map $\tilde{F}$ which makes the following diagram

$$\begin{tikzcd}
(V^*)^{\otimes^k} \arrow[r, "\pi"] \arrow[dr, "F"'] & \Lambda^k(V^*) \arrow[d, "\tilde{F}"] \\
& W
\end{tikzcd}$$

commutative, where π is the projection

$$\pi(\alpha^1, \ldots, \alpha^k) = \alpha^1 \wedge \cdots \wedge \alpha^k \tag{1.15}$$

We could just stick with this definition and try and prove all results using only this definition. However, for calculational purposes, it is convenient to think of differential forms as alternating linear maps from $V^{\otimes^k} \to \mathbb{R}$. For this, one has to choose a pairing

$$\Lambda^k(V^*) \cong (\Lambda^k(V))^*. \tag{1.16}$$

The pairing we will choose is as follows. If $\alpha = \alpha^1 \wedge \cdots \wedge \alpha^k$ and $v = v_1 \wedge \cdots \wedge v_k$, then

$$\alpha(v) = \det(\alpha^i(v_j)). \tag{1.17}$$

For example,

$$\alpha^1 \wedge \alpha^2(v_1 \wedge v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1). \tag{1.18}$$

Then to view as a mapping from $V^{\otimes^k} \to \mathbb{R}$, we specify that if $\alpha \in (\Lambda^k(V))^*$, then

$$\alpha(v_1, \ldots, v_k) \equiv \alpha(v_1 \wedge \cdots \wedge v_k). \tag{1.19}$$

For example

$$\alpha^1 \wedge \alpha^2(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1). \tag{1.20}$$

With this convention, if $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ then

$$\alpha \wedge \beta(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sign}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\beta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}). \tag{1.21}$$

This then agrees with the definition of the wedge product given in [Spi79, Chapter 7].

Some important properties of the wedge product

- The wedge product is bilinear $(\alpha^1 + \alpha^2) \wedge \beta = \alpha^1 \wedge \beta + \alpha^2 \wedge \beta$, and $(c\alpha) \wedge \beta = c(\alpha \wedge \beta)$ for $c \in \mathbb{R}$.
- If $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$, then $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$.
- The wedge product is associative $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.

It is convenient to have our 2 definitions of the wedge product because the proofs of these properties can be easier using one of the definitions, but harder using the other.

2 Lecture 2

2.1 Differential forms and the d operator

A differential form is a section of $\Lambda^p(T^*M)$. I.e., a differential form is a smooth mapping $\omega : M \to \Lambda^p(T^*M)$ such that $\pi\omega = Id_M$, where $\pi : \Lambda^p(T^*M) \to M$ is the bundle projection map. We will write $\omega \in \Gamma(\Lambda^p(T^*M))$, or $\omega \in \Omega^p(M)$.

Note that for a smooth mapping $f : M \to N$, we have

$$f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta). \tag{2.1}$$

Given a coordinate system $x^i : U \to \mathbb{R}$, $i = 1 \dots n$, a local basis of T^*M is given by $dx^1, \dots, dx^n$. Then $\alpha \in \Omega^p(U)$ can be written as

$$\alpha = \sum_{1 \le i_1 < i_2 < \dots < i_p \le n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}. \tag{2.2}$$

Then we also have

$$\alpha = \frac{1}{p!} \sum_{1 \le i_1, i_2, \dots, i_p \le n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}, \tag{2.3}$$

where the sum is over ALL indices.

However, if we want to think of α as a multilinear mapping from $TM^{\otimes^p} \to \mathbb{R}$, then we extend the coefficients $\alpha_{i_1 \dots i_p}$, which are only defined for strictly increasing sequences $i_1 < \dots < i_p$, to ALL indices by skew-symmetry. Then we have

$$\alpha = \sum_{1 \le i_1, i_2, \dots, i_p \le n} \alpha_{i_1 \dots i_p} dx^{i_1} \otimes \dots \otimes dx^{i_p}. \tag{2.4}$$

This convention is slightly annoying because then the projection to the exterior algebra of this is $p!$ times the original α, but has the positive feature that coefficients depending upon p do not enter into various formulas.

The exterior derivative operator [War83, Theorem 2.20],

$$d : \Omega^p(T^*M) \to \Omega^{p+1}(T^*M) \tag{2.5}$$

is the unique anti-derivation satisfying

- For $\alpha \in \Omega^p(M)$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.
- $d^2 = 0$.
- If $f \in C^\infty(M)$ then df is the differential of f. (I.e., $f_* : TM \to \mathbb{R}$ is a element of $Hom(TM, \mathbb{R})$ which is unambiguously an element of $\Gamma(T^*M) = \Omega^1(M)$.)

Next, letting $Alt^p(TM)$ denote the alernating multilinear maps from $TM^{\otimes^p} \to \mathbb{R}$, then d can be considered as a mapping

$$d : Alt^p(TM) \to Alt^{p+1}(TM) \tag{2.6}$$

given by the formula

$$d\omega(X_0,\dots,X_p)=\sum_{j=0}^{p}(-1)^j X_j\Big(\omega(X_0,\dots,\hat{X}_j,\dots,X_p)\Big) \\ +\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p), \tag{2.7}$$

which agrees with the formula for d given in [Spi79, Chapter 7].

Note that in a coordinate system, d is given by

$$(d\alpha)_{i_0\dots i_p}=\sum_{j=0}^{p}(-1)^j\partial_{i_j}\alpha_{i_0\dots\hat{i}_j\dots i_p}. \tag{2.8}$$

(Note this is indeed skew-symmetric in all indices.)

An important fact is that d commutes with pull-back.

Proposition 2.1. *If $f: M\to N$ is a smooth mapping, and $\omega\in\Omega^p(N)$, then*

$$f^*(d\omega)=d(f^*\omega). \tag{2.9}$$

Another important fact is that we can integrate top-dimensional differential forms on a compact manifold. But we need to recall orientability. First, an orientation on a n-dimensional vector space V is a choice of ordered basis $(v_1,\dots,v_n)$ with equivalence relation if 2 ordered bases are related by a change of basis matrix with positive determinant. There are exactly 2 such equivalence classes, and if M is a manifold, the oriented double cover of M denoted by $\tilde{M}$ is the double cover obtained by replacing a point p with the 2 orientations on T_pM.

Definition 2.2. A manifold M is orientable if any of the following equivalent conditions are satisfied.

- M admits an coordinate atlas (U_α,ϕ_α) such that the overlap maps are orientation-preserving $\phi_\alpha\circ\phi_\beta^{-1}$, that is, the Jacobian $(\phi_\alpha\circ\phi_\beta^{-1})_*$ has positive determinant.
- M admits a nowhere-zero n-form.
- The oriented double cover $\tilde{M}\to M$ is trivial, i.e., it has 2 components.

If M is orientable, the choice of one of the components of $\tilde{M}$ is called an *orientation* on M.

On an oriented n-dimensional manifold, the integral of $\omega\in\Omega^n(M)$ is defined as follows. Choose an oriented coordinate atlas (U_α,ϕ_α). First, assume that $\omega\in\Omega^n(M)$ has compact support in a single coordinate system U_α. Then

$$(\phi_\alpha)_*(\omega)=f\,dx^1\wedge\cdots\wedge dx^n, \tag{2.10}$$

where $f : \phi_\alpha(U_\alpha) \to \mathbb{R}$ has compact support. Define

$$\int_M \omega \equiv \int_{\phi_\alpha(U_\alpha)} f dx^1 \dots dx^n. \tag{2.11}$$

By the change-of-variables formula for integrals, this definition is independent of coordinate system containing the support of ω.

Next, if M is compact, or if ω has compact support, let χ_α be a partition of unity subordinate to U_α, and define

$$\int_M \omega = \sum_\alpha \int_M \chi_\alpha \omega. \tag{2.12}$$

Since the sum is finite, this definition is independent of the choice of coordinate atlas and choice of partition of unity.

Integration by parts on manifolds is the following.

Theorem 2.3 (Stokes' Theorem). *Let $(M, \partial M)$ be a compact oriented manifold with boundary of dimension n. If $\omega \in \Omega^{n-1}(M)$, then*

$$\int_{\partial M} \omega = \int_M d\omega, \tag{2.13}$$

where the boundary has the orientation induced from the outer normal, i.e., if $v_i \in T_p(\partial M)$, then the ordered basis $(v_1, \dots v_{n-1})$ is oriented if $(v, v_1, \dots, v_{n-1})$ is positively oriented, for any outward pointing normal vector v.

3 Lecture 3

3.1 Classical tensor calculus

A vector field is a section of the tangent bundle, $X \in \Gamma(TM)$, and the components of X with respect to a coordinate system $x : U \to \mathbb{R}^n$ are functions $X^i : U \to \mathbb{R}$, $i = 1 \dots n$, defined by

$$X = X^i \frac{\partial}{\partial x^i} \tag{3.1}$$

on U, where $\frac{\partial}{\partial x^i}$ is the ith coordinate partial, which is a vector field on TU. Given another overlapping coordinate system $\tilde{x} : U \to \mathbb{R}^n$, we can write

$$X = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}. \tag{3.2}$$

Proposition 3.1. *The components of a vector field are related by*

$$\tilde{X}^j = \frac{\partial \tilde{x}^j}{\partial x^i} X^i. \tag{3.3}$$

Conversely, any collection of locally-defined functions satisfying this relation gives a well defined vector field $X \in \Gamma(TM)$.

Proof. Since vector fields are derivations on germs of functions, plug in the function $\tilde{x}^j$ to the equality

$$X^i \frac{\partial}{\partial x^i} = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}, \tag{3.4}$$

to obtain

$$X^i \frac{\partial}{\partial x^i}(\tilde{x}^j) = \tilde{X}^j. \tag{3.5}$$

□

Similarly, a 1-form is a section of the cotangent bundle, $\omega \in \Gamma(T^*M)$, and the components of ω with respect to a coordinate system $x : U \to \mathbb{R}^n$ are functions $\omega_i : U \to \mathbb{R}$, $i = 1 \dots n$, defined by

$$\omega = \omega_i dx^i \tag{3.6}$$

on U. Given another overlapping coordinate system $\tilde{x} : U \to \mathbb{R}^n$, we can write

$$\omega = \tilde{\omega}_i d\tilde{x}^i. \tag{3.7}$$

Proposition 3.2. *The components of a 1-form are related by*

$$\tilde{\omega}_j = \frac{\partial x^i}{\partial \tilde{x}^j}\omega_i. \tag{3.8}$$

Conversely, any collection of locally-defined functions satisfying this relation gives a well defined 1-form $\omega \in \Gamma(T^*M)$.

Proof. Plug in the vector field $\frac{\partial}{\partial \tilde{x}^j}$ to the equality

$$\omega_i dx^i = \tilde{\omega}_i d\tilde{x}^i, \tag{3.9}$$

to obtain

$$\omega_i dx^i \left(\frac{\partial}{\partial \tilde{x}^j} \right) = \tilde{\omega}_j. \tag{3.10}$$

But recall the definition of df, where $f : U \to \mathbb{R}$ is a function. We claim that

$$df(X) = X(f). \tag{3.11}$$

To see this, the left hand side is

$$df(X) = \frac{\partial f}{\partial x^i} dx^i \left(X^j \frac{\partial}{\partial x^j} \right) = \frac{\partial f}{\partial x^i} X^i. \tag{3.12}$$

For the right hand side, let $\gamma : (-\epsilon, \epsilon) \to M$ satisfy $\gamma(0) = p$, $\gamma'(0) = X_p$, then

$$X(f) = \frac{d}{dt}(f \circ \gamma)|_{t=0} = \frac{\partial f}{\partial x^i} \frac{d\gamma^i}{dt}_{t=0} = \frac{\partial f}{\partial x^i} X^i_p. \tag{3.13}$$

Then plugging (3.11) into (3.10), we have

$$\tilde{\omega}_j = \omega_i \frac{\partial x^i}{\partial \tilde{x}^j}. \tag{3.14}$$

□

For a general tensor $T \in \Gamma((TM)^{\otimes p} \otimes (T^*M)^{\otimes q})$ we can locally write

$$T = T^{j_1 \dots j_p}_{i_1 \dots i_q} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_p}} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_q}, \tag{3.15}$$

and in another coordinate system

$$T = \tilde{T}^{j_1 \dots j_p}_{i_1 \dots i_q} \frac{\partial}{\partial \tilde{x}^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial \tilde{x}^{j_p}} \otimes d\tilde{x}^{i_1} \otimes \dots \otimes d\tilde{x}^{i_q}, \tag{3.16}$$

The above transformation formulas combine to give the following.

Proposition 3.3. *The components of T satisfy the transformation formulas*

$$\tilde{T}^{j_1 \dots j_p}_{i_1 \dots i_q} = \frac{\partial \tilde{x}^{j_1}}{\partial x^{l_1}} \cdots \frac{\partial \tilde{x}^{j_p}}{\partial x^{l_p}} \frac{\partial x^{k_1}}{\partial \tilde{x}^{i_1}} \cdots \frac{\partial x^{k_q}}{\partial \tilde{x}^{i_q}} T^{l_1 \dots l_p}_{k_1 \dots k_q} \tag{3.17}$$

Conversely, any collection of locally-defined functions satisfying this relation gives a well defined tensor $T \in \Gamma(TM^{\otimes p} \otimes T^*M^{\otimes q})$.

Exercise 3.4. Show that the Kronecker δ symbol, defined by

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{3.18}$$

defines a tensor. Consequently,

$$T = \delta^i_j \frac{\partial}{\partial x^i} \otimes dx^j \tag{3.19}$$

is a well-defined global tensor. Show that under the canonical isomorphisms

$$TM \otimes T^*M \cong T^*M \otimes T \cong Hom(TM, TM), \tag{3.20}$$

the tensor T corresponds to the identity transformation $Id : TM \to TM$.

We note that for a n-form, we can write

$$\omega = \omega_{1 \dots n} dx^1 \wedge \dots \wedge dx^n, \tag{3.21}$$

In another coordinate system, we can write

$$\omega = \tilde{\omega}_{1 \dots n} d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n, \tag{3.22}$$

These components are related by

$$\tilde{\omega}_{1 \dots n} = \det \left(\frac{\partial x^i}{\partial \tilde{x}^j} \right) \omega_{1 \dots n}, \tag{3.23}$$

which is why the integral is well-defined. If M is not orientable, we can define a *density* to be a collection of function so that under coordinate changes,

$$\tilde{\omega}_{1 \dots n} = \left| \det \left(\frac{\partial x^i}{\partial \tilde{x}^j} \right) \right| \omega_{1 \dots n}, \tag{3.24}$$

It turns out that these quantities are sections of a trivial 1-dimension line bundle, but their integral is well-defined, even on a non-orientable manifold.

Remark 3.5. Since a density bundle is just a trivial bundle, it seems we could define an integral for section of any trivial line bundle. But this is not possible: to define the integral of densities you need to look at how these behave under changes of coordinates systems on the *base* manifold, not for an arbitrary trivialization of the bundle.

4 Lecture 4

4.1 Lie derivatives

Given a vector field $X \in \Gamma(TM)$, the Lie derivative of Y with respect to X is

$$\mathcal{L}_X Y = [X, Y], \tag{4.1}$$

where $[X, Y]f = X(Yf) - Y(Xf)$

Proposition 4.1. *For $X, Y \in \Gamma(TM)$, the bracket $[X, Y] \in \Gamma(TM)$.*

Proof. In a local coordinate system, write

$$X = X^i \frac{\partial}{\partial x^i}, \ Y = Y^i \frac{\partial}{\partial x^i}, \tag{4.2}$$

then

$$\begin{aligned} [X, Y]f &= X^i \frac{\partial}{\partial x^i} \left(Y^j \frac{\partial f}{\partial x^j} \right) - Y^j \frac{\partial}{\partial x^j} \left(X^i \frac{\partial f}{\partial x^i} \right) \\ &= X^i \left(\frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} + Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} \right) - Y^j \left(\frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} + X^i \frac{\partial^2 f}{\partial x^j \partial x^i} \right). \end{aligned} \tag{4.3}$$

Since f is smooth, we have equality of the mixed partials, so

$$\begin{aligned} [X, Y]f &= X^i \left(\frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} \right) - Y^j \left(\frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} \right) \\ &= \left(X^i \frac{\partial Y^l}{\partial x^i} - Y^j \frac{\partial X^l}{\partial x^j} \right) \frac{\partial f}{\partial x^l}. \end{aligned} \tag{4.4}$$

This shows that $[X, Y]$ is a derivation on germs of function, so is a well-defined vector field. Alternatively, using the classical method, we can prove this directly as follows,

$$\begin{aligned} \tilde{X}^i \frac{\partial \tilde{Y}^l}{\partial \tilde{x}^i} - \tilde{Y}^j \frac{\partial \tilde{X}^l}{\partial \tilde{x}^j} &= \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i} \left(Y^k \frac{\partial \tilde{x}^l}{\partial x^k} \right) - \tilde{Y}^j \frac{\partial}{\partial \tilde{x}^j} \left(X^k \frac{\partial \tilde{x}^l}{\partial x^k} \right) \\ &= X^i \frac{\partial}{\partial x^i} \left(Y^k \frac{\partial \tilde{x}^l}{\partial x^k} \right) - Y^j \frac{\partial}{\partial x^j} \left(X^k \frac{\partial \tilde{x}^l}{\partial x^k} \right) \\ &= \left(X^i \frac{\partial Y^k}{\partial x^i} - Y^j \frac{\partial X^k}{\partial x^j} \right) \frac{\partial \tilde{x}^l}{\partial x^k}, \end{aligned} \tag{4.5}$$

since the mixed partial terms cancel out, thus showing $[X, Y]$ is a globally defined vector field. □

Next, for $X, Y \in \Gamma(TM)$, and $\omega \in \Gamma(T^*M)$, define

$$\mathcal{L}_X \omega(Y) = X(\omega(Y)) - \omega(\mathcal{L}_X Y). \tag{4.6}$$

Proposition 4.2. *If $X \in \Gamma(TM)$ and $\omega \in \Gamma(T^*M)$, then $\mathcal{L}_X \omega \in \Gamma(T^*M)$.*

Proof. Let $f : M \to \mathbb{R}$. Then

$$\begin{aligned}\mathcal{L}_X\omega(fY) &= X(\omega(fY)) - \omega(\mathcal{L}_X(fY))\\ &= X(f\omega(Y) - \omega([X, fY])\\ &= (Xf)\omega(Y) + fX(\omega(Y)) - \omega(f[X.Y] - (Xf)Y)\\ &= fX(\omega(Y)) - \omega(f[X,Y]) = f\mathcal{L}_X\omega(Y).\end{aligned} \tag{4.7}$$

Since this expression is linear over C^∞ functions, it is a well-defined tensor. To see this, let $\alpha : \Gamma(TM) \to C^\infty(M)$ be a mapping which is linear over C^∞-functions. It sufices to show that $\alpha(X)(p) = 0$ if $X_p = 0$. This is because if we let X and $\tilde{X}$ be any smooth extensions of X_p, then since $X - \tilde{X}$ vanishes at p

$$\omega(X - \tilde{X})(p) = 0, \tag{4.8}$$

so $\omega(X)(p) = \omega(\tilde{X})(p)$ has a well-defined value, independent of the extension of X_p. To proceed, given a coordinate system around p, choose a cutoff function which is 1 in a coordinate neighborhood of p, and 0 outside. Then

$$X = (\phi X^i)(\phi\frac{\partial}{\partial x^i}) + (1 - \phi^2)X. \tag{4.9}$$

Both terms in the above are smooth vector fields on M, so using linearity,

$$\alpha(X)(p) = (\phi(p)X^i(p))\alpha(\phi\frac{\partial}{\partial x^i})(p) + (1 - \phi^2)(p)\alpha(X)(p) = 0. \tag{4.10}$$

□

Next, consider a (p, q)-*tensor field*

$$\Omega \in \Gamma\Big((TM)^{\otimes p} \otimes (T^*M)^{\otimes q}\Big). \tag{4.11}$$

We define $\mathcal{L}_X\Omega$ as follows. For any tensor product of tensors, define

$$\nabla_X(s \otimes s') = (\nabla_X s) \otimes s' + s \otimes (\nabla'_X s'). \tag{4.12}$$

For example,

$$\begin{aligned}\mathcal{L}_X(Y \otimes \omega) &= \mathcal{L}_X(Y) \otimes \omega + Y \otimes \mathcal{L}_X\omega\\ &= [X, Y] \otimes \omega + Y \otimes \mathcal{L}_X\omega,\end{aligned} \tag{4.13}$$

where the last term is defined in (4.6).

We can also define a Lie derivative operator on differential forms in $\Lambda^p(M)$ by

$$\mathcal{L}_X(\omega_1 \wedge \cdots \wedge \omega_p) = \sum_{i=1}^{p} \omega_1 \wedge \cdots \wedge (\mathcal{L}_X\omega_i) \wedge \cdots \wedge \omega_p, \tag{4.14}$$

for $\omega_i \in \Gamma(T^*M)$. There is a analogous formula for the Lie derivative as (2.7)

$$\begin{aligned}(\mathcal{L}_X\omega)(X_1,\dots,X_p) = X\Big(\omega(X_1,\dots,\dots,X_p)\Big)\\ + \sum_{i=1}^{p}(-1)^i\omega([X,X_i],X_1,\dots,\hat{X}_i,\dots,X_p).\end{aligned} \tag{4.15}$$

The Lie derivative operator can be defined by using the 1-parameter group of diffeomorphisms generated by X via

$$\mathcal{L}_X Y = \frac{d}{dt}(\Phi_{-t})_* Y\Big|_{t=0} \tag{4.16}$$

$$\mathcal{L}_X\omega = \frac{d}{dt}(\Phi_t)^*\omega\Big|_{t=0}, \tag{4.17}$$

with similar formulas for higher tensor fields.

An important formula is Cartan's formula relating the Lie derivative and the exterior derivative: if $\omega \in \Omega^p(M)$, then

$$\mathcal{L}_X\omega = d(X\lrcorner\omega) + X\lrcorner d\omega, \tag{4.18}$$

where the interior product $X\lrcorner : \Omega^r(M) \to \Omega^{r-1}(M)$ is defined by

$$X\lrcorner\alpha(X_1,\dots,X_{r-1}) = \alpha(X,X_1,\dots,X_{r-1}). \tag{4.19}$$

Note Cartan's formula implies that

$$\mathcal{L}_X(d\omega) = d(\mathcal{L}_X\omega). \tag{4.20}$$

Here is an important point: the expression $\mathcal{L}_X\omega$ is NOT tensorial in the variable X. In fact, we have the formula

$$\mathcal{L}_{f\omega} = f\mathcal{L}_X\omega + df \wedge (X\lrcorner\omega), \tag{4.21}$$

To obtain a derivative which is tensorial in X will lead us to the concept of a *connection*.

5 Lecture 5

5.1 Riemannian metrics

Let (M,g) be a Riemannian manifold, with metric $g \in \Gamma(S^2(T^*M))$. In coordinates,

$$g = \sum_{i,j=1}^{n} g_{ij}(x)dx^i \otimes dx^j,\ \ g_{ij} = g_{ij}, \tag{5.1}$$

and $g_{ij} >> 0$ is a positive definite matrix. The symmetry condition is of course invariantly

$$g(X,Y) = g(Y,X). \tag{5.2}$$

Note that any manifold admits a Riemannian metric, by using a partition of unity to patch together the Euclidean metric in local coordinates.

5.2 The musical isomorphisms

The metric gives an isomorphism between TM and T^*M,

$$\flat : TM \to T^*M \tag{5.3}$$

defined by

$$\flat(X)(Y) = g(X, Y). \tag{5.4}$$

The inverse map is denoted by $\sharp : T^*M \to TM$. The cotangent bundle is endowed with the metric

$$\langle \omega_1, \omega_2 \rangle = g(\sharp\omega_1, \sharp\omega_2). \tag{5.5}$$

Note that if g has components g_{ij}, then $\langle \cdot, \cdot \rangle$ has components g^{ij}, the inverse matrix of g_{ij}.

If $X \in \Gamma(TM)$, then

$$\flat(X) = X_i dx^i, \tag{5.6}$$

where

$$X_i = g_{ij} X^j, \tag{5.7}$$

so the flat operator "lowers" an index. If $\omega \in \Gamma(T^*M)$, then

$$\sharp(\omega) = \omega^i \partial_i, \tag{5.8}$$

where

$$\omega^i = g^{ij} \omega_j, \tag{5.9}$$

thus the sharp operator "raises" an index.

5.3 Inner product on tensor bundles

The metric induces a metric on $\Lambda^k(T^*M)$. We give 3 definitions, all of which are equivalent:

- Definition 1: If

$$\begin{aligned} \omega^1 &= \alpha^1 \wedge \cdots \wedge \alpha^k \\ \omega^2 &= \beta^1 \wedge \cdots \wedge \beta^k, \end{aligned} \tag{5.10}$$

  then

$$\langle \omega^1, \omega^2 \rangle = \det(\langle \alpha^i, \beta^j \rangle), \tag{5.11}$$

  and extend linearly. This is well-defined.

- Definition 2: If $\{e_i\}$ is an ONB of T_pM, let $\{e^i\}$ denote the dual basis, defined by $e^i(e_j) = \delta^i_j$. Then declare that

$$e^{i_1} \wedge \cdots \wedge e^{i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n, \tag{5.12}$$

is an ONB of $\Lambda^k(T^*_pM)$.

- Definition 3: If $\omega \in \Lambda^k(T^*M)$, then in coordinates

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{5.13}$$

Then

$$\|\omega\|^2_{\Lambda^k} = \langle \omega, \omega \rangle = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega^{i_1 \ldots i_k} \omega_{i_1 \ldots i_k}, \tag{5.14}$$

where

$$\omega^{i_1 \ldots i_k} = \sum_{1 \leq l_1 < \cdots < l_k \leq n} g^{i_1 l_i} g^{i_2 l_2} \ldots g^{i_k l_k} \omega_{l_1 \ldots l_k}. \tag{5.15}$$

To define an inner product on the full tensor bundle, we let

$$\Omega \in \Gamma\Big((TM)^{\otimes p} \otimes (T^*M)^{\otimes q} \Big). \tag{5.16}$$

We call such Ω a (p, q)-*tensor field*. As above, we can define a metric by declaring that

$$e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_q} \tag{5.17}$$

to be an ONB. If in coordinates,

$$\Omega = \Omega^{i_1 \ldots i_p}_{j_1 \ldots j_q} \partial_{i_1} \otimes \cdots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}, \tag{5.18}$$

then

$$\|\Omega\|^2 = \langle \Omega, \Omega \rangle = \Omega^{j_1 \ldots j_q}_{i_1 \ldots i_p} \Omega^{i_1 \ldots i_p}_{j_1 \ldots j_q}, \tag{5.19}$$

where the term $\Omega^{j_1 \ldots j_q}_{i_1 \ldots i_p}$ is obtained by raising all of the lower indices and lowering all of the upper indices of $\Omega^{j_1 \ldots j_q}_{i_1 \ldots i_p}$, using the metric. By polarization, the inner product is given by

$$\langle \Omega_1, \Omega_2 \rangle = \frac{1}{2} \Big(\|\Omega_1 + \Omega_2\|^2 - \|\Omega_1\|^2 - \|\Omega_2\|^2 \Big). \tag{5.20}$$

Remark 5.1. Recall we are using (1.17) to identify forms and alternating tensors. If $\omega \in \Lambda^p(T^*M)$, then if we view ω as an alternating p-tensor, then

$$\|\omega\|_{(T^*M)^{\otimes p}} = \sqrt{p!} \|\omega\|_{\Lambda^p}. \tag{5.21}$$

For example, as an element of $\Lambda^2(T^*M)$, $e^1 \wedge e^2$ has norm 1 if e^1, e^2 are orthonormal in T^*M. But under our identification with tensors, $e^1 \wedge e^2$ is identified with $e^1 \otimes e^2 - e^2 \otimes e^1$, which has norm $\sqrt{2}$ with respect to the tensor inner product. Thus our identification in (1.17) is *not* an isometry, but is a constant multiple of an isometry.

We remark that one may reduce a (p,q)-tensor field into a $(p-1,q-1)$-tensor field for $p \geq 1$ and $q \geq 1$. This is called a *contraction*, but one must specify which indices are contracted. For example, the contraction of Ω in the first contrvariant index and first covariant index is written invariantly as

$$Tr_{(1,1)}\Omega, \tag{5.22}$$

and in coordinates is given by

$$\delta^{j_1}_{i_1}\Omega^{i_1\dots i_p}_{j_1\dots j_q} = \Omega^{li_2\dots i_p}_{lj_2\dots j_q}. \tag{5.23}$$

6 Lecture 6

6.1 Review of theory of vector bundles

We will next define real vector bundles, but note that everything we will say works for complex bundles, by replacing $\mathbb{R}$ with $\mathbb{C}$.

Definition 6.1. A smooth real vector bundle of rank k over a smooth manifold M^n is a topological space E together with a smooth projection

$$\pi : E \to M \tag{6.1}$$

such that

- For $p \in M$, $\pi^{-1}(p)$ is a vector space of dimension k over $\mathbb{R}$.
- There exists local trivializations, that is, there are smooth mappings

  $$\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to E \tag{6.2}$$

  which maps $p \times \mathbb{R}^k$ linearly onto the fiber $\pi^{-1}(p)$ for every $p \in U_\alpha$.

The transition functions of a bundle are defined as follows.

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k,\mathbb{R}) \tag{6.3}$$

defined by

$$\varphi_{\alpha\beta}(x)(v) = \pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x,v)), \tag{6.4}$$

for $v \in \mathbb{R}^k$.

On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, we have the identity

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}. \tag{6.5}$$

Conversely, given a covering U_α of M and transition functions $\varphi_{\alpha\beta}$ satifsying (6.5), there is a vector bundle $\pi : E \to M$ with transition functions given by $\varphi_{\alpha\beta}$. (It turns out this bundle is uniquely defined up to bundle equivalence, which we will define below.) If the transitions function $\varphi_{\alpha\beta}$ are C^∞, then we say that E is a smooth vector bundle.

Example 6.2. (The tangent bundle) Given a coordinate system (U_α, x_α) on a smooth manifold M, let

$$\Phi_\alpha(x, (v^1, \ldots, v^n)) = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x^i_\alpha}. \tag{6.6}$$

On U_β, we have

$$\Phi_\beta(x, (\tilde{v}^1, \ldots, \tilde{v}^n)) = \sum_{i=1}^{n} \tilde{v}_i \frac{\partial}{\partial x^i_\beta}. \tag{6.7}$$

Recall that

$$\frac{\partial}{\partial x^i_\beta} = \sum_{j=1}^{n} \frac{\partial x^j_\alpha}{\partial x^i_\beta} \frac{\partial}{\partial x^j_\alpha}, \tag{6.8}$$

so then

$$\begin{aligned} \Phi_\alpha^{-1} \circ \Phi_\beta(x, (\tilde{v}^1, \ldots, \tilde{v}^n)) &= \Phi_\alpha^{-1}\left(\sum_{i=1}^{n} \tilde{v}^i \frac{\partial}{\partial x^i_\beta} \right) \\ &= \Phi_\alpha^{-1}\left(\sum_{i=1}^{n} \tilde{v}^i \sum_{j=1}^{n} \frac{\partial x^j_\alpha}{\partial x^i_\beta} \frac{\partial}{\partial x^j_\alpha} \right) \\ &= \Phi_\alpha^{-1}\left(\sum_{j=1}^{n} \left(\sum_{i=1}^{n} \tilde{v}^i \frac{\partial x^j_\alpha}{\partial x^i_\beta} \right) \frac{\partial}{\partial x^j_\alpha} \right). \end{aligned} \tag{6.9}$$

Consequently,

$$\varphi_{\alpha\beta}(x)(v^1, \ldots, v^n) = \sum_{i=1}^{n} \tilde{v}^i \frac{\partial x^j_\alpha}{\partial x^i_\beta} \tag{6.10}$$

A vector bundle mapping is a mapping $f : E_1 \to E_2$ which is linear on fibers, and covers the identity map. Assume we have a covering U_α of M such that E_1 has trivializations Φ_α and E_2 has trivializations Ψ_α. Then any vector bundle mapping gives locally defined functions

$$f_\alpha : U_\alpha \to Hom(\mathbb{R}^{k_1}, \mathbb{R}^{k_2}) \tag{6.11}$$

defined by

$$f_\alpha(x)(v) = \pi_2(\Psi_\alpha^{-1} \circ F \circ \Phi_\alpha(x, v)). \tag{6.12}$$

It is easy to see that on overlaps $U_\alpha \cap U_\beta$,

$$f_\alpha = \varphi^{E_2}_{\alpha\beta} f_\beta \varphi^{E_1}_{\beta\alpha}, \tag{6.13}$$

equivalently,

$$\varphi^{E_2}_{\beta\alpha} f_\alpha = f_\beta \varphi^{E_1}_{\beta\alpha}. \tag{6.14}$$

We say that two bundles are E_1 and E_2 are equivalent if there exists an invertible bundle mapping $f : E_1 \to E_2$. This is equivalent to non-singularity of the local representatives, that is, $\det(f_\alpha) \neq 0$. A vector bundle is *trivial* if it is equivalent to the trivial product bundle. That is, E is trivial if there exist functions

$$f_\alpha : U_\alpha \to GL(k, \mathbb{R}) \tag{6.15}$$

such that

$$\varphi_{\beta\alpha} = f_\beta f_\alpha^{-1}. \tag{6.16}$$

7 Lecture 7

7.1 Operations on bundles

The direct sum $E_1 \oplus E_2$ of bundles E_1 and E_2 is a vector bundle with transition functions

$$\varphi^{E_1 \oplus E_2}_{\alpha\beta} = \varphi^{E_1}_{\alpha\beta} \oplus \varphi^{E_2}_{\alpha\beta}. \tag{7.1}$$

The tensor product $E_1 \otimes E_2$ of bundles E_1 and E_2 is again a bundle, and has transition functions

$$\varphi^{E_1 \otimes E_2}_{\alpha\beta} = \varphi^{E_1}_{\alpha\beta} \otimes \varphi^{E_2}_{\alpha\beta}. \tag{7.2}$$

The dual E^* of any bundle E, is a bundle, and has transition functions

$$\varphi^{E^*}_{\alpha\beta} = \left((\varphi^{E}_{\alpha\beta})^{-1}\right)^T = (\varphi^{E}_{\beta\alpha})^T. \tag{7.3}$$

Note that for any linear map $f : \mathbb{R}^k \to \mathbb{R}^k$, there is a naturally induced mapping

$$\Lambda^p f : \Lambda^p(\mathbb{R}^k) \to \Lambda^p(\mathbb{R}^k) \tag{7.4}$$

therefore for any vector bundle E, the pth exterior power $\Lambda^p(E)$ is defined to be the bundle with transition functions

$$\varphi^{\Lambda^p(E)}_{\alpha\beta} = \Lambda^p(\varphi^{E}_{\alpha\beta}). \tag{7.5}$$

For a complex vector bundle $\pi : E \to M$, there is another operation called the conjugate bundle $\overline{E}$ which is the complex vector bundle obtained by replacing each fiber of E with the complex conjugate vector space. The transition functions are simply

$$\varphi^{\overline{E}}_{\alpha\beta} = \overline{\varphi^{E}_{\alpha\beta}}. \tag{7.6}$$

Remark 7.1. In the above, we only defined morphisms in the category of vector bundle to be mappings covering the identity map. We could have instead morphisms to cover arbitrary diffeomorphisms. This would lead to a coarser notion of equivalence. More on this later.

7.2 Riemannian metrics on real vector bundles

If $\pi : E \to M$ is a real vector bundle, a Riemannian metric on E is a choice of smoothly varying positive definite symmetric inner product on each fiber. That is $g \in \Gamma(E^* \otimes E^*)$ satisfying

$$g(e_1, e_2) = g(e_2, e_1), \tag{7.7}$$

and

$$g(e, e) > 0 \text{ for } e \neq 0. \tag{7.8}$$

Proposition 7.2. *If E is any real vector bundle, then E admits a Riemannian metric.*

Proof. Take the Euclidean metric on trivializations, and patch together using a partition of unity. □

Corollary 7.3. *For any real vector bundle E, $E^* \cong E$.*

Proof. Choose a Riemannian metric g on E. Then the mapping $\flat : E \to E^*$ defined by

$$\flat(e_1)(e_2) = g(e_1, e_2) \tag{7.9}$$

is an ismorphism on fibers, and covers the identity map. □

In bundle terms, existence of a Riemannian metric implies that there is always a non-zero section of $S^2(E^*)$, which says that

$$S^2(E^*) = A \oplus B \tag{7.10}$$

always admits a trivial 1-dimensional subbundle. Of course, the metric gives a isomorphism

$$E^* \otimes E^* \cong E^* \otimes E \cong Hom(E, E), \tag{7.11}$$

and the latter bundle always admits the identity section. The latter choice is canonical, but the sub-bundle A is not.

Note the following corollary.

Corollary 7.4. *If $E_1 \subset E$ is a sub-bundle, then there exists a subbundle $E_2 \subset E$ such that*

$$E \cong E_1 \oplus E_2. \tag{7.12}$$

Furthermore, the quotient bundle $(E/E_1) \cong E_2$.

Proof. Choose a Riemannian metric g on E, and let $E_2 = (E_1)^\perp$. Use Gram-Schmidt to construct local trivializations for $(E_1)^\perp$ to show this is indeed a subbundle. □

7.3 Hermitian metrics on complex vector bundles

If $\pi : E \to M$ is a complex vector bundle, a Hermitian metric on E is a choice of smoothly varying Hermitian inner product on each fiber. That is $h \in \Gamma(E^* \otimes \overline{E}^*)$ satisfying

$$g(e_1, e_2) = \overline{g(e_2, e_1)} \tag{7.13}$$

and

$$g(e, \overline{e}) > 0 \text{ for } e \neq 0. \tag{7.14}$$

Proposition 7.5. *If E is any complex vector bundle, then E admits a hermitian metric.*

Proof. Take the Euclidean metric on $\mathbb{C}^n$, i.e.,

$$h_{Euc}(v, w) = \sum v_j \overline{w}_j \tag{7.15}$$

on trivializations, and patch together using a partition of unity. □

Corollary 7.6. *For any complex vector bundle E, we have $\overline{E}^* \cong E$. Equivalently, $\overline{E} \cong E^*$.*

Proof. Choose a hermitian metric h on E. Define the mapping $\flat : E \to \overline{E}^*$ by

$$\flat(e_1)(e_2) = h(e_1, e_2) \tag{7.16}$$

Note that $\bar{\flat}(e_1)$ is a complex anti-linear mapping E to $\mathbb{C}$, and thus in indeed an element of $\overline{E}^*$. It is easy to see this is an isomorphism. □

8 Lecture 8

8.1 Reduction of Structure group

Definition 8.1. If a bundle $\pi : E \to M$ is equivalent to a bundle which has transition functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to K$, where K is a subgroup of $GL(k, \mathbb{R})$ (or $GL(k, \mathbb{C})$), then we say that the structure group of E can be *reduced* to K.

Another way to state the results from the previous section is as follows.

Proposition 8.2. *We have the following.*

- *A bundle is trivial if and only if its structure group can be reduced to $\{Id\}$.*
- *The structure group of any real vector bundle $\pi : E \to M$ of rank k can be reduced to $O(k)$ if and only if E admits a Riemannian metric.*
- *The structure group of any complex vector bundle $\pi : E \to M$ of rank k can be reduced to $U(k)$ if and only if E admits a Hermitian metric.*

Proof. The first case is clear. For the second cases, if E admits a Riemannian metric, then consider only bundle charts given by local orthonormal frames. Then overlaps maps then necessarily lie in $O(k)$. Conversely, if the overlap maps lie in $O(k)$, then just patch together the Euclidean metric using the corresponding bundle charts. The complex case is analogous using Hermitian frames. □

8.2 Real line bundles

Note for a real 1-dimensional line bundle $\pi : L \to M$, we have that the structure group can be reduced to $O(1) = \{\pm 1\}$, or equivalently, there exists a Riemannian metric g on L. Consider the set

$$\tilde{M} = \{v \in L \,|g(v,v) = 1\} \tag{8.1}$$

Since there are exactly two unit norm vectors in any fiber, we have that $\pi : \tilde{M} \to M$ is a 2-fold covering space. So any real line bundle give an associated 2-fold covering space. Conversely, any 2-fold covering space gives a real line bundle, which is uniquely determined up to equivalence. To see this, note that a 2-fold covering space can be viewed as a fiber bundle with group $\mathbb{Z}_2$, and viewing $\mathbb{Z}_2 = \{\pm 1\} \subset GL(1, \mathbb{R})$, we naturally obtain an associated real line bundle.

Using some basic topology, we have the isomorphisms

$$H^1(M, \mathbb{Z}_2) \cong Hom(H_1(M), \mathbb{Z}_2) \cong Hom(\pi_1(M), \mathbb{Z}_2). \tag{8.2}$$

The latter space corresponds to index 2 subgroups of $\pi_1(M)$, so corresponds to 2-fold coverings of M. Consequently, we have proved the following.

Proposition 8.3. *The real line bundles on M up to bundle equivalence, are in one-one correspondence with $H^1(M, \mathbb{Z}_2)$.*

8.3 Orientability of real bundles

Proposition 8.4. *Let $\pi : E \to M$ be a real vector bundle of rank k. The following are equivalent.*

- *The line bundle $\Lambda^k(E)$ is trivial.*
- *$\Lambda^k(E)$ admits a non-zero section.*
- *The double cover $\tilde{M}$ corresponding to $\Lambda^k(E)$ is a trivial 2-fold covering space.*
- *The structure group of E can be reduced to*

$$GL_+(k, \mathbb{R}) \equiv \{A \in GL(k, \mathbb{R}) \mid \det(A) > 0\} \tag{8.3}$$

- *The structure group of E can be reduced to $SO(k)$*

Proof. The proof follows from the above discussion. □

Definition 8.5. We say that a real vector bundle $\pi : E \to M$ is *orientable* if any of the equivalent conditions in Proposition 8.4 are satisfied.

We can restate the above as follows. Given any real rank k vector bundle E over M, we let $w_1(E) \in H^1(M, \mathbb{Z}_2)$ be the cohomology class associated to $\Lambda^k(E)$ using the isomorphisms (8.2) above. We call $w_1(E)$ the *first Stiefel-Whitney class* of E. We can state the above as follows.

Proposition 8.6. *A real vector bundle $\pi : E \to M$ is orientable if and only if $w_1(E) = 0$.*

This immediately implies the following.

Corollary 8.7. *If $H^1(M, \mathbb{Z}_2) = 0$, then every vector bundle over E is orientable.*

Example 8.8. Thus, every vector bundle over S^n is orientable for $n \geq 2$. But for $n = 1$, we have $H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$, so there is exactly one non-orientable line bundle over S^1, called the Möbius bundle.

Exercise 8.9. We have $H^1(T^2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, so there are exactly 4 real line bundles over T^2 up to equivalence. Describe these bundles in terms of open covers and transition functions.

9 Lecture 9

9.1 Tensor product of line bundles

The set of real line bundles form a group with operation the tensor product, i.e., for line bundles $\pi_1 : L_1 \to M$ and $\pi_2 : L_2 \to M$ then

$$\pi : L_1 \otimes L_2 \to M \tag{9.1}$$

is also a line bundle. We claim that inverses exists. So let $\pi : L \to M$ be any line bundle, with transition functions $\varphi^L_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(1, \mathbb{R}) = \mathbb{R}^*$. Then the transition functions of the dual bundle are given by

$$\varphi^{L^*}_{\alpha\beta} = ((\varphi^L_{\alpha\beta})^{-1})^T = (\varphi^L_{\alpha\beta})^{-1}. \tag{9.2}$$

So the transition functions of $L \otimes L^*$ are

$$\varphi^{L\otimes L^*}_{\alpha\beta} = \varphi^L_{\alpha\beta} \cdot \varphi^{L^*}_{\alpha\beta} = \varphi^L_{\alpha\beta} \cdot (\varphi^L_{\alpha\beta})^{-1} = 1. \tag{9.3}$$

But we know that $L^* \cong L$, so any line bundle L is its own inverse.

Proposition 9.1. *For any two line bundles $\pi_1 : L_1 \to M$ and $\pi_2 : L_2 \to M$ we have*

$$w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2). \tag{9.4}$$

Proof. Recall the definition of the first Stiefel-Whitney class. Given any line bundle $\pi : L \to M$, associate the double covering $\tilde{M}$ of M and use the isomorphisms

$$H^1(M, \mathbb{Z}_2) \cong Hom(H_1(M), \mathbb{Z}_2) \cong Hom(\pi_1(M), \mathbb{Z}_2). \tag{9.5}$$

Given an element $[\gamma]$ in $\pi_1(M)$, choose a representative $\gamma \in \pi_1(M)$. We can assume that $\gamma : S^1 \to M$ is a smooth imbedding. Then L restricted to $\gamma(S^1)$ is a line bundle over S^1. Recall that since $H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$, there are exactly 2 line bundles over S^1, $S^1 \times \mathbb{R} = E_0$ and the Mobius bundle, which we call E_1. Equivalently, there are exactly 2 double covers of S^1. Note that the double cover $\tilde{M}$ restricted to the image of γ is a trivial covering if and only if the corresponding homomorphism from $\pi_1(M)$ to $\mathbb{Z}_2$ maps γ to 0. Otherwise, it is the nontrivial covering of S^1. Consequently, we have

$$w_1(L)(\gamma) = w_1(L|_{\gamma(S^1)}). \tag{9.6}$$

Returning to the tensor product $L_1 \otimes L_2$. Assume that $L_1|_{\gamma(S^1)} = E_i$, where $i = 0$ or $i = 1$, and $L_2|_{\gamma(S^1)} = E_j$, where $j = 0$ or $j = 1$. Then we have

$$w_1(L_1 \otimes L_2)(\gamma) = w_1((L_1 \otimes L_2)|_{\gamma S^1}) = w_1(E_i \otimes E_j). \tag{9.7}$$

But note that

$$w_1(E_i \otimes E_j) = w_1(E_i) + w_1(E_j), \tag{9.8}$$

because $E_0 \otimes E_0 \cong E_0$, $E_0 \otimes E_1 \cong E_1 \otimes E_0 \cong E_1$ and $E_1 \otimes E_1 \cong E_0$. So therefore we have

$$\begin{aligned} w_1(L_1 \otimes L_2)(\gamma) &= w_1(E_i) + w_1(E_j) \\ &= w_1(L_1|_{\gamma(S^1)}) + w_1(L_2|_{\gamma(S^1)}) = w_1(L_1)(\gamma) + w_1(L_2)(\gamma). \end{aligned} \tag{9.9}$$

□

Consequently, w_1 gives an isomorphism from the multiplicative group of line bundles with the tensor product operation to the additive group $H^1(M, \mathbb{Z}_2)$.

9.2 First Stiefel-Whitney class of direct sums

Now we can prove the following.

Proposition 9.2. *Let $\pi_1 : E_1 \to M$ be a real vector bundle of rank k_1 and $\pi_2 : E_2 \to M$ be a real vector bundle of rank k_2. Then*

$$w_1(E_1 \oplus E_2) = w_1(E_1) + w_1(E_2) \tag{9.10}$$

Proof. Recall that if $\varphi^{E_1}_{\alpha\beta}$ and $\varphi^{E_2}_{\alpha\beta}$ are transition functions for E_1, E_2 respectively, then the transition functions for $E_1 \oplus E_2$ are given by

$$\varphi^{E_1 \oplus E_2}_{\alpha\beta} = \begin{pmatrix} \varphi^{E_1}_{\alpha\beta} & 0 \\ 0 & \varphi^{E_2}_{\alpha\beta} \end{pmatrix}. \tag{9.11}$$

Since the determinant of a block diagonal matrix is the product of the determinants of the blocks, we have that

$$\Lambda^{k_1+k_2}(E_1 \oplus E_2) \cong \Lambda^{k_1}(E_1) \otimes \Lambda^{k_2}(E_2). \tag{9.12}$$

Consequently,

$$\begin{aligned} w_1(E_1 \oplus E_2) &= w_1\big(\Lambda^{k_1+k_2}(E_1 \oplus E_2)\big) \\ &= w_1\big(\Lambda^{k_1}(E_1) \otimes \Lambda^{k_2}(E_2)\big) \\ &= w_1\big(\Lambda^{k_1}(E_1)\big) + w_1\big(\Lambda^{k_2}(E_2)\big) = w_1(E_1) + w_1(E_2), \end{aligned} \tag{9.13}$$

where in the middle line we used Proposition 9.1. □

Note that the formulas in Propositions 9.1 and 9.2 look very similar. But Proposition 9.1 only hold for line bundles. To see they must be different in general, we state the following.

Proposition 9.3. *Let $\pi_1 : E \to M$ be a real vector bundle of rank k, and $\pi_2 : L \to M$ be a real line bundle. Then*

$$w_1(E \otimes L) = w_1(E) + k w_1(L). \tag{9.14}$$

Proof. We first show that

$$\Lambda^k(E \otimes L) \cong \Lambda^k(E) \otimes L^k. \tag{9.15}$$

To see this, note that the transition functions for $E \otimes L$ are given by

$$\varphi_{\alpha\beta}^{E\otimes L} = \varphi_{\alpha\beta}^{E} \cdot \varphi_{\alpha\beta}^{L}, \tag{9.16}$$

and therefore

$$\begin{aligned} \varphi_{\alpha\beta}^{\Lambda^k(E\otimes L)} &= \det\left(\varphi_{\alpha\beta}^{E}\right) \cdot \varphi_{\alpha\beta}^{L}\big) \\ &= \det\left(\varphi_{\alpha\beta}^{E}\right) \cdot (\varphi_{\alpha\beta}^{L})^k \\ &= \varphi_{\alpha\beta}^{\Lambda^k(E)} \cdot \varphi_{\alpha\beta}^{L^k} \end{aligned} \tag{9.17}$$

So we have

$$\begin{aligned} w_1(E \otimes L) &= w_1\big(\Lambda^k(E \otimes L)\big) \\ &= w_1\big(\Lambda^k(E) \otimes L^k\big) \\ &= w_1\big(\Lambda^k(E)\big) + w_1\big(L^k\big) \\ &= w_1(E) + k w_1(L). \end{aligned} \tag{9.18}$$

□

10 Lecture 10

10.1 Pull-back bundles

If $\pi : E \to M$ is a vector bundle, and $f : N \to M$ is a smooth mapping, then we define

$$f^*E = \{(p, v) \in N \times E \mid f(p) = \pi(v)\}. \tag{10.1}$$

Proposition 10.1. *We have $\pi_1 : f^*E \to N$ is a vector bundle over N such that $\pi_1^{-1}(p) = \pi^{-1}f(p)$.*

Proof. Given an open covering U_α of M with local trivializations $\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to E_{U_\alpha}$, and transition function $\varphi_{\alpha\beta}$ of E, then $V_\alpha = f^{-1}(U_\alpha)$ is an open covering of M, and

$$f^*\Phi_\alpha : V_\alpha \times \mathbb{R}^k \to (f^*E)_{V_\alpha} \tag{10.2}$$

defined by

$$f^*\Phi_\alpha(p, v) = (p, \Phi_\alpha(f(p), v)) \tag{10.3}$$

gives a system of local trivializations for f^*E. The transition functions for f^*E with respect to the covering V_α are

$$\varphi_{\alpha\beta}^{f^*E} = \varphi_{\alpha\beta} \circ f. \tag{10.4}$$

$\square$

Above, we defined bundles to be equivalent if $F : E_1 \to E_2$ is a mapping which is an isomorphism on fibers and covers the identity mapping, that is, the following diagram commutes

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_{E_1}"] & E_2 \arrow[d, "\pi_{E_2}"] \\ M \arrow[r, "id"] & M. \end{tikzcd} \tag{10.5}$$

Let us consider the more general situation where F is a mapping which is an isomorphism on fibers and covers a diffeomorphism $f : M \to M$,

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_{E_1}"] & E_2 \arrow[d, "\pi_{E_2}"] \\ M \arrow[r, "f"] & M. \end{tikzcd} \tag{10.6}$$

Proposition 10.2. *In this setting, the bundle $\pi_1 : E_1 \to M$ is isomorphic to f^*E_2.*

Proof. We need to find a mapping H, which is an isomorphism on fibers, such that the following diagram commutes

$$\begin{tikzcd} E_1 \arrow[r, "H"] \arrow[d, "\pi_{E_1}"] & f^*E_2 \arrow[d, "\pi_{E_2}"] \\ M \arrow[r, "id"] & M. \end{tikzcd} \tag{10.7}$$

Define $H : E_1 \to f^*E_2$ by

$$H(e_1) = (\pi_{E_1}(e_1), F(e_1)). \tag{10.8}$$

Then H covers the identity map, and is an isomorphism on fibers. □

So if we had defined bundle equivalence using the coarser notion of covering a diffeomorphism, then we also need to mod out the first notion of equivalence by the pull-back operation.

Proposition 10.3 (Naturality). *If $f : N \to M$ is a smooth mapping, and $\pi : E \to M$ is a real vector bundle, then*

$$w_1(f^*E) = f^*(w_1(E)). \tag{10.9}$$

Proof. If E is rank k, then since $\Lambda^k(f^*E) \cong f^*(\Lambda^k(E))$, we just need to assume that E is a line bundle.

To get $w_1(f^*E)$, we view this as an element of $Hom(\pi_1(N), \mathbb{Z}_2)$. Then

$$w_1(f^*E)(\gamma) = w_1(\gamma^* f^* E) \tag{10.10}$$

which is 0 if the pull-back bundle is trivial on S^1, and 1 if it is the nontrivial bundle.

To get $f^*(w_1(E))$, we have

$$f^*(w_1(E))(\gamma) = w_1(E)(f \circ \gamma) = w_1((f \circ \gamma)^* E). \tag{10.11}$$

It is clear that

$$(f \circ \gamma)^* E \cong \gamma^*(f^*E), \tag{10.12}$$

so these must be the same. □

Example 10.4. (Line bundles over T^2). Recall that since $H^1(T^2, \mathbb{Z}_2) \cong \mathbb{Z}_4$, there are 4 line bundles on T^2 up to equivalence. We can describe them more easily using the pullback operation. Let $\pi_i : S^1 \times S^1 \to S^1 \to S^1$ be the ith projection. Let E denote the Mobius bundle over S^1. Let E_1 denote the trivial line bundle over T^2,

$$E_2 = \pi_1^* E,\ E_3 = \pi_2^* E,\ E_4 = \pi_1^* E \otimes \pi_2^* E. \tag{10.13}$$

Then these represent the 4 bundles up to equivalence. To see this, identify the cohomology group $H^1(T^2, \mathbb{Z}_2)$ with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, by letting the element $(1, 0)$ be Poincaré dual to $S^1 \times \{pt\}$ and the element $(0, 1)$ be Poincarè dual to $\{pt\} \times S^1$. Then using Propositions 10.3 and 9.1, we have

$$w_1(E_2) = w_1(\pi_1^* E) = \pi_1^* w_1(E) = (1, 0), \tag{10.14}$$
$$w_1(E_3) = w_1(\pi_2^* E) = \pi_2^* w_1(E) = (0, 1), \tag{10.15}$$
$$w_1(E_4) = w_1(\pi_1^* E \otimes \pi_1^* E) = \pi_1^* w_1(E) + \pi_2^* w_1(E) = (1, 1). \tag{10.16}$$

Example 10.5. (A manifold M, and a diffeomorphism $f : M \to M$, and a bundle $\pi : E \to M$ such that f^*E is not isomorphic to E.) Let $f : S^1 \times S^1 \to S^1 \times S^1$ be the mapping $f(\theta_1, \theta_2) = f(\theta_2, \theta_1)$. Then $f^*E_2 = E_3$, but E_2 is not equivalent to E_3. This is because

$$w_1(f^*E_2) = f^*w_1(E_2) = f^*(1, 0) = (0, 1) = w_1(E_3). \tag{10.17}$$

Therefore f^*E_2 is equivalent to E_3, which is not equivalent to E_2.

There are only 3 line bundles over T^2 if we consider pull-back bundles to be equivalent allowing diffeomorphisms of the base.

11 Lecture 11

Example 11.1. (Tautological bundle on $\mathbb{RP}^n$) Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Equivalently, $\mathbb{RP}^n$ is the space of vectors in $\mathbb{R}^{n+1}$ modulo the equivalence relation

$$(v_1, \dots v_{n+1}) \sim (cv_1, \dots, cv_{n+1}),\ c \neq 0. \tag{11.1}$$

Define

$$\gamma^1_n = \{([x], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in [x]\} \tag{11.2}$$

Since $H^1(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2$, there are only 2 line bundles over $\mathbb{RP}^n$. We claim that γ^1_n is the nontrivial one. Assume by contradiction that it were the trivial bundle. Then there would exists a nowhere vanishing section $\sigma : \mathbb{RP}^n \to \gamma^1_n$. This is a mapping

$$\sigma : \mathbb{RP}^n \to \mathbb{RP}^n \times \mathbb{R}^{n+1} \tag{11.3}$$

of the form for $x \in S^n$,

$$\sigma([x]) = ([x], c(x) \cdot x) \tag{11.4}$$

For this to be well-defined, we require that $c(x) : S^n \to \mathbb{R}$ is a function satisfying $c(-x) = -c(x)$. Since c must take negative and positive values, by the intermediate value theorem, $c(x_0) = 0$ for some x_0, which is a contradiction.

Consequently, we have shown that $w_1(\gamma^1_n) = 1 \in \mathbb{Z}_2 = H^1(\mathbb{RP}^n, \mathbb{Z}_2)$.

Example 11.2. (Universal bundle on $G(k, n)$) Recall that the Grassmannian $G(k, n)$ is the space of k-planes through the origin in $\mathbb{R}^{n+1}$. Define

$$\gamma^k_n = \{([x], v) \in G(k, n) \times \mathbb{R}^{n+1} \mid v \in [x]\} \tag{11.5}$$

Proposition 11.3. *For any real rank k vector bundle $\pi : E \to M$ over a compact n-manifold M, there exists a mapping $f : M \to G(k, N)$ for some N so that $E \cong f^*\gamma^k_N$.*

Proof. Cover M by charts U_α, $1 \leq \alpha \leq N$, for which there are local trivializations of E over U_α,

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to E\big|_{U_\alpha}, \tag{11.6}$$

for which the mapping $h_\alpha = \pi_2 \Phi_\alpha^{-1} : E\big|_{U_\alpha} \to \mathbb{R}^k$ is linear on fibers.

Let χ_α be a partition of unity subordinate to the covering $\{U_\alpha\}$. Define a mapping $h'_\alpha : E \to \mathbb{R}^k$ by

$$h'_\alpha(e) = \begin{cases} 0 & e \notin U_\alpha \\ \chi_\alpha(\pi(e)) h_\alpha & e \in U_\alpha. \end{cases} \tag{11.7}$$

Then h_α is smooth, and is linear on fibers.

Define $f : E \to \mathbb{R}^{Nk}$ by

$$f(e) = (h'_1(e), \dots, h'_N(e)). \tag{11.8}$$

Then f is a mapping which is linear and injective on fibers.

Define $F : E \to \gamma^k_{kN}$ by

$$F(e) = (f(\pi^{-1}\pi(e)), f(e)). \tag{11.9}$$

Then F is a mapping which makes the following diagram commute

$$\begin{array}{ccc} E & \xrightarrow{F} & \gamma^k_N \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f'} & G(k, kN), \end{array} \tag{11.10}$$

where F is linear and injective on fibers.

Define $H : E \to (f')^*\gamma^k_N$ by

$$H(e) = (\pi(e), F(e)) \tag{11.11}$$

Then H makes the following diagram commute

$$\begin{array}{ccc} E & \xrightarrow{H} & (f')^*\gamma^k_{kN} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{id} & M, \end{array} \tag{11.12}$$

and H is an isomorphism on fibers, so $E \cong (f')^*\gamma^k_{kN}$. $\square$

Definition 11.4. Let $\pi : E \to M$ be a real line bundle over M. If $f : M \to G(k, N)$ is any map such that $E \cong f^*\gamma^k_N$ then f is called a *classifying map* for E.

12 Lecture 12

We review some homology and cohomology theory. Define the standard n-simplex to be

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1}, \sum_{i=0}^{n} t_i = 1, t_i \geq 0\}. \tag{12.1}$$

The ith face of Δ^n is the $(n-1)$-simplex Δ^n_i defined by

$$(t_0, \dots, \hat{t}_i, \dots t_p). \tag{12.2}$$

For a topological space X and an abelian group G, define the pth singular chain group $C_p(X, G)$ to be the free abelian group over G generated by a singular p-simplices, which is a continuous mapping,

$$T : \Delta^p \to X. \tag{12.3}$$

Define the boundary operator $\partial : C_p(X, G) \to C_{p-1}(X, G)$ by the following given a singular p-simplex $T : \Delta^p \to X$, let

$$\partial T = \sum_{i=0}^{p} (-1)^i T \circ \Delta^p_i, \tag{12.4}$$

and extend to all chains by linearity. It is not hard to see that $\partial^2 = 0$ thus we have a chain complex

$$\cdots \xrightarrow{\partial} C_{p+1}(X, G) \xrightarrow{\partial} C_p(X, G) \xrightarrow{\partial} C_{p-1}(X, G) \xrightarrow{\partial} \cdots . \tag{12.5}$$

Define the pth singular homology group by

$$H^p(X, G) = \frac{Ker\{\partial : C_p(X, G) \to C_{p-1}(X, G)\}}{Im\{\partial : C_{p+1}(X, G) \to C_p(X, G)\}} \tag{12.6}$$

To define singular cohomology groups, let $C^p(X, G)$ denote the singular cochains, which are dual elements to singular chain, i.e.,

$$C^p(X, G) = Hom(C_p(X, G), G), \tag{12.7}$$

and let $\delta : C^p(X, G) \to C^{p+1}(X, G)$ denote the dual to the boundary operator. This satisfies $\delta^2 = 0$, so we have a cochain complex

$$\cdots \xrightarrow{\delta} C^{p-1}(X, G) \xrightarrow{\delta} C^p(X, G) \xrightarrow{\delta} C^{p+1}(X, G) \xrightarrow{\delta} \cdots . \tag{12.8}$$

Define the pth singular cohomology group by

$$H^p(X, G) = \frac{Ker\{\delta : C^p(X, G) \to C^{p+1}(X, G)\}}{Im\{\delta : C^{p-1}(X, G) \to C^p(X, G)\}}. \tag{12.9}$$

Cochains have some extra ring structure: we next define the cup product

$$\cup : C^p(X,G) \otimes C^q(X,G) \to C^{p+q}(X,G), \tag{12.10}$$

by the following. If T is a singular $(p+q)$-simplex, and c^p and c^q are singular cohains, then define

$$(c^p \cup c^q)(T) = c^p(T \circ (t_0, \cdots, t_p, 0, \ldots, 0)) \cdot c^q(T \circ (0, \ldots, 0, t_p, \ldots, t_{p+q})). \tag{12.11}$$

It can be shown that

$$c^p \cup c^q = (-1)^{pq} c^q \cup c^q, \tag{12.12}$$

thus the cup product is anti-commutative, and that

$$\delta(c^p \cup c^q) = (\delta c^p) \cup c^q + (-1)^p c^p \cup (\delta c^q). \tag{12.13}$$

(Note the similarity with the wedge product of forms). The latter relation shows that the the cup product descends to cohomology, i.e.,

$$\cup : H^p(X,G) \otimes H^q(X,G) \to H^{p+q}(X,G). \tag{12.14}$$

12.1 De Rham's Theorem

In the special case of real coefficients, we have the following

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d} & \Omega^{p-1}(X) & \xrightarrow{d} & \Omega^p(X) & \xrightarrow{d} & \Omega^{p+1}(X) & \xrightarrow{d} & \cdots \\
 & & \big\downarrow{\scriptstyle \int} & & \big\downarrow{\scriptstyle \int} & & \big\downarrow{\scriptstyle \int} & & \\
\cdots & \xrightarrow{\delta} & C^{p-1}(X,\mathbb{R}) & \xrightarrow{\delta} & C^p(X,\mathbb{R}) & \xrightarrow{\delta} & C^{p+1}(X,\mathbb{R}) & \xrightarrow{\delta} & \cdots,
\end{array} \tag{12.15}$$

where the vertical maps are defined as follows. If $\omega \in \Omega^P(X)$, and c_p is a p-chain, then let

$$(f\omega)(c_p) = \int_{c_p} \omega = \int_{\Delta^p} c_p^* \omega. \tag{12.16}$$

Stokes' Theorem can be stated in the form

$$\int_{c_{p+1}} d\omega = \int_{\partial c_{p+1}} \omega, \tag{12.17}$$

so the above diagram commutes. So have mappings

$$H^p_{DR}(X) \xrightarrow{\int} H^p_{sing}(X,\mathbb{R}), \tag{12.18}$$

and De Rham's Theorem says that this mapping is an isomorphism if X is a smooth manifold.

13 Lecture 13

13.1 Exact sequences of chain complexes

Let C^i be a complex of G-modules for $i = 1, 2, 3$.

$$\cdots \xrightarrow{\partial^i_{p+2}} C^i_{p+1} \xrightarrow{\partial^i_{p+1}} C^i_p \xrightarrow{\partial^i_p} C^i_{p-1} \xrightarrow{\partial^i_{p-1}} \cdots . \tag{13.1}$$

with $\partial^2 = 0$. A morphism from C^i to C^j are mappings $\alpha_k : C^i_k \to C^j_k$ such that the following diagram commutes for every p

$$\begin{tikzcd} C^i_{p+1} \arrow[r, "\partial^i_{p+1}"] \arrow[d, "\alpha_{p+1}"] & C^i_p \arrow[d, "\alpha_p"] \\ C^j_{p+1} \arrow[r, "\partial^j_{p+1}"] & C^j_p \end{tikzcd} \tag{13.2}$$

For complexes C^1, C^2, C^3, and morphisms $\alpha : C^1 \to C^2$ and $\beta : C^2 \to C^3$. We say that a sequence of complexes is exact if

$$0 \longrightarrow C^1 \xrightarrow{\alpha} C^2 \xrightarrow{\beta} C^3 \longrightarrow 0 \tag{13.3}$$

if the sequence

$$0 \longrightarrow C^1_p \xrightarrow{\alpha_p} C^2_p \xrightarrow{\beta_p} C^3_p \longrightarrow 0 \tag{13.4}$$

is exact for every p.

Lemma 13.1 (The zig-zag lemma)**.** *If*

$$0 \longrightarrow C^1 \xrightarrow{\alpha} C^2 \xrightarrow{\beta} C^3 \longrightarrow 0 \tag{13.5}$$

is a short exact sequence of complexes, then there exists mappings

$$\partial_p : C^3_p \to C^1_{p-1} \tag{13.6}$$

for every p such that the sequence

$$\cdots \xrightarrow{\partial_{p+1}} H_p(C_1) \xrightarrow{\alpha_p} H_p(C_2) \xrightarrow{\beta_p} H_p(C_3) \xrightarrow{\partial_p} H_{p-1}(C_1) \longrightarrow \cdots \tag{13.7}$$

is exact.

Proof. We look at the huge commutative diagram

$$
\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C^1_{p+1} \arrow[r, "\alpha_{p+1}"] \arrow[d, "\partial^1_{p+1}"] & C^2_{p+1} \arrow[r, "\beta_{p+1}"] \arrow[d, "\partial^2_{p+1}"] & C^3_{p+1} \arrow[r] \arrow[d, "\partial^3_{p+1}"] & 0 \\
0 \arrow[r] & C^1_p \arrow[r, "\alpha_p"] \arrow[d, "\partial^1_p"] & C^2_p \arrow[r, "\beta_p"] \arrow[d, "\partial^2_p"] & C^3_p \arrow[r] \arrow[d, "\partial^3_p"] & 0 \\
0 \arrow[r] & C^1_{p-1} \arrow[r, "\alpha_{p-1}"] \arrow[d] & C^2_{p-1} \arrow[r, "\beta_{p-1}"] \arrow[d] & C^3_{p-1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}
\tag{13.8}
$$

which has all horizontal rows exact.

To define the boundary operator, take $c^3_p \in C^3_p$ with $\partial^3_p c^3_p = 0$. By exactness of the middle row, β_p is surjective, so $c^3_p = \beta_p(c^2_p)$ for some $c^2_p \in C^2_p$. Then since the diagram commutes, we have

$$\beta_{p-1}\partial^2_p c^2_p = \partial^3_p \beta_p c^2_p = \partial^3_p c^3_p = 0. \tag{13.9}$$

By exactness of the bottow row, we have $\partial^2_p c^2_p = \alpha_{p-1} c^1_{p-1}$ for some $c^1_{p-1} \in C^1_{p-1}$.

$$0 = \partial^2_{p-1}\partial^2_p c^2_p = \partial^2_{p-1}\alpha_{p-1}c^1_{p-1} = \alpha_{p-1}\partial^1_{p-1}c^1_{p-1}, \tag{13.10}$$

which implies that $\partial^1_{p-1}c^1_{p-1} = 0$. Consequently, $c^1_{p-1} \in H_{p-1}(C^1)$.

To prove this mapping is well-defined, assume that we started with $c^3_p \in C^3_p$ which was of the form $c^3_p = \partial^3_{p+1}c^3_{p+1}$. Then we can write $c^3_{p+1} = \beta_{p+1}c^2_{p+1}$, and the element $\tilde{c}^2_p = \partial^2_{p+1}c^2_{p+1}$ satisfies $\beta_p(\tilde{c}^2_p) = c^3_p$. But this this element is exact, the next step clearly gives zero. Independence of the choice of c^2_p is similarly established.

Exactness of the resulting sequence is left as an exercise in diagram chasing.

□

14 Lecture 14

14.1 Exact sequences of cochain complexes

Let C_i be a co-complex of G-modules for $i = 1, 2, 3$.

$$\cdots \xrightarrow{d_i^{p-2}} C_i^{p-1} \xrightarrow{d_i^{p-1}} C_i^p \xrightarrow{d_i^p} C_i^{p+1} \xrightarrow{d_i^{p+1}} \cdots. \tag{14.1}$$

with $d^2 = 0$. A morphism from C^i to C^j are mappings $\alpha^k : C_i^k \to C_j^k$ such that the following diagram commutes for every p

$$\begin{tikzcd} C_i^p \arrow[r, "d_i^p"] \arrow[d, "\alpha^p"] & C_i^{p+1} \arrow[d, "\alpha^{p+1}"] \\ C_j^p \arrow[r, "d_j^p"] & C_j^{p+1} \end{tikzcd} \tag{14.2}$$

For co-complexes C_1, C_2, C_3, and morphisms $\alpha : C_1 \to C_2$ and $\beta : C_2 \to C_3$. We say that a sequence of co-complexes is exact if

$$0 \longrightarrow C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \longrightarrow 0 \tag{14.3}$$

if the sequence

$$0 \longrightarrow C_1^p \xrightarrow{\alpha_p} C_2^p \xrightarrow{\beta_p} C_3^p \longrightarrow 0 \tag{14.4}$$

is exact for every p.

Lemma 14.1 (The zig-zag lemma)**.** *If*

$$0 \longrightarrow C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \longrightarrow 0 \tag{14.5}$$

is a short exact sequence of co-complexes, then there exists mappings

$$\delta^p : C_3^p \to C_1^{p+1} \tag{14.6}$$

for every p such that the sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p(C_1) \xrightarrow{\alpha^p} H^p(C_2) \xrightarrow{\beta^p} H^p(C_3) \xrightarrow{\delta^p} H^{p+1}(C_1) \longrightarrow \cdots \tag{14.7}$$

is exact.

Proof. Same as before, with arrows reversed. □

14.2 Mayer-Vietoris for singular chains

Write $M = U \cup V$ as the union of two open sets in M. Then the following sequence is exact:

$$0 \longrightarrow C_p(U \cap V) \xrightarrow{\alpha_p} C_p(U) \oplus C_p(V) \xrightarrow{\beta_p} C_p(U) + C_p(V) \longrightarrow 0 \tag{14.8}$$

where

$$\alpha(c_p) = \big((i_{U\cap V \hookrightarrow U})_* c_p, (i_{U\cap V \hookrightarrow V})_* c_p\big) \tag{14.9}$$

and

$$\beta(a_p, b_p) = (i_{U\hookrightarrow M})_* a_p - (i_{V\hookrightarrow M})_* b_p. \tag{14.10}$$

It is not hard to see this sequence is exact. Furthermore, by a barycentric subdivision argument, the homology $H_*(C_p(U) + C_p(V))$ is ismorphic to $H_*(U \cup V)$. (Roughly, keep subdividing simplices until their images are contained in U or V.) Consequently, we obtain a long exact sequence

$$\cdots \xrightarrow{\partial_{p+1}} H_p(U \cap V) \xrightarrow{\alpha_p} H_p(U) \oplus H_p(V) \xrightarrow{\beta_p} H_p(U \cup V) \xrightarrow{\partial_p} \cdots \tag{14.11}$$

14.3 Mayer-Vietoris for singular co-chains

Write $M = U \cup V$ as the union of two open sets in M. Then the following sequence is exact:

$$0 \longrightarrow (C_p(U) + C_p(V))^* \xrightarrow{\beta^p} C^p(U) \oplus C^p(V) \xrightarrow{\alpha^p} C^p(U \cap V) \longrightarrow 0 \tag{14.12}$$

where $\beta^p = (\beta_p)^*$ and $\alpha^p = (\alpha_p)^*$. This sequence is exact because the original sequence consisted of free abelian groups, so the tensored sequence is also exact.

Consequently, we obtain a long exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p(U \cup V) \xrightarrow{\beta^p} H^p(U) \oplus H^p(V) \xrightarrow{\alpha^p} H^p(U \cap V) \xrightarrow{\delta^p} \cdots \tag{14.13}$$

14.4 Mayer-Vietoris for de Rham cohomology

Write $M = U \cup V$ as the union of two open sets in M. Then the following sequence is exact:

$$0 \longrightarrow \Omega^p(U \cup V) \xrightarrow{\beta^p} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{\alpha^p} \Omega^p(U \cap V) \longrightarrow 0 \tag{14.14}$$

where

$$\beta^p(\omega) = \big((i_{U\hookrightarrow M})^*\omega, (i_{V\hookrightarrow M})^*\omega\big). \tag{14.15}$$

and

$$\alpha^p(\omega_U, \omega_V) = (i_{U\cap V\hookrightarrow U})^*\omega_U - (i_{U\cap V\hookrightarrow V})^*\omega_V \tag{14.16}$$

To see this, β^p is obviously injective. For exactness at the middle step, obviously $\alpha^p\beta^p\omega = 0$. If $\beta^p(\omega_U, \omega_V) = 0$, then $\omega_U = \omega_V$ on $U \cap V$, so then (ω_U, ω_V) is a well-defined global form on M.

To show that α is onto, let $\omega \in \Omega^p(U \cap V)$. Let ϕ_U, ϕ_V be a partition of unity subordinate to the covering $\{U, V\}$. Then $\omega = \alpha(\phi_V\omega, -\phi_U\omega)$.

Consequently, we obtain a long exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{DR}(U \cup V) \xrightarrow{\beta^p} H^p_{DR}(U) \oplus H^p_{DR}(V) \xrightarrow{\alpha^p} H^p_{DR}(U \cap V) \xrightarrow{\delta^p} \cdots \tag{14.17}$$

Let us review the definition of the mapping δ^p. Given a cohomology class $[\omega] \in H^p_{c,dR}(U \cap V)$, represented by $\omega \in \Omega^p_c(U \cap V)$ with $d\omega = 0$, we first write $\omega = \alpha^p(\phi_V \omega, -\phi_U \omega)$, then we apply the exterior derivative to get

$$(d(\phi_V \omega), -d(\phi_U \omega)) = (d\phi_V \wedge \omega, -d\phi_U \wedge \omega) \in \Omega^p(U) \oplus \Omega^p(V). \tag{14.18}$$

Note that on $U \cap V$, we have $(\phi_U + \phi_V)\omega = \omega$, so applying d to this equation, we have that $d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0$ on $U \cap V$, so together these define a global form

$$\delta^p \omega = \begin{cases} d\phi_V \wedge \omega & \text{in } U \\ -d\phi_U \wedge \omega & \text{in } V \end{cases} \tag{14.19}$$

and we take the cohomology class of this form.

Remark 14.2. This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

15 Lecture 15

15.1 The Poincaré Lemma

Let M be a differentiable n-manifold, and consider $N = M \times [0,1]$. Define the inclusion maps $\iota_0, \iota_1 : M \hookrightarrow N$ by $\iota_0(p) = (p, 0), \iota_1(p) = (p, 1)$.

Lemma 15.1 (The Poincaré Lemma)**.** *There exist mappings $I^k : \Omega^k(N) \to \Omega^{k-1}(M)$ such that if $\omega \in \Omega^k(N)$, then*

$$\iota_1^* \omega - \iota_0^* \omega = d_M(I^k \omega) + I^{k+1}(d_N \omega). \tag{15.1}$$

Proof. Let $\pi : N \to M$ be the projection $\pi(p,t) = p$. Then any k-form on N can be written as

$$\omega = h(p,t)\pi^* \phi_k + f(p,t) dt \wedge \pi^* \phi_{k-1}, \tag{15.2}$$

where $\phi_k \in \Omega^k(M)$ and $\phi_{k-1} \in \Omega^{k-1}(M)$. Define

$$I^k(\omega) = \Big(\int_0^1 f(p,t) dt \Big) \phi_{k-1}, \tag{15.3}$$

proof of (15.1) was outlined in lecture. $\square$

Definition 15.2. Mappings $f, g : X \to Y$ are said to be smoothly homotopic if there exists a smooth mapping $F : X \times [0,1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

Proposition 15.3. *If $f, g : X \to Y$ are smoothly homotopic then*

$$f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X) \tag{15.4}$$

Proof. Let $F : X \times [0,1] \to Y$ be a smooth homotopy between f and g. The Poincaré Lemma implies that

$$\iota_0^* = \iota_1^* : H^*_{dR}(M \times [0,1]) \to H^*_{dR}(M). \tag{15.5}$$

Since $f = F \circ \iota_0$ and $g = F \circ \iota_1$, we have

$$f^* = \iota_0^* \circ F^*, \ \ g^* = \iota_1^* \circ F^*, \tag{15.6}$$

therefore $f^* = g^*$ as mappings between de Rham cohomology. □

Corollary 15.4. *The de Rham cohomology groups of $\mathbb{R}^n$ are given by*

$$H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \leq n \end{cases}. \tag{15.7}$$

Proof. For $k = 0$, the result is obvious, since $df = 0$ implies that f is constant. The mapping $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ defined by

$$F(x,t) = tx. \tag{15.8}$$

is a homotopy from the zero mapping O to the identity map $Id : \mathbb{R}^n \to \mathbb{R}^n$. The corollary says that

$$O^* = Id^* : H^k_{dR}(\mathbb{R}^n) \to H^k_{dR}(\mathbb{R}^n). \tag{15.9}$$

But for $k > 0$, the mapping O^* is the zero mapping on cohomology, and Id^* is the identity mapping, and these are only equal if the vector space is 0-dimensional. □

Proposition 15.5. *We have the cohomology groups*

$$H^k_{sing}(\mathbb{R}^n, \mathbb{R}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \leq n \end{cases} \tag{15.10}$$

Proof. The analog of the Poincaré lemma is the following. There is a mapping

$$I_k : C_k(M) \to C_{k+1}(M \times [0,1]) \tag{15.11}$$

such that if $c_k \in C_k(M)$ then

$$(\iota_1)_* c_k - (\iota_0)_* c_k = \partial_N (I_k c_k) + I_{k-1}(\partial_M c_k). \tag{15.12}$$

This mapping is defined by the following. If $c_k \in C_k(M)$ then $c_k : \Delta^k \to M$. Then

$$c_k \times id : \Delta^k \times [0,1] \to M \times [0,1]. \tag{15.13}$$

The left hand side of this is a "prism", and can be subdivided into $k+1$ copies of a $(k+1)$-simplex. Roughly if $k=0$, then $\Delta^0 \times [0,1] = [0,1]$,which is a 1-simplex. If $k=1$, then $\Delta^1 \times [0,1] = [0,1] \times [0,1]$ is a square, and cutting along the diagonal gives 2 2-simplices. The precise definition of this mapping and verification of (15.12) will be left as an exercise.

Similarly to above, this implies that homotopic maps induce the same mapping on singular homology. Dualizing, this yields that homotopic maps also induce the same mapping on singular cohomology, and the proof is the same as above since the identity map of $\mathbb{R}^n$ is homotopic to the zero mapping.

□

15.2 Proof of de Rham's Theorem

The following lemma is crucual for the proof.

Lemma 15.6 (The Five Lemma)**.** *Assume the diagram*

$$\begin{tikzcd} V_1 \arrow[r, "\alpha_1"] \arrow[d, "\phi_1"] & V_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & V_3 \arrow[r, "\alpha_3"] \arrow[d, "\phi_3"] & V_4 \arrow[r, "\alpha_4"] \arrow[d, "\phi_4"] & V_5 \arrow[d, "\phi_5"] \\ W_1 \arrow[r, "\beta_1"] & W_2 \arrow[r, "\beta_2"] & W_3 \arrow[r, "\beta_3"] & W_4 \arrow[r, "\beta_4"] & W_5 \end{tikzcd} \tag{15.14}$$

commutes, and has exact rows. If $\phi_1, \phi_2, \phi_4, \phi_5$ are isomorphisms, then ϕ_3 is also an isomorphism.

Proof. Injectivity of ϕ_3: If $\phi_3(v_3) = 0$, then $\beta_3(\phi_3(v_3) = 0 = \phi_4\alpha_3(v_3)$. Since ϕ_4 is injective, $\alpha_3(v_3) = 0$. By exactness, $v_3 = \alpha_2(v_2)$. Then $\phi_3\alpha_2(v_2) = 0 = \beta_2\phi_2(v_2)$. By exactness, $\phi_2(v_2) = \beta_1(w_1)$. By surjectivity of ϕ_1, $w_1 = \phi_1(v_1)$. Then

$$\phi_2(v_2) = \beta_1\phi_1(v_1) = \phi_2\alpha_1(v_1), \tag{15.15}$$

but since ϕ_2 is injective, this implies that $v_2 = \alpha_1(v_1)$. Finally, $v_2 = \alpha_2(v_2) = \alpha_2\alpha_1(v_1) = 0$, by exactness.

The proof of surjectivity is similar, and left to the reader. □

Theorem 15.7 (de Rham)**.** *If X has a finite good cover, i.e., a finite covering so that all intersections are diffeomorphic to $\mathbb{R}^n$, then the mappings*

$$H^p_{DR}(X) \xrightarrow{\int} H^p_{sing}(X, \mathbb{R}), \tag{15.16}$$

are isomorphisms.

Proof. If there is only 1 element in the covering, then we are done by the above results. Next, consider the following diagram

$$\begin{tikzcd}[column sep=small] H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) \arrow[r, "\alpha^{k-1}"] \arrow[d, "\int \oplus \int"] & H^{k-1}_{dR}(U \cap V) \arrow[r, "\delta^k"] \arrow[d, "\int"] & H^k_{dR}(U \cup V) \arrow[r, "\beta^k"] \arrow[d, "\int"] & H^k_{dR}(U) \oplus H^k_{dR}(V) \arrow[r, "\alpha^k"] \arrow[d, "\int \oplus \int"] & H^k_{dR}(U \cap V) \arrow[d, "\int"] \\ H^{k-1}_s(U) \oplus H^{k-1}_s(V) \arrow[r, "\alpha^{k-1}"] & H^{k-1}_s(U \cap V) \arrow[r, "\delta^k"] & H^k_s(U \cup V) \arrow[r, "\beta^k"] & H^k_s(U) \oplus H^k_s(V) \arrow[r, "\alpha^k"] & H^k_s(U \cap V) \end{tikzcd} \tag{15.17}$$

By the Five Lemma, if the result is true for U, V and $U \cap V$, then it is also true for $U \cup V$. By induction, the theorem is then true for any manifold which admits a finite good cover. $\square$

One can show also the following:

$$[\alpha \wedge \beta] = \textstyle\int[\alpha] \cup \int[\beta], \tag{15.18}$$

so the mapping between cohomology rings

$$H^*_{DR}(X) \xrightarrow{\int} H^*_{sing}(X, \mathbb{R}), \tag{15.19}$$

is moreover a ring isomorphism.

Remark 15.8. Using a Riemannian metric, there exists a covering by geodesically convex neighborhoods, so it follows that every compact manifold admits a finite good cover. Furthermore, the Mayer-Vietoris sequence shows that the de Rham cohomology of any compact manifold is finite-dimensional.

16 Lecture 16

16.1 Cohomology with compact supports

Let M be a manifold, possibly noncompact. Let $\Omega^p_c(M)$ denote the smooth p-forms with compact support. We have a complex

$$\cdots \xrightarrow{\ d\ } \Omega^{p-1}_c(M) \xrightarrow{\ d\ } \Omega^p_c(M) \xrightarrow{\ d\ } \Omega^{p+1}_c(M) \xrightarrow{\ d\ } \cdots, \tag{16.1}$$

and $H^p_{c,dR}(M)$ is defined to be the cohomology of this complex.

Let M be a differentiable n-manifold, and consider $N = M \times \mathbb{R}$. Let $\pi : N \to M$ be the projection $\pi(p,t) = p$. We next define a mapping

$$\pi_* : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M) \tag{16.2}$$

by the following. Any k-form on N can be written as

$$\omega = h(p,t)\pi^*\phi_k + f(p,t)(\pi^*\phi_{k-1}) \wedge dt, \tag{16.3}$$

where $\phi_k \in \Omega^k(M)$ and $\phi_{k-1} \in \Omega^{k-1}(M)$, but $h, f \in \Omega^0_c(M \times \mathbb{R})$. Define

$$\pi_*(\omega) = \Big(\int_{-\infty}^{\infty} f(p,t)dt \Big) \phi_{k-1}, \tag{16.4}$$

noting that the integral is defined because ω is assumed to have compact support, and this form has compact support since f has compact support.

We claim that

$$d_M \circ \pi_* = \pi_* \circ d_N. \tag{16.5}$$

To see this, the left hand side of (16.5) is

$$
\begin{aligned}
d_M \circ \pi_* \omega &= d_M \Big(\Big(\int_{-\infty}^{\infty} f(p,t) dt \Big) \phi_{k-1} \Big) \\
&= \Big(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} dt \Big) dx \wedge \phi_{k-1} + \Big(\int_{-\infty}^{\infty} f(p,t) dt) \Big) d_M \phi_{k-1}.
\end{aligned}
\tag{16.6}
$$

The right hand side of (16.5) is

$$
\begin{aligned}
\pi_* \circ d_N \omega &= \pi_* \Big(\frac{\partial h}{\partial t} dt \wedge \pi^* \phi_k + \frac{\partial f}{\partial x} dx \wedge \pi^* \phi_{k-1} \wedge dt + f(p,t) \pi^* (d_M \phi_{k-1}) \wedge dt \Big) \\
&= \pi_* \Big(\frac{\partial f}{\partial x} dx \wedge \pi^* \phi_{k-1} \wedge dt + f(p,t) \pi^* (d_M \phi_{k-1}) \wedge dt \Big) \\
&= \Big(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} dt \Big) dx \wedge \phi_{k-1} + \Big(\int_{-\infty}^{\infty} f(p,t) dt) \Big) d_M \phi_{k-1},
\end{aligned}
\tag{16.7}
$$

since the term involving h is zero because h has compact support, and using the fundamental theorem of calculus. Therefore π_* induces a mapping

$$
\pi_* : H^k_{c,dR}(M \times \mathbb{R}) \to H^{k-1}_{c,dR}(M). \tag{16.8}
$$

Next, we choose $e \in \Omega^1_c(\mathbb{R})$ with $\int_R e = 1$, and define

$$
e_* : \Omega^k_c(M) \to \Omega^{k+1}_c(M \times \mathbb{R}) \tag{16.9}
$$

by

$$
e_*(\omega) = (\pi^* \omega) \wedge e. \tag{16.10}
$$

We claim that

$$
d_N \circ e_* = e_* \circ d_M. \tag{16.11}
$$

To see this,

$$
d_N \circ e_*(\omega) = d_N \pi^* \omega \wedge e = (d_N \pi^* \omega) \wedge e = \pi^*(d_M \omega) \wedge e = e_* \circ d_M(\omega). \tag{16.12}
$$

Therefore e_* induces a mapping

$$
e_* : H^k_{c,dR}(M) \to H^{k+1}_{c,dR}(M \times \mathbb{R}). \tag{16.13}
$$

Let us write $e = \chi dt$, then

$$
\pi_* \circ e_*(\omega) = \pi_* \Big(\chi(t) (\pi^* \omega) \wedge dt \Big) = \Big(\int_{-\infty}^{\infty} \chi(t) dt \Big) \omega = \omega \tag{16.14}
$$

Therefore, we have $\pi_* \circ e_* = 1$ on $\Omega^k_c(M)$, so $\pi_* \circ e_* = 1$ on $H^k_{c,dR}(M)$.

Proposition 16.1. *We have $e_* \circ \pi_* = 1$ on $H^k_{c,dR}(M \times \mathbb{R})$. Consequently, π_* and e_* are isomorphisms on compactly supported cohomology.*

Proof. Again writing

$$\omega = h(p,t)\pi^*\phi_k + f(p,t)(\pi^*\phi_{k-1}) \wedge dt, \tag{16.15}$$

define a mapping

$$K : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M \times \mathbb{R}) \tag{16.16}$$

by

$$K(\omega) = \pi^*\phi_{k-1}\Big(\int_{-\infty}^t f(x,s)ds - \Big(\int_{-\infty}^t e\Big)\int_{-\infty}^\infty f(x,s)ds\Big). \tag{16.17}$$

Note that the right hand side is indeed a $(k-1)$-form on $M \times \mathbb{R}$ with compact support, the dt-s are not 1-forms in this formula. We claim that if $\omega \in \Omega^k_c(M \times \mathbb{R})$ then

$$(1 - e_*\pi_*)\omega = (-1)^{k-1}(dK - Kd)\omega, \tag{16.18}$$

which can be separately verified for $\omega = h(p,t)\pi^*\phi_k$, and for forms of type $\omega = f(p,t)dt \wedge \pi^*\phi_{k-1}$.

For forms of the first type, we obviously have

$$(1 - e_*\pi_*)h(p,t)\pi^*\phi_k = h(p,t)\pi^*\phi_k. \tag{16.19}$$

On the other hand, since K is zero on forms of this type,

$$\begin{aligned}
(dK - Kd)(h(p,t)\pi^*\phi_k) &= -K\Big(\Big(\frac{\partial h}{\partial x}\Big)dx \wedge \pi^*\phi_k + \Big(\frac{\partial h}{\partial t}\Big)dt \wedge \pi^*\phi_k + h(p,t)\pi^*d\phi_k\Big)\\
&= -K\Big(\Big(\frac{\partial h}{\partial t}\Big)dt \wedge \pi^*\phi_k\Big)\\
&= (-1)^{k-1}K\Big(\Big(\frac{\partial h}{\partial t}\Big)(\pi^*\phi_k) \wedge dt\Big)\\
&= (-1)^{k-1}\pi^*\phi_k\Big(\int_{-\infty}^t \frac{\partial h}{\partial t}ds - \Big(\int_{-\infty}^t e\Big)\int_{-\infty}^\infty \frac{\partial h}{\partial t}ds\Big)\\
&= (-1)^{k-1}(\pi^*\phi_k)h(p,t).
\end{aligned} \tag{16.20}$$

For forms of the second type, we have

$$\begin{aligned}
(1 - e_*\pi_*)f(p,t)\pi^*\phi_{k-1} \wedge dt &= f(p,t)\pi^*\phi_{k-1} \wedge dt - \Big(\int_{-\infty}^\infty f(p,t)dt\Big)(\pi^*\phi_{k-1}) \wedge e\\
&= \pi^*\phi_{k-1} \wedge \Big(f(p,t)dt - \Big(\int_{-\infty}^\infty f(p,t)dt\Big)e\Big)\\
&= (-1)^{k-1}\Big(f(p,t) - \Big(\int_{-\infty}^\infty f(p,t)dt\Big)\chi(t)\Big)\pi^*\phi_{k-1} \wedge dt
\end{aligned} \tag{16.21}$$

The verification that this is equal to $(-1)^{k-1}(dK - Kd)$ is left as an exercise.

This formula then implies that $e_* \circ \pi_* = 1$ as a mapping on $H^k_{c,dR}(M \times \mathbb{R})$, and the proposition follows. □

Corollary 16.2. *We have*

$$H^k_{c,dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases} \tag{16.22}$$

and a generator for $H^n_{c,dR}(\mathbb{R}^n)$ *is given by any compactly supported* n*-form* μ *with* $\int_{\mathbb{R}^n} \mu = 1$.

Notice that $H^k_{c,dR}(\mathbb{R}^n) \cong H^{n-k}_{dR}(\mathbb{R}^n)$. Furthermore, we have an isomorphism

$$PD : H^k_{dR}(\mathbb{R}^n) \to (H^{n-k}_{c,dR}(\mathbb{R}^n))^* \tag{16.23}$$

given by by $PD(\alpha)(\beta) = \int_{\mathbb{R}^n} \alpha \wedge \beta$.

17 Lecture 17

17.1 Mayer-Vietoris for cohomology with compact supports

Write $M = U \cup V$ as the union of two open sets in M. Note that if $U_1 \subset U_2$ and $\omega \in \Omega^k_c(U_1)$ then ω extends to be a compactly supported form in U_2. Letting $\iota : U_1 \hookrightarrow U_2$ denote the inclusion mapping, we denote by $i_*\omega$ this extension map on forms. We claim that the following sequence is exact:

$$0 \longrightarrow \Omega^p_c(U \cap V) \xrightarrow{\tilde{\alpha}^p} \Omega^p_c(U) \oplus \Omega^p_c(V) \xrightarrow{\tilde{\beta}^p} \Omega^p_c(U \cup V) \longrightarrow 0 \tag{17.1}$$

where

$$\tilde{\alpha}^p(\omega_{U\cap V}) = \big((i_{U\cap V \hookrightarrow U})_*\omega_{U\cap V}, -(i_{U\cap V\hookrightarrow V})_*\omega_{U\cap V}\big) \tag{17.2}$$

and

$$\tilde{\beta}^p(\omega_U, \omega_V) = (i_{U\hookrightarrow M})_*\omega_U + (i_{V\hookrightarrow M})_*\omega_V. \tag{17.3}$$

To see this, $\tilde{\alpha}^p$ is obviously injective. For exactness at the middle step, obviously $\tilde{\beta}^p\tilde{\alpha}^p\omega = 0$. If $\tilde{\beta}^p(\omega_U, \omega_V) = 0$, then $\omega_U = -\omega_V$. This implies that the support of both forms is contained in $U \cap V$, and since they are equal there, take $\omega_{U\cap V} = \omega_U$, and then $(\omega_U, \omega_V) = \tilde{\alpha}^p(\omega_U)$.

To show that $\tilde{\beta}$ is onto, let $\omega \in \Omega^p_c(M)$. Let ϕ_U, ϕ_V be a partition of unity subordinate to the covering $\{U, V\}$. Then $\omega = \tilde{\beta}^p(\phi_U\omega, \phi_V\omega)$.

Consequently, from the ziz-zag Lemma, we obtain a long exact sequence

$$\cdots \xrightarrow{\tilde{\delta}^{p-1}} H^p_{c,dR}(U \cap V) \xrightarrow{\tilde{\alpha}^p} H^p_{c,dR}(U) \oplus H^p_{c,dR}(V) \xrightarrow{\tilde{\beta}^p} H^p_{c,dR}(U \cup V) \xrightarrow{\tilde{\delta}^p} \cdots \tag{17.4}$$

Let us review the definition of the mapping $\tilde{\delta}^p$. Given a cohomology class $[\omega] \in H^p_{c,dR}(U \cup V)$, represented by $\omega \in \Omega^p_c(U \cup V)$ with $d\omega = 0$, we first write $\omega = \tilde{\beta}^p(\phi_U \omega, \phi_V \omega)$, then we apply the exterior derivative to get

$$(d(\phi_U \omega), d(\phi_V \omega)) = (d\phi_U \wedge \omega, d\phi_V \wedge \omega) \in \Omega^p_c(U) \oplus \Omega^p_c(V) \tag{17.5}$$

Either of these elements is supported in $U \cap V$ and then since $d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0$,

$$\tilde{\delta}^p \omega = [d\phi_U \wedge \omega] = [-d\phi_V \wedge \omega] \in H^{p+1}_{c,dR}(U \cap V). \tag{17.6}$$

Remark 17.1. This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

17.2 Poincaré Duality

Lemma 17.2. *If the sequence*

$$W_1 \xrightarrow{\alpha} W_2 \xrightarrow{\beta} W_3 \tag{17.7}$$

is exact at W_2, then the dual sequence

$$W_3^* \xrightarrow{\beta^*} W_2^* \xrightarrow{\alpha^*} W_1^* \tag{17.8}$$

is exact at W_2^*.

Proof. First, if $w_3^* \in W_3^*$, and $w_1 \in W_1$, then

$$\alpha^*(\beta^* w_3^*)(w_1) = (\beta^* w_3^*)(\alpha(w_1)) = w_3^*(\beta\alpha(w_1)) = 0, \tag{17.9}$$

since $\beta \circ \alpha = 1$ by assumption. This proves that $Im(\beta^*) \subset Ker(\alpha^*)$. For the other direction, if $w_2^* \in Ker(\alpha^*)$, then for all $w_1 \in W_1$, $\alpha^*(w_2^*)(w_1) = w_2^*(\alpha(w_1))$. So the element $0 = w_2^* \circ \alpha \in W_1^*$. We want to find $w_3^* \in W_3^*$ such that $w_2^* = \beta^* w_3^*$. For all $w_2 \in W_2$, this is $w_2^*(w_2) = w_3^* \beta w_2$, which is just $w_2^* = w_3^* \circ \beta$. So if $w_3 \in W_3$ is of the form $\beta(w_2)$ then define

$$w_3^*(w_3) \equiv w_2^*(w_2). \tag{17.10}$$

If $w_3 = \beta(w_2')$, then $\beta(w_2 - w_2') = 0$, so $w_2 - w_2' = \alpha(w_1)$. Then

$$w_2^*(w_2 - w_2') = w_2^*(\alpha(w_1)) = (w_2^* \alpha)(w_1) = 0. \tag{17.11}$$

So we have defined w_3^* on the subspace $Im(\beta) \subset W_3$. To extend to a linear mapping on all of W_3, just take any subspace so that $W_3 = Im(\beta) \oplus W$, and define w_3^* to vanish on W. Then the condition $w_2^* = w_3^* \circ \beta$ is obviously satisifed. $\square$

Theorem 17.3. *If M^n is orientable and has a finite good cover, then*

$$PD : H^k_{dR}(M) \to (H^{n-k}_{c,dR})^* \tag{17.12}$$

is an isomorphism for all $0 \leq k \leq n$.

Proof. Let $m = n - k$, and consider the diagram

$$\begin{tikzcd}
H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) \arrow[r, "\alpha^{k-1}"] \arrow[d, "PD \oplus PD"] & H^{k-1}_{dR}(U \cap V) \arrow[r, "\delta^k"] \arrow[d, "PD"] & H^k_{dR}(U \cup V) \arrow[r, "\beta^k"] \arrow[d, "PD"] & H^k_{dR}(U) \oplus H^k_{dR}(V) \arrow[r, "\alpha^k"] \arrow[d, "PD \oplus PD"] & H^k_{dR}(U \cap V) \arrow[d, "PD"] \\
\left(H^{m+1}_{c,dR}(U) \oplus H^{m+1}_{d,dR}(V)\right)^* \arrow[r, "(\tilde{\alpha}^{m+1})^*"] & H^{m+1}_{c,dR}(U \cap V)^* \arrow[r, "(\tilde{\delta}^m)^*"] & H^m_{c,dR}(U \cup V)^* \arrow[r, "(\tilde{\beta}^m)^*"] & \left(H^m_{c,dR}(U) \oplus H^m_{c,dR}(V)\right)^* \arrow[r, "(\tilde{\alpha}^m)^*"] & H^m_{c,dR}(U \cap V)^*
\end{tikzcd} \tag{17.13}$$

The top horizontal row is exact since it is the usual Mayer-Vietoris sequence. The bottom horizontal row is exact since is the dual exact sequence of the Mayer-Vietoris sequence with compact support. We next claim that this diagram commutes up to sign, so by changing some of the vertical maps to their negatives if necessary, we obtain a commutative diagram. (Proof done in lecture).

By the five lemma, if the outer 4 vertical maps are isomorphisms, then so is the central vertical map. The proof is completed by induction on the number of open sets in the good cover, since we know it is true for $\mathbb{R}^n$ from the previous lecture. □

Corollary 17.4. *If M^n is a connected and orientable n-manifold with a finite good cover, then $H^n_{c,dR}(M) \cong \mathbb{R}$. If M is moreover compact, then $H^n_{dR}(M) \cong \mathbb{R}$.*

Corollary 17.5. *If M^n is a connected and orientable n-manifold with a finite good cover then $H^k_{dR}(M)$ and $H^{n-k}_{c,dR}(M)$ have the same dimension. If M is moreover compact, then $H^k_{dR}(M)$ and $H^{n-k}_{dR}(M)$ have the same dimension.*

Corollary 17.6. *If M^n is a compact oriented odd-dimensional manifold, then the Euler characteristic $\chi(M) = 0$.*

Remark 17.7. Poincaré duality is also true for singular homology with $\mathbb{Z}$ coefficients on a orienable manifold. If M is not orientable, then it is still true for $\mathbb{Z}/2\mathbb{Z}$ coefficients.

18 Lecture 18

Theorem 18.1 (Künneth formula)**.** *We have*

$$H^k(M \times N) = \bigoplus_{p+q=k} H^p(M) \otimes H^q(N). \tag{18.1}$$

Proof. Proof done in lecture, using the Mayer-Vietoris argument, the Poincaré Lemma, and the five lemma. □

Corollary 18.2. *Let*

$$T^n = \overbrace{S^1 \times \cdots \times S^1}^{n}, \tag{18.2}$$

then

$$\dim(H^k(T^n)) = \binom{n}{k} \tag{18.3}$$

19 Lecture 19

Definition 19.1. If $\Sigma^k \subset M^n$ is a closed submanifold then the closed Poincaré dual is denoted η_Σ and the compact Poincaré dual is η'_Σ, to be completed.

From Poincaré duality, we have that if M is orientable and connected, then

$$H^n(M) = \begin{cases} \mathbb{R} & \text{M compact} \\ 0 & \text{M non-compact} \end{cases}, \tag{19.1}$$

and

$$H^n_c(M) = \begin{cases} \mathbb{R} & \text{M compact} \\ \mathbb{R} & \text{M non-compact} \end{cases}, \tag{19.2}$$

Examples of computations of homology and cohomology using Mayer-Vietoris.

Example 19.2. S^n: Cover with 2 open sets U, V, with $U \cong \mathbb{R}^n \cong V$ a and $U \cap V \cong S^{n-1}$, use induction to get

$$H^k(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & 0 < k < n \end{cases} \tag{19.3}$$

Example 19.3. T^2: cover with 2 open sets U, V, with $U \cong V \cong S^1 \times \mathbb{R} \cong S^1$ and $U \cap V \cong S^1 \coprod S^1$, to get

$$H^k(T^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & k = 1 \end{cases} \tag{19.4}$$

20 Lecture 20

Recall definition of orientation bundle $L = \Lambda^n(T^*M)$, integration of densities. Only use bundle transition functions arising from coordinate systems.

Definition 20.1. Define $H^k(M, L)$ de Rham cohomology with coefficients in L.

Theorem 20.2 (Poincaré duality for densities)**.** *We have*

$$H^k(M) \cong H^{n-k}_c(M, L), \tag{20.1}$$

and

$$H^k_c(M) \cong H^{n-k}(M, L). \tag{20.2}$$

Proof. Proof by Mayer-Vietoris sequence, integration of densities to get duality, five lemma, and induction on the cardinality of a good cover. □

Corollary 20.3. *If M is non-orientable, then*

$$H^n(M) = H^n_c(M) = 0, \tag{20.3}$$

Example 20.4. $\mathbb{RP}^n$: Orientable if and only if n is odd. Cover with 2 open sets U, V, with $U \cong \mathbb{R}^n$, $V \cong \mathbb{RP}^{n-1}$ and $U \cap V \cong S^{n-1}$, use induction to get

$$H^k(\mathbb{RP}^n) = \begin{cases} \mathbb{R} & k = 0, \text{ or } k = n \text{ is odd} \\ 0 & \text{otherwise,} \end{cases} \tag{20.4}$$

Example 20.5. $\mathbb{CP}^n$

21 Lecture 21

Proposition 21.1. *For any N, $H^k(\mathbb{RP}^N, \mathbb{Z}_2) = \mathbb{Z}_2$ for $k \leq N$. Denote the nontrivial element of $H^1(\mathbb{RP}^N, \mathbb{Z}_2)$ by a. Then a generator of $H^k(\mathbb{RP}^N, \mathbb{Z}_2)$ is given by a^k, where the operation is the cup product. The cohomology algebra $H^*(\mathbb{RP}^\infty, \mathbb{Z}_2)$ is a polynomial algebra generated by $1, a$.*

Proof. We use the fact that singular homology is isomorphic to cellular homology for a CW-complex. The space $\mathbb{RP}^N$ is a CW-complex with a single cell in each dimension less than or equal to N. To see this: the top dimensional cell is $\mathbb{R}^N = [1, x_1, \ldots, x_N] \subset \mathbb{RP}^N$. The missing set is $\mathbb{RP}^{N-1} = [0, x_1, \ldots, x_N] \subset \mathbb{RP}^N$, then use induction. Therefore, the CW chain complex with $\mathbb{Z}_2$ coefficients

$$0 \to \mathcal{C}_N \xrightarrow{\partial} \mathcal{C}_{N-1} \xrightarrow{\partial} \cdots \to \mathcal{C}_0 \to 0. \tag{21.1}$$

is just

$$0 \to \mathbb{Z}_2 \xrightarrow{\partial} \mathbb{Z}_2 \xrightarrow{\partial} \cdots \to \mathbb{Z}_2 \to 0. \tag{21.2}$$

The boundary of any n-cell is always twice the $(n-1)$-cell or 0, so the boundary maps are all zero since the coefficients are ZZ_2. This shows that $H_k(\mathbb{RP}^N, \mathbb{Z}_2) \cong \mathbb{Z}_2$. By the universal coefficient theorem, the cohomology groups are the same.

To determine the ring structure, we need some more information, because cellular cohomology loses the ring structure. The proof proceeds by induction. For $n = 2$, Poincaré duality (which still works for $\mathbb{Z}_2$ coefficients on a non-orientable manifold), says that for $[a] \in H^1(\mathbb{RP}^2, \mathbb{Z}_2)$, there is a dual element $[a'] \in H^1(\mathbb{RP}^2, \mathbb{Z}_2)$ such that $[a] \cup [a']$ is the generator of $H^2(\mathbb{RP}^2, \mathbb{Z}_2)$. Clearly, the only possbility is that $[a'] = [a]$, so we have that $a^2 = [a] \cup [a]$ is the generator of $H^2(\mathbb{RP}^2, \mathbb{Z}_2)$. Next, use that fact that the inclusion $\mathbb{RP}^{n-1} \subset \mathbb{RP}^n$ induces isomorphisms in homology and cohomology in dimensions strictly less than n. So then $a, a^2, \ldots, a^{n-1}$ are all non-zero. To show that a^n is a generator, again use Poincaré duality. This says that there is an element $[a'] \in H^1(\mathbb{RP}^n, \mathbb{Z}_2)$ such that $[a^{n-1}] \cup [a']$ is a generator of $H^n(\mathbb{RP}^n, \mathbb{Z}_2)$. Again, clearly $[a'] = [a]$ so $[a^n]$ is a generator of $H^n(\mathbb{RP}^n, \mathbb{Z}_2)$. $\square$

Definition 21.2. Let $\pi : L \to M$ be a real line bundle over M. Define $w_1'(L) = f^*a$, where f is any classifying map for L.

We need to prove that this is well-defined. For this, we will prove a much more general fact.

Proposition 21.3. *If $f_1 : M \to N$ is homotopic to $f_2 : M \to N$ then for any vector bundle $\pi : E \to N$, we have $f_1^*E \cong f_2^*E$.*

Conversely, if vector bundles E_1, E_2 over M are equivalent, then the classifying maps $f_1 : M \to G(k, N_1)$ and $f_2 : M \to G(k, N_2)$ are homotopic in $G(k, \infty)$ (the infinite Grassmannian).

Proof. (Outline) For the first part, a homotopy is a smooth mapping $H : [0,1] \times M \to N$ such that $H(0,p) = f_1(p)$, and $H(1,p) = f_2(p)$.

One proves that if H^*E can be trivialized by a covering of the form $(\epsilon_i, \epsilon_{i+1}) \times V_\alpha$ where $V_\alpha = f^{-1}(U_\alpha)$, and where E is trivial on U_α.

Next, one shows that if a bundle in trivial on $(a,b) \times U$ and also on $(c,d) \times U$, where $a < c < b < d$, then the bundle is trivial on $(a,d) \times U$.

From this it follows that H^*E is trivial on $(0,1) \times V_\alpha$. This then implies that $H^*E\big|_{\{0\}\times M}$ is isomorphic that $H^*E\big|_{\{1\}\times M}$, which implies that $f_1^*E \cong f_2^*E$.

For the second part, we saw that a classifying map $f : M \to \mathbb{RP}^N$ gives a mapping $\hat{f} : L \to \mathbb{R}^N$, which is linear and injective on fibers, and conversely, such a map determines f. So if we have another such map $\hat{g} : L \to \mathbb{R}^N$, then just take the homotopy $\hat{H}(t,p) = (1-t)\hat{f} + t\hat{g}$, which gives a homotopy between f and g. The problem with this is that $\hat{f}(e)$ might be equal to $-\hat{g}(e)$ for some vector e. To get around this, let $d_1 : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the mapping

$$(x_0, x_1, x_2 \dots) \mapsto (x_0, 0, x_1, 0, x_2, \dots), \tag{21.3}$$

and $d_2 : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the mapping

$$(x_0, x_1, x_2 \dots) \mapsto (0, x_0, 0, x_1, 0, x_2, \dots). \tag{21.4}$$

Note that we must have the following homotopies

$$\hat{f} \cong d_1 \circ \hat{f} \cong d_2 \circ \hat{g} \cong \hat{g}, \tag{21.5}$$

so that f is homotopic to g.

□

Corollary 21.4. *The vector bundles of rank k over M up to equivalence are in bijection with the homotopy classes $[M, G(k, \infty)]$.*

We claim this new definition w_1', is well-defined, and is equivalent to the first definition of w_1. To see this, note that for any mapping, we have

$$w_1'(f^*L) = f^*(w_1'(L)), \tag{21.6}$$

and

$$w_1'(\gamma_N^1) = a. \tag{21.7}$$

Apply this to a classifying map $f : M \to \mathbb{RP}^N$, we have

$$w_1(L) = w_1(f^*\gamma_N^1) = f^*w_1(\gamma_N^1) = f^*[a] = w_1'(L). \tag{21.8}$$

21.1 Lorentzian metrics

If we instead specify that g is non-degenerate, but with a 1-dimensional maximally negative definite subspace at each point, then g is called a Lorentzian metric.

Proposition 21.5. *If M is compact, then M admits a Lorentzian metric if and only $\chi(M) = 0$. If M is non-compact, then M admits a Lorentzian metric.*

Proof. If M is non-compact, then M admits a nowhere vanishing vector field. This means that

$$TM = A \oplus B, \tag{21.9}$$

where $\dim(A_p) = 1$ for every $p \in M$, and A is a trivial bundle. The bundle A admits a Riemannian metric g_A, and B admits a Riemannian metric g_B. Then $g = -g_A + g_B$ is a Lorentzian metric.

If M is compact then M admits a nowhere vanishing vector field if and only if $\chi(M) = 0$. So if $\chi(M) = 0$, M admits a Lorentzian metric by the same argument. Conversely, if M admit a Lorenztian metric g, then the negative definite subspace defines a 1-dimensional sub-bundle of the tangent bundle, i.e.,

$$TM = A \oplus B, \tag{21.10}$$

where $\dim(A_p) = 1$ for every $p \in M$. There is a double cover $\pi : \tilde{M} \to M$ such that $\pi^* A$ is a trivial bundle. So then

$$T\tilde{M} = \pi^* TM = \pi^* A \oplus \pi^* B. \tag{21.11}$$

Since $\pi^* A$ is trivial, $\tilde{M}$ admits a non-zero vector field, which implies that $\chi(\tilde{M}) = 0$. But the Euler number is multiplicative under coverings, so $\chi(M) = \chi(\tilde{M})/2 = 0$. □

Corollary 21.6. *S^n admits a Lorenztian metric if and only if $n \leq 3$ and n is odd.*

The only compact surfaces which admit a Lorenztian metric are T^2 and the Klein bottle.

22 Lecture 22

22.1 Realification of complex bundles

If we view $\mathbb{C}$ as a 2-dimensional vector space over $\mathbb{R}$, then we can view any complex rank k vector bundle as a real rank $2k$ vector bundle. This corresponds to an embedding

$$GL(k, \mathbb{C}) \hookrightarrow GL(2k, \mathbb{R}) \tag{22.1}$$

given by

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{22.2}$$

Proposition 22.1. *Any complex vector bundle, when viewed as a real vector bundle, is orientable.*

Proof. The matrix

$$\begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix} \tag{22.3}$$

is related to the matrix on the right hand side of (22.2) by a change of basis. Therefore,

$$\det \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \det(A+iB)\det(A-iB) = |\det(A+iB)|^2 > 0, \tag{22.4}$$

which shows that above imbedding maps

$$GL(k,\mathbb{C}) \hookrightarrow GL_+(2k,\mathbb{R}). \tag{22.5}$$

Alternatively, by choosing a Hermitian metric, we can reduce the structure group to $U(k)$. Then we have

$$(A+iB)(\overline{A+iB})^T = Id_k, \tag{22.6}$$

which yields

$$AA^T + BB^T = Id_k \tag{22.7}$$

$$BA^T - AB^T = 0. \tag{22.8}$$

It follows that

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \cdot \begin{pmatrix} A & -B \\ B & A \end{pmatrix}^T = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \cdot \begin{pmatrix} A^T & B^T \\ -B & A^T \end{pmatrix} = \begin{pmatrix} Id_k & 0 \\ 0 & Id_k \end{pmatrix}, \tag{22.9}$$

which shows that $U(n) \hookrightarrow SO(2n)$ under the above imbedding. $\square$

23 Lecture 23

23.1 Connections on vector bundles

A connection is a mapping $\Gamma(TM) \times \Gamma(E) \to \Gamma(E)$, with the properties

- $\nabla_X s \in \Gamma(E)$,
- $\nabla_{f_1 X_1 + f_2 X_2} s = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s$,
- $\nabla_X (fs) = (Xf)s + f\nabla_X s$.

In coordinates, letting $s_i, i = 1 \dots p$, be a local basis of sections of E,

$$\nabla_{\partial_i} s_j = \Gamma^k_{ij} s_k. \tag{23.1}$$

If E carries an inner product, then ∇ is *compatible* if

$$X\langle s_1, s_2\rangle = \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle. \tag{23.2}$$

For a connection in TM, ∇ is called *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall X, Y \in \Gamma(TM). \tag{23.3}$$

Theorem 23.1. *(Fundamental Theorem of Riemannian Geometry) There exists a unique symmetric, compatible connection in TM.*

Invariantly, the connection is defined by

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}\Big(&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &-\langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle\Big).\end{aligned} \tag{23.4}$$

Letting $X = \partial_i, Y = \partial_j, Z = \partial_k$, we obtain

$$\begin{aligned}\Gamma^l_{ij} g_{lk} &= \langle \Gamma^l_{ij}\partial_l, \partial_k\rangle = \langle \nabla_{\partial_i}\partial_j, \partial_k\rangle \\ &= \frac{1}{2}\Big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\Big),\end{aligned} \tag{23.5}$$

which yields the formula

$$\boxed{\Gamma^k_{ij} = \frac{1}{2} g^{kl}\Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\Big)} \tag{23.6}$$

for the Riemannian Christoffel symbols.

23.2 Pull-back bundles

Let $\pi : E \to M$ be a real vector bundle of rank k over M, and $f : N \to M$ be a smooth mapping. Recall that

$$f^*E = \{(p, v) \in N \times E \mid f(p) = \pi(v)\} \tag{23.7}$$

is a vector bundle over N, called the pull-back bundle of E along f. Note the following:

- If X is a vector field on N, then in general $f_* X$ is not a vector field on M. However, $f_* X$ is a well-defined section of f^*TM.
- For a vector $v \in E_{f(p)}$, we define $f^*v \in (f^*E)_p$ by $f^*v = (p, v)$.
- Given a section $s : M \to E$, we define the pull-back $f^*s \in \Gamma(f^*E)$ by $f^*s(p) = (p, s \circ f)$.

Proposition 23.2 (Pull-back connection). *If $f : N \to M$, and ∇ is a connection in $\pi : E \to M$, then there is a unique connection $f^*\nabla$ in the pull-back bundle f^*E over N such that for any section $s : M \to E$, and $X \in TN$,*

$$(f^*\nabla)_X(f^*s) = f^*(\nabla_{f_*X}s). \tag{23.8}$$

Proof. To define the pullback connection, fix a $p \in N$, and choose a local frame $s_1, \ldots s_k$ near $f(p)$ for the bundle E. Then locally, write a section $s \in \Gamma(f^*E)$ as

$$s = \sum_{i=1}^{k} s^i(f^*s_i), \tag{23.9}$$

where s^i is a smooth function defined in a neighborhood of p. Then for $X \in T_pN$, define

$$(f^*\nabla)_X s = X(s^i)f^*s_i + s^i f^*(\nabla_{f_*X}s_i). \tag{23.10}$$

Observe that the connection, if it exists, is locally unique. This formula then yields a well-defined global connection on f^*E over N, which is the unique one satisfying (23.8) □

Note that if E admits a Riemannian metric g, then f^*E admits a Riemmannian metric f^*g defined by

$$(f^*g)(v, w) \equiv g(\pi_2 v, \pi_2 w). \tag{23.11}$$

Note that this is really a "restriction" of the metric to the pull-back bundle, it is not the same as the pull-back of a tensor field. For the restriction, we have the following

Proposition 23.3. *If ∇ is compatible with g then $f^*\nabla$ is compatible with f^*g.*

Proof. Let $f^*s_1, f^*s_2 \in \Gamma(f^*E)$ for sections s_1 and s_2 in $\Gamma(E)$ and $X \in T_pN$. Then

$$\begin{aligned}
X_p\big(f^*g(f^*s_1, f^*s_2)\big) &= X_p\Big(\big(g(s_1, s_2)\big) \circ f\Big)\\
&= (f_*X)(g(s_1, s_2))\\
&= g(\nabla_{f_*X}s_1, s_2) + g(s_1, \nabla_{f_*X}s_2)\\
&= g(\pi_2((f^*\nabla)_X f^*s_1), \pi_2 f^*s_2) + g(\pi_2 f^*s_1, \pi_2((f^*\nabla)_X f^*s_2))\\
&= f^*g((f^*\nabla)_X f^*s_1, f^*s_2) + f^*g(f^*s_1, (f^*\nabla)_X f^*s_2).
\end{aligned} \tag{23.12}$$

The general case follows since any section may locally be written in the form (23.9). □

24 Lecture 24

We will show some properties of pullback connection which will be useful later.

Proposition 24.1. *Let $f : N \to M$ be smooth. If ∇ is a symmetric connection in $E = TM$, then the pull-back connection $f^*\nabla$ on f^*TM satifies for any $X, Y \in \Gamma(TN)$,*

$$(f^*\nabla)_X(f_*Y) - (f^*\nabla)_Y(f_*X) = f_*([X,Y]). \tag{24.1}$$

Proof. Choose a local coordinate $x^i : U \to \mathbb{R}^m$ near p and $y^i : V \to \mathbb{R}^n$ near $f(p)$. Write $X = X^i\frac{\partial}{\partial x^i}$ and $Y = Y^i\frac{\partial}{\partial x^i}$, and $y \circ f = (f^1, \dots, f^n)$, where $f^i : U \to \mathbb{R}$. Note that

$$f_*X = \sum_{j=1}^{m}\sum_{\alpha=1}^{n} X^j\Big(\frac{\partial}{\partial x^j}f^\alpha\Big)f^*\Big(\frac{\partial}{\partial y^\alpha}\Big). \tag{24.2}$$

Then

$$\begin{aligned}
&(f^*\nabla)_X(f_*Y) - (f^*\nabla)_Y(f_*X) = (f^*\nabla)_X\Big(Y^j\Big(\frac{\partial}{\partial x^j}f^\alpha\Big)f^*\Big(\frac{\partial}{\partial y^\alpha}\Big)\Big) - (X \leftrightarrow Y)\\
&= (f^*\nabla)_{X^i\frac{\partial}{\partial x^i}}\Big(Y^j\Big(\frac{\partial}{\partial x^j}f^\alpha\Big)f^*\Big(\frac{\partial}{\partial y^\alpha}\Big)\Big) - (X \leftrightarrow Y)\\
&= X^i(f^*\nabla)_{\frac{\partial}{\partial x^i}}\Big(Y^j\Big(\frac{\partial}{\partial x^j}f^\alpha\Big)f^*\Big(\frac{\partial}{\partial y^\alpha}\Big)\Big) - (X \leftrightarrow Y)\\
&= X^i\frac{\partial}{\partial x^i}\Big(Y^j\Big(\frac{\partial}{\partial x^j}f^\alpha\Big)\Big)f^*\Big(\frac{\partial}{\partial y^\alpha}\Big) + X^iY^j\Big(\frac{\partial}{\partial x^j}f^\alpha\Big)(f^*\nabla)_{\frac{\partial}{\partial x^i}}\Big(f^*\frac{\partial}{\partial y^\alpha}\Big) - (X \leftrightarrow Y)\\
&= X^i\frac{\partial}{\partial x^i}\Big(Y^j\frac{\partial}{\partial x^j}f^\alpha\Big)f^*\Big(\frac{\partial}{\partial y^\alpha}\Big) + X^iY^j\Big(\frac{\partial}{\partial x^j}f^\alpha\Big)f^*\Big(\nabla_{f_*\frac{\partial}{\partial x^i}}\frac{\partial}{\partial y^\alpha}\Big) - (X \leftrightarrow Y)\\
&= f_*([X,Y]).
\end{aligned} \tag{24.3}$$

This is because the covariant derivative terms vanish. To see this,

$$\begin{aligned}
&X^iY^j\Big(\frac{\partial f^\alpha}{\partial x^j}\Big)f^*\Big(\nabla_{f_*\frac{\partial}{\partial x^i}}\frac{\partial}{\partial y^\alpha}\Big) - (X \leftrightarrow Y)\\
&= X^iY^j\Big(\frac{\partial f^\alpha}{\partial x^j}\Big)f^*\Big(\nabla_{\frac{\partial f^\beta}{\partial x^i}\cdot\frac{\partial}{\partial y^\beta}}\frac{\partial}{\partial y^\alpha}\Big) - (X \leftrightarrow Y)\\
&= X^iY^j\Big(\frac{\partial f^\alpha}{\partial x^j}\Big)\Big(\frac{\partial f^\beta}{\partial x^i}\Big)f^*\Big(\nabla_{\frac{\partial}{\partial y^\beta}}\frac{\partial}{\partial y^\alpha}\Big) - (X \leftrightarrow Y)\\
&= X^iY^j\Big(\frac{\partial f^\alpha}{\partial x^j}\Big)\Big(\frac{\partial f^\beta}{\partial x^i}\Big)f^*\Big(\nabla_{\frac{\partial}{\partial y^\beta}}\frac{\partial}{\partial y^\alpha} - \nabla_{\frac{\partial}{\partial y^\alpha}}\frac{\partial}{\partial y^\beta}\Big) = 0,
\end{aligned} \tag{24.4}$$

because ∇ is symmetric. □

Definition 24.2. The curvature of a connection ∇ on a vector bundle E over M is $R_\nabla \in \Gamma(T^*M \otimes T^*M \otimes E^* \otimes E)$ defined by

$$R_\nabla(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s. \tag{24.5}$$

Exercise 24.3. Show that R_∇ is tensorial in all variables.

Proposition 24.4. *If $f : N \to M$ and ∇ is a connection on E over M and $X, Y \in \Gamma(TN)$, $s \in \Gamma(E)$ then*

$$R_{f^*\nabla}(X,Y)f^*s = f^*\big(R_\nabla(f_*X, f_*Y)s\big). \tag{24.6}$$

Proof. Choose a local coordinate $x^i : U \to \mathbb{R}^m$ near p and $y^i : V \to \mathbb{R}^n$ near $f(p)$. Write $y \circ f = (f^1, \dots, f^n)$, where $f^i : U \to \mathbb{R}$. Let s be a section of $\Gamma(E)$, then we compute

$$\begin{aligned} R_{f^*\nabla}\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)f^*s &= (f^*\nabla)_{\frac{\partial}{\partial x^i}}(f^*\nabla)_{\frac{\partial}{\partial x^j}} f^*s - (i \leftrightarrow j) \\ &= (f^*\nabla)_{\frac{\partial}{\partial x^i}} f^*(\nabla_{f_*\frac{\partial}{\partial x^j}} s) - (i \leftrightarrow j) \\ &= (f^*\nabla)_{\frac{\partial}{\partial x^i}} \Big(\frac{\partial}{\partial x^j} f^\alpha\Big) f^*\Big(\nabla_{\frac{\partial}{\partial y^\alpha}} s\Big) - (i \leftrightarrow j) \\ &= \Big(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f^\alpha\Big) f^*\Big(\nabla_{\frac{\partial}{\partial y^\alpha}} s\Big) + \Big(\frac{\partial}{\partial x^j} f^\alpha\Big)(f^*\nabla)_{\frac{\partial}{\partial x^i}} f^*\Big(\nabla_{\frac{\partial}{\partial y^\alpha}} s\Big) - (i \leftrightarrow j). \end{aligned} \tag{24.7}$$

Since the Hessian is symmetric in i and j, we have

$$\begin{aligned} R_{f^*\nabla}\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)f^*s &= \Big(\frac{\partial}{\partial x^j} f^\alpha\Big)(f^*\nabla)_{\frac{\partial}{\partial x^i}} f^*\Big(\nabla_{\frac{\partial}{\partial y^\alpha}} s\Big) - (i \leftrightarrow j) \\ &= \Big(\frac{\partial}{\partial x^j} f^\alpha\Big) f^*\Big(\nabla_{f_*\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial y^\alpha}} s\Big) - (i \leftrightarrow j) \\ &= \Big(\frac{\partial}{\partial x^j} f^\alpha\Big)\Big(\frac{\partial}{\partial x^i} f^\beta\Big) f^*\Big(\nabla_{\frac{\partial}{\partial y^\beta}} \nabla_{\frac{\partial}{\partial y^\alpha}} s\Big) - (i \leftrightarrow j) \\ &= \Big(\frac{\partial}{\partial x^j} f^\alpha\Big)\Big(\frac{\partial}{\partial x^i} f^\beta\Big) f^*\Big(\nabla_{\frac{\partial}{\partial y^\beta}} \nabla_{\frac{\partial}{\partial y^\alpha}} s\Big) - (\alpha \leftrightarrow \beta) \\ &= \Big(\frac{\partial}{\partial x^j} f^\alpha\Big)\Big(\frac{\partial}{\partial x^i} f^\beta\Big) f^* R_\nabla\Big(\frac{\partial}{\partial y^\beta}, \frac{\partial}{\partial y^\alpha}\Big)s \\ &= f^*\big(R_\nabla(f_*X, f_*Y)s\big). \end{aligned} \tag{24.8}$$

□

25 Lecture 25

25.1 Parallel Transport

As above, let ∇ be a connection in the bundle $\pi : E \to M$.

Definition 25.1. A section $s \in \Gamma(E)$ is *parallel* if $\nabla s \in \Gamma(T^*M \otimes E)$ satisfies $\nabla s \equiv 0$.

Choose a local basis of section $s_i, i = 1 \dots k$ of E, and local coordinates x^i on M, then by definition

$$\nabla_{\frac{\partial}{\partial x^i}} s_j = \Gamma_{ij}^k s_k, \tag{25.1}$$

so then for a section $s = s^j s_j$, we have

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^i}} s &= \Big(\frac{\partial}{\partial x^i} s^j \Big) s_j + s^j \Gamma_{ij}^k s_k \\ &= \Big(\frac{\partial}{\partial x^i} s^j + \Gamma_{ik}^j s^k \Big) s_j, \end{aligned} \tag{25.2}$$

so s being parallel implies the system of first order linear differental equations

$$\frac{\partial}{\partial x^i} s^j + \Gamma_{ik}^j s^k = 0. \tag{25.3}$$

A parallel section does not necessarily exist, even locally. In general, existence of a parallel section is an extremely restrictive condition. Note that if the Christoffel symbols vanish, the functions s^j must be constant.

Example 25.2. The space of parallel vector fields for a flat metric on a torus is 2-dimensional. If Σ is any compact orientable surface of genus $g \neq 1$, then there are no nontrivial parallel vector fields on Σ with respect to any metric. If there were, then this would be a nonzero vector field which implies the Euler characteristic vanishes.

Let $\gamma : I \to M$ be a smooth curve, where I is an interval. Then the pull-back bundle $\gamma^* E$ is a bundle over I and carries the connection $\gamma^* \nabla$.

Definition 25.3. A section $s \in \gamma^* E$ is *parallel along* γ if $(\gamma^* \nabla)_{\frac{d}{dt}} s = 0$ for every $t \in I$. Given a vector $V_{t_0} \in E_{\gamma(t_0)}$, there exists a unique parallel section $V \in \gamma^* E$ such that $V(t_0) = (t_0, V_{t_0})$. The section V is called the *parallel translate* of V_{t_0} along γ.

Proposition 25.4. *Let $\gamma : I \to M$, and choose a coordinate system $x : U \to \mathbb{R}^n$ and a coordinate neighborhood of $\gamma(t_0), t_0 \in I$. Assume also that $E|_U$ is trivial, and let $s_1, \ldots, s_k$ be a local basis of E over U Write $x \circ \gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$. Write $s \in \Gamma(\gamma^* E)$ as $s = \sum_{l=1}^k s^i \gamma^* s_i$, where $s^i : U \to \mathbb{R}$. Then then equation for $s \in \Gamma(\gamma^* E)$ to be parallel along γ is locally*

$$\frac{d}{dt} s^l + \Gamma_{ij}^l (\gamma(t)) s^j \Big(\frac{d\gamma^i}{dt} \Big) = 0. \tag{25.4}$$

Proof. First, using the chain rule, we write

$$\gamma_* \Big(\frac{d}{dt} \Big) = \frac{d\gamma^i}{dt} \cdot \frac{\partial}{\partial x^i}. \tag{25.5}$$

Next, we calculate

$$\begin{aligned} (\gamma^* \nabla)_{\frac{d}{dt}} s &= (\gamma^* \nabla)_{\frac{d}{dt}} \Big(s^j (\gamma^* s_j) \Big) \\ &= \frac{ds^j}{dt} (\gamma^* s_j) + s^j (\gamma^* \nabla)_{\frac{d}{dt}} (\gamma^* s_j). \end{aligned} \tag{25.6}$$

Note that by the definition of the pullback connection

$$\begin{aligned}(\gamma^*\nabla)_{\frac{d}{dt}}(\gamma^* s_j) &= \gamma^*\Big(\nabla_{\gamma_*\frac{d}{dt}} s_j\Big)\\ &= \gamma^*\Big(\nabla_{\frac{d\gamma^i}{dt}\cdot\frac{\partial}{\partial x^i}} s_j\Big)\\ &= \frac{d\gamma^i}{dt}\gamma^*\Big(\nabla_{\frac{\partial}{\partial x^i}} s_j\Big)\\ &= \frac{d\gamma^i}{dt}\Gamma^l_{ij}(\gamma(t))(\gamma^* s_l).\end{aligned}\tag{25.7}$$

Substituting this into the above, we obtain

$$\begin{aligned}(\gamma^*\nabla)_{\frac{d}{dt}} s &= \frac{ds^l}{dt}(\gamma^* s_l) + s^j\frac{d\gamma^i}{dt}\Gamma^l_{ij}(\gamma(t))(\gamma^* s_l)\\ &= \Big(\frac{ds^l}{dt} + \frac{d\gamma^i}{dt}\Gamma^l_{ij}(\gamma(t))s^j\Big)(\gamma^* s_l),\end{aligned}\tag{25.8}$$

and since the $\gamma^* s_l$ are a local basis of sections of $\gamma^* E$, the proposition follows. □

Since this is a first order linear ODE, the parallel translate of a vector at any point is a globally defined section along γ, there is no obstruction. This is closely related to the fact that the curvature tensor of $\gamma^*\nabla$ is identically zero, which follows from skew-symmetry of the curvature tensor in the first two indices.

Exercise 25.5. Prove that if $\gamma : [a, b] \to M$, then $P_{a,b} : E_{\gamma(a)} \to E_{\gamma(b)}$ is an invertible linear mapping.

Lemma 25.6. *If M is connected, and $s \in \Gamma(E)$ is parallel, then $s(p) = 0$ at a single point $p \in M$ implies that $s \equiv 0$. Equivalently, if s is non-zero at a point p, then s is non-zero everywhere.*

Proof. Take $q \in M$, and let $\gamma : I \to M$ be a path between p and q. Consider $\gamma^* s \in \Gamma(\gamma^* E)$. Define $\gamma' \in \Gamma(\gamma^* E)$ by $\gamma' \equiv \gamma_*\Big(\frac{\partial}{\partial t}\Big)$. Then by the definition of the pull-back connection

$$(\gamma^*\nabla)_{\frac{d}{dt}}(\gamma^* s) = \gamma^*(\nabla_{\gamma'} s) = 0. \tag{25.9}$$

Therefore $\gamma^* s$ is parallel along γ. Since $s(p) = 0$, by the uniqueness theorem for ODEs, $\gamma^* s \equiv 0$, so $s(q) = 0$. □

Proposition 25.7. *If a connection ∇ on $\pi : E \to M$ is compatible with a metric g, then parallel translation along any curve is an isometry.*

Proof. Given a curve $\gamma : I \to M$, and $a, b \in I$, and $V_a, W_a \in E_{\gamma(a)}$ then V_a, W_a extend uniquely to parallel vector fields along γ, $V, W \in \Gamma(\gamma^* E)$. Recall that $\gamma^* g$ is a metric on the bundle $\gamma^* E \to I$, so $\gamma^* g(V, W) : I \to \mathbb{R}$ is a function on I. By Proposition 23.3, the connection $\gamma^*\nabla$ is compatible with $\gamma^* g$. In the definition of compatibility, let X be the vector field $d/dt \in \Gamma(TI)$, then

$$\frac{d}{dt}\Big((\gamma^* g)(V, W)\Big) = \gamma^* g\Big((\gamma^*\nabla)_{\frac{d}{dt}} V, W\Big) + \gamma^* g\Big(V, (\gamma^*\nabla)_{\frac{d}{dt}} W\Big) = 0, \tag{25.10}$$

since both V and W are parallel. Since inner product is constant, the proposition follows. $\square$

26 Lecture 26

We begin with the following lemma.

Lemma 26.1 (Independence of parametrization). *Let $\gamma : [a, b] \to M$ be a smooth curve, and let $P_{a,b} : E_{\gamma(a)} \to E_{\gamma(b)}$ be parallel translation along γ from a to b. Let $\alpha : [c, d] \to [a, b]$ be a diffeomorphism with $\alpha(c) = a$ and $\alpha(d) = b$. Consider $\tilde\gamma : [c, d] \to M$ defined by $\tilde\gamma = \gamma \circ \alpha$. Let $\tilde P_{c,d} : E_{\gamma(a)} \to E_{\gamma(b)}$ be parallel translation along $\tilde\gamma$ from c to d. Then $P_{a,b} = \tilde P_{c,d}$.*

Proof. Take $V_a \in E_{\gamma(a)}$ and extend V_a to a section $V \in \Gamma(\gamma^* E)$ such that V is parallel along γ. Also, we can extend V_a to a section $\tilde V \in \Gamma(\tilde\gamma^* E)$ such that V is parallel along $\tilde\gamma$. Noting that $\tilde\gamma^* E \cong \alpha^*\gamma^* E$, we have the diagram

$$\begin{CD} \tilde\gamma^* E @>>> \gamma^* E @>>> E \\ @VVV @VVV @VV{\pi}V \\ [c, d] @>{\alpha}>> [a, b] @>{\gamma}>> M, \end{CD} \tag{26.1}$$

Note that V is a section of the middle bundle, and $\tilde V$ is a section of the leftmost bundle. Consider the section $\hat V = \alpha^* V \in \Gamma(\alpha^*\gamma^* E) \cong \Gamma(\tilde\gamma^* E)$. Then

$$\begin{aligned} \big(\alpha^*(\gamma^*\nabla)\big)_{\frac{d}{dt}}(\hat V) &= \alpha^*\big((\gamma^*\nabla)_{\alpha_* \frac{d}{dt}} V\big) \\ &= \alpha^*\big((\gamma^*\nabla)_{\alpha' \cdot \frac{d}{dt}} V\big) \\ &= \alpha^*\big(\alpha'(\gamma^*\nabla)_{\frac{d}{dt}} V\big) = 0, \end{aligned} \tag{26.2}$$

since V is parallel along γ. So $\hat V$ is parallel along $\tilde\gamma$, and $\hat V(c) = \alpha^*(V(a)) = V_a$. But $\tilde V$ is by definition a parallel section along $\tilde\gamma$ with the same initial value. By the uniqueness theorem for ODEs, we conclude that $\tilde V = \hat V = \alpha^* V$. Finally, we have that $\tilde V(d) = \alpha^*(V(b)) = V_b$, so the parallel translations are the same. $\square$

26.1 Holonomy

Notice that we can obviously extend parallel translation to piecewise smooth curves, and this will also be independent of parametrization. Given piecewise smooth curves

$\gamma_1 : [a, b] \to M$ and $\gamma_2 : [b, c] \to M$, with $\gamma_1(b) = \gamma_2(b)$, define the composition $\gamma_1 * \gamma_2 : [a, b] \to M$ to be the curve

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t) & t \in [b, c] \end{cases} \tag{26.3}$$

Given $\gamma : [a, b] \to M$, define the reverse curve $\gamma^{-1} : [a, b] \to M$ by $\gamma^{-1}(t) = \gamma(a+b-t)$. By independence of parametrization, we can always reparametrize all curves to be defined on $[0, 1]$, and for the composition, make the first one from $[0, 1/2]$ and the second one from $[1/2, 1]$. Given $p_1, p_2 \in M$, parallel translation from E_{p_1} to E_{p_2} along piecewise smooth paths is associative, that is, parallel transport along $\alpha * (\beta * \gamma)$ equals parallel transport along $(\alpha * \beta) * \gamma$.

Definition 26.2. Given $p \in M$, the holonomy group of a connection $\nabla : E \to M$ at p is the subgroup $Hol_p(\nabla) \subset GL(E_p)$ consisting of all the parallel transport maps $E_p \to E_p$ along all piecewise smooth loops based at p, with group operation induced from the composition of paths.

If we take a path $\gamma : [a, b] \to M$ with $\gamma(a) = p_1$ and $\gamma(b) = p_2$, then sending a loop α based at p_1 to $\beta = \gamma * \alpha * \gamma^{-1}$ gives an isomorphism $Hol_{p_1}(\nabla) \cong Hol_{p_2}(\nabla)$, so if M is connected then we can talk about the holonomy group $Hol(\nabla) \subset GL(n, \mathbb{R})$. Note that if ∇ is compatible with a Riemannian metric g on E, then by Proposition 25.7, we can view $Hol(\nabla) \subset O(n, \mathbb{R})$.

27 Lecture 27

We begin with the following.

Proposition 27.1. *The holonomy group $Hol(\nabla) = \{e\}$ if and only if $E \to M$ is a trivial bundle, and there is a global basis of parallel sections of E, in other words ∇ is the trivial connection. In this case, we have $R_\nabla \equiv 0$.*

Proof. If $Hol(\nabla)$ is trivial, then fix any point $p_0 \in M$. For any other point $p \in M$, choose a path γ from p to p_0, and let $P : E \to E_{p_0}$ be the mapping given by parallel transport of $V_p \in E_p$ along γ. Since $Hol(\nabla)$ is trivial, this mapping is independent of the choice of γ, and gives a global trivialization of E, with a basis of parallel sections. The converse is obvious. To see the last statement, choose a basis of parallel sections $s_i, i = 1 \ldots k$, then

$$R_\nabla(X, Y)s_i = \nabla_X \nabla_Y s_i - \nabla_Y \nabla_X s_i - \nabla_{[X.Y]} s_i \equiv 0, \tag{27.1}$$

since the s_i are parallel. Since R_∇ is a tensor, this implies that $R_\nabla \equiv 0$. $\square$

Remark 27.2. In general, vanishing of curvature $R_\nabla = 0$ does not imply the connection is trivial, we will say more about this below.

Proposition 27.3. *For $\ell \leq k$, if there is a trivial subbundle $\tilde{E} \subset E$ of rank ℓ which is spanned by parallel sections $s_1, \dots s_\ell$ (in which case we say that $\tilde{E}$ is a parallel subbundle) then the holonomy group $Hol(\nabla) \subset GL(k-\ell, \mathbb{R}) \subset GL(k, \mathbb{R})$. Conversely, if there is an ℓ-dimensional subspace of $V_p \subset E_p$ which is invariant under the holonomy group action, then there is a parallel subbundle $\tilde{E} \subset E$ such that $\tilde{E}_p = V_p$. In this case, we have $R_\nabla(X, Y)s = 0$ if $s(p) \in \tilde{E}_p$ at $p \in M$.*

Proof. Clearly, $\tilde{E} = \text{span}\{s_1, \dots, s_l\}$ defines a trivial rank ℓ subbundle of E, which is preserved under parallel translation. Conversely, $\tilde{E}$ is obatined by parallel translating V_p to any other point along a curve, with the resulting subspace being independent of choice of curve. The curvature statement is similar to above. □

Definition 27.4. A homotopy between piecewise smooth curves $\gamma_0 : [0,1] \to M$ and $\gamma_1 : [0,1] \to M$ is a continuous mapping $H : [0,1] \times [0,1] \to M$ such that $H(0,t) = \gamma_0(t)$, $H(1,t) = \gamma_1(t)$ and there exists a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ so that $H : [0,1] \times (t_i, t_{i+1}) \to M$ is smooth. If $\gamma_0(0) = \gamma_1(0) = p$ and $\gamma_0(1) = \gamma_1(1) = q$, then we say the homotopy *fixes endpoints* if $H(s,0) = p$ and $H(s,1) = q$ for all $s \in [0,1]$.

Definition 27.5. The restricted holonomy group of ∇ at p is the subgroup of $Hol_{o,p}(\nabla) \subset Hol_p(\nabla)$ given by parallel translates along all contractible curves (curves which are homotopic to a constant path $\{p\}$).

Again if M is connected, there is a well-defined group $Hol_o(\nabla)$ up to isomorphism.

Proposition 27.6. *The group $Hol(\nabla)$ is a Lie group and $Hol_o(\nabla)$ is the identity component (which is a normal subgroup), and there exists a homomorphism from $\pi_1(M)$ onto the quotient group $Hol(\nabla)/Hol_o(\nabla)$.*

Proof. First, fixing a basepoint, we have the embedding $Hol_o(\nabla) \subset GL(E_p)$, so $Hol_o(\nabla)$ is a subgroup of a Lie group. We claim that $Hol_0(\nabla)$ is path connected. To see this, let H be a homotopy between γ and a constant path $\{p\}$, and let $\gamma_s(t)$ be the curve $H(s,t)$. Let $P_{\gamma_s} : E_p \to E_p$ be parallel translation along γ_s. Then P_{γ_s} is a path between the restricted holonomy group element determined by γ and the identity map $Id : E_p \to E_p$, since parallel translation along a constant curve is trivial. By a Theorem of Yamabe [**?**], $Hol_o(\nabla)$ is a Lie group. It is a normal subgroup of $Hol(\nabla)$ because for all $h \in Hol(\nabla)$, $h^{-1} \circ P_{\gamma_s} \circ h$ is a path from $h^{-1} \circ P_\gamma \circ h$ to Id. Then $Hol(\nabla)$ is a Lie group, with $Hol(\nabla)/Hol_0(\nabla)$ in one-one correspondence with the connected components. This is countable because there is clearly a well-defined homomorphism from $\pi_1(M) \to Hol(\nabla)/Hol_0(\nabla)$ which is surjective, and because $\pi_1(M)$ is countable. □

28 Lecture 28

28.1 Group actions

First, we recall some definitions. A left action of a Lie group G on a manifold M is a smooth mapping $F : G \times M \to M$ satisfying

$$F(g_1 g_2, p) = F(g_1, F(g_2, p)), \quad F(e, p) = p \tag{28.1}$$

We will also sometimes denote the action as $p \mapsto g \cdot p$. Given a group action of G on M the orbit of $p \in M$ is

$$F(G, p) = \{g \cdot p \mid g \in G\}. \tag{28.2}$$

Being in the same orbit defines an equivalence relation, and the space of orbits M/G carries the quotient topology such that

$$\pi : M \to M/G \tag{28.3}$$

is open.

- The action is *effective* is $F(g, p) = p$ for all $p \in M$ implies that $g = e$.
- The action is *transitive* if for all $p, q \in M$, there exists $g \in G$ such that $F(g, p) = q$.
- The action is *free* if the only diffeomorphism $p \mapsto F(g, p)$ with a fixed point is with $g = e$.
- The action is *properly discontinuous* if for $p \in M$, there exists a neighborhood U_p of p such that $F(g, U_p) \cap U_p \neq \emptyset$ if and only if $g = e$.

Note that

$$\text{properly discontinuous} \Rightarrow \text{free} \Rightarrow \text{effective}. \tag{28.4}$$

The first basic theorem we will need is the following.

Theorem 28.1 ([**?**]). *If G acts properly discontinuously on M then M/G is a manifold, and $\pi : M \to M/G$ is a covering space of M/G.*

For $p \in M$ the *isotropy group* at p is

$$H = \{g \in G \mid F(g, p) = p\} \tag{28.5}$$

The second basic theorem we will need is the following.

Theorem 28.2 ([War83]). *Assume that G acts transitively on M. For $p_0 \in M$, let H denote the isotropy group at p_0, and let G/H be the space of left cosets of H with the quotient topology. Then the mapping $\beta : G/H \to M$ defined by*

$$\beta(gH) = F(g, p_0) \tag{28.6}$$

is a diffeomorphism.

28.2 Examples

Take a basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$, and consider the lattice

$$L = \{a_1 v_1 + \cdots + a_n v_n | (a_1, \ldots, a_n) \in \mathbb{Z}^n\} \tag{28.7}$$

Let $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ by integers translations in the lattice directions. This is properly discontinuous, so by Theorem 28.1 then quotient $\mathbb{R}^n/\mathbb{Z}^n$ is a manifold, and it is not hard to see that

$$\mathbb{R}^n/\mathbb{Z}^n \cong \overbrace{S^1 \times \cdots \times S^1}^{n} = T^n \tag{28.8}$$

is an n-dimensional torus. Note the Euclidean metric descends to a metric on T^n called the flat metric.

The next example is the unit n-sphere $S^n \subset \mathbb{R}^{n+1}$, with the Riemannian metric induced from Euclidean space. It is not hard to see that $SO(n+1)$ acts transitively on S^n, with stabilizer subgroup of any point isomorphic to $SO(n)$, so by Theorem 28.2 we have

$$S^n = SO(n+1)/SO(n) \tag{28.9}$$

- For an n-torus T^n with a flat metric g, the holonomy of the Riemannian connection is trivial, and there are n linearly independent parallel sections.

- For S^n with the round metric and Riemannian connection, $Hol(\nabla) = SO(n)$ (details given in lecture). To see this, prove for S^2 first. Parallel translation along great circles is the identity map after rotating to identity tangent spaces. From the north pole, take a path down a longitude to the equator, then travel along the equator, and then go back up to the north pole along another longitude. This shows that the holonomy group at the north pole is $S^1 = SO(2)$. For higher dimensions, assume that the isotropy group is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & SO(n) \end{pmatrix} \tag{28.10}$$

  Using the canonical form for orthogonal matrices, we can assume that the $SO(n)$ piece is block diagonal with 2×2 rotation matrices, and possibly an identity block. For each 2×2 block, we can obtain this map by the above parallel translation argument on S^2, and this completes the proof. Note that by Proposition 27.3, there are no parallel vector fields, even locally.

- For $\mathbb{RP}^n$ with the round metric and Riemannian connection, $Hol(\nabla) = O(n)$, $Hol_o(\nabla) = SO(n)$, and $\pi_1(\mathbb{RP}^n) = Hol(\nabla)/Hol_o(\nabla) = \mathbb{Z}_2$.

29 Lecture 29

Today, we will show the following.

Proposition 29.1. *If $R_\nabla \equiv 0$, and $p, q \in M$, and γ_0, γ_1 are paths from p to q which are homotopic with fixed endpoints, then parallel tranport along γ_0 and γ_1 from p to q is the same.*

Proof. Let $H : [0,1] \times [0,1] \to M$ be a homotopy between $\gamma_0 : [0,1] \to M$ and $\gamma_1 : [0,1] \to M$ $H(0,t) = \gamma_0(t)$, $H(1,t) = \gamma_1(t)$ which fixed endpoints, that is, $H(s,0) = p$ and $H(s,1) = q$ for all $s \in [0,1]$. Let $P_s(t)$ be parallel transport along $\gamma_s(t) = H(s,t)$ from E_p to $E_{\gamma_s(t)}$, and let

$$\mathcal{P}_s \equiv P_s(1) : E_p \to E_q. \tag{29.1}$$

Take $V_0 \in E_p$ and define $V(s,t) = P_s(t)V_0$. This is a piecewise smooth section along H, that is, $V \in \Gamma(H^*E)$. Consider the pull-back connection $H^*\nabla$ which is a connection on $H^*E \to [0,1] \times [0,1]$. Since V is by definition parallel along the curve $\gamma_s(t)$, we have

$$(H^*\nabla)_{\frac{\partial}{\partial t}} V(s,t) = 0 \tag{29.2}$$

$$\tag{29.3}$$

with initial conditions $V(s,0) = V_0$, which makes sense since the fiber of H^*E over $I \times \{0\}$ is the fixed fiber E_p.

Next, define

$$W(s,t) = (H^*\nabla)_{\frac{\partial}{\partial s}} V(s,t) \tag{29.4}$$

Since the homotopy fixes endpoints, it is not hard to see that

$$W(s,0) = \frac{\partial}{\partial s} V(s,t)|_{t=0} = 0 \tag{29.5}$$

$$W(s,1) = \frac{\partial}{\partial s} V(s,t)|_{t=1} = \frac{\partial}{\partial s}(\mathcal{P}_s V_0). \tag{29.6}$$

By definition of the curvature tensor, we have

$$(H^*\nabla)_{\frac{\partial}{\partial s}}(H^*\nabla)_{\frac{\partial}{\partial t}} V - (H^*\nabla)_{\frac{\partial}{\partial t}}(H^*\nabla)_{\frac{\partial}{\partial s}} V = R_{H^*\nabla}\Big(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\Big) V \tag{29.7}$$

By the assumption that $R_\nabla = 0$, and Proposition 24.4, the right hand side is zero. Also, from (29.2), the first term on the left hand side is zero, so we have

$$(H^*\nabla)_{\frac{\partial}{\partial t}}(H^*\nabla)_{\frac{\partial}{\partial s}} V = (H^*\nabla)_{\frac{\partial}{\partial t}} W = 0. \tag{29.8}$$

Since $W(s,0) = 0$ and W is also parallel along the $\gamma_s(t)$-curves, we conclude that

$$W(s,1) = \frac{\partial}{\partial s}(\mathcal{P}_s V_0), \tag{29.9}$$

which shows that $\mathcal{P}_s V_0$ is independent of s. $\square$

This implies the following.

Corollary 29.2. *If $R_\nabla = 0$, then the restricted holonomy group at any $p \in M$ is trivial, and there is a surjective homomorphism from $\pi_1(M)$ onto $Hol(\nabla)$. Consequently, if $R_\nabla = 0$ and $\pi_1(M) = \{e\}$, then E is a trivial bundle and ∇ is the trivial connection on E.*

Proof. The first part follows from Propositions 27.6 and 29.1. The second part then follows from Proposition 27.1. □

Remark 29.3. As an example that this is sharp, the flat metric on T^2 descends to a flat metric on the Klein bottle $K^2 = T^2/\mathbb{Z}_2$. The Riemannian connection is a flat connection on TK^2, but this is not a trivial bundle (since K^2 is non-orientable). We have $Hol_o(\nabla) = \{e\}$, and $Hol(\nabla) = \mathbb{Z}_2$.

30 Lecture 30

Today we will prove a few more items about holonomy. The first is the following.

Proposition 30.1. *If ∇ is a connection on $\pi : E \to M$ then the structure group of E can be reduced to $Hol(\nabla)$.*

Proof. A local trivialization $\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to E|_{U_\alpha}$ is equivalent to choosing a local basis of sections $s_i \in \Gamma(E|_{U_\alpha})$ for $i = 1 \dots k$. Choose a coordinate system $x : U_\alpha \to \mathbb{R}^n$, and assume that $x(p) = 0$ and that $x(U_\alpha)$ is a ball centered at the origin, and choose any frame $e_1, \dots, e_k$ at p. Choose radial coordinate on $\mathbb{R}^n$, and parallel translate along radial rays to extend the frame at p to a frame $s_1, \dots, s_k$ in U_α. It is not hard to see that this is a smooth frame field over U_α, and thus gives a local trivialization of E over U_α. The overlap maps must lie in

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to Hol_p(\nabla) \subset GL(k, \mathbb{R}). \tag{30.1}$$

However, since the holonomy groups at different points are conjugate, we can choose functions $f_\alpha : U_\alpha \to GL(k, \mathbb{R})$ and $f_\beta : U_\beta \to GL(k, \mathbb{R})$ so that

$$f_\alpha^{-1} \cdot \varphi_{\alpha\beta} \cdot f_\beta \in Hol_p(\nabla), \tag{30.2}$$

which is a reduction of the structure group. □

Another proposition which will be useful later.

Proposition 30.2. *Given any $p \in M$, there exists a local frame field $s_1, \dots, s_k$ in a neighborhood of p such that $\nabla s_i(p) = 0$, and $\Gamma^l_{ij}(p) = 0$.*

Proof. As in the previous proposition, parallel translate a fixed frame along radial curves. Then clearly $\nabla s_j = 0$ at p. Next, choosing a local coordinate system x near p, we have

$$0 = \nabla_{\frac{\partial}{\partial x^i}} s_j(p) = \Gamma^k_{ij}(p) s_k. \tag{30.3}$$

The right hand side is a linear combination of the s_k which form a basis, so by linear independence, $\Gamma^k_{ij}(p) = 0$. □

We return to flat connections.

Proposition 30.3. *The space of flat connections on $\pi : E \to M$ modulo pullback under bundle equivalence can be identified with the k-dimensional representations of $\pi_1(M)$ modulo equivalence of representations.*

Proof. By the corollary above, if ∇ is a flat connection, then one obtains a homomorphism

$$\rho : \pi_1(M) \to Hol(\nabla) \subset GL(k, \mathbb{R}) \tag{30.4}$$

which is a representation of $\pi_1(M)$.

Conversely, given such a representation, one can build a flat connection by taking flat connection on the the trivial bundle $\tilde{M} \times \mathbb{R}^k$, where $\tilde{M}$ is the universal cover of M, and quotienting by the action of $\pi_1(M)$

$$\gamma \cdot (\tilde{p}, v) = (\gamma\tilde{p}, \rho(\gamma)v), \tag{30.5}$$

where $\gamma \in \pi_1(M)$, which acts on $\tilde{p} \in \tilde{M}$, and $v \in \mathbb{R}^k$, to get a bundle $\tilde{E} \to M$. Note the following. If $\alpha : \tilde{M} \to M$ denotes the universal cover, then $\alpha^*\tilde{E}$ is the trivial bundle. $\square$

Note that if ∇ is a flat connection, then by Proposition ?, $Hol(\nabla)$ is a discrete group. By Proposition ?, the transition functions of the bundle can be taken to be constant.

In this case, we have the complex

$$\cdots \xrightarrow{d^\nabla} \Omega^{l-1}(M, E) \xrightarrow{d^\nabla} \Omega^l(M, E) \xrightarrow{d^\nabla} \Omega^{l+1}(M, E) \xrightarrow{d^\nabla} \cdots, \tag{30.6}$$

where $\Omega^l(M, E) = \Gamma(\Lambda^l(T^*M) \otimes E)$, and

$$d^\nabla(\alpha \otimes s) = d\alpha \otimes s + \alpha \wedge \nabla s. \tag{30.7}$$

Note that this is a complex by flatness of the bundle.

So for a flat connection on a bundle, one can define a cohomology theory

$$H^p_\nabla(M, E) = \frac{Ker\{\delta : \Omega^p(M, E) \to \Omega^{p+1}(M, E)\}}{Im\{\delta : \Omega^{p-1}(M, E) \to \Omega^p(M, E)\}}. \tag{30.8}$$

By the Proposition above, this can really be thought of as a fancy invariant depending upon representation theory of $\pi_1(M)$.

31 Lecture 31

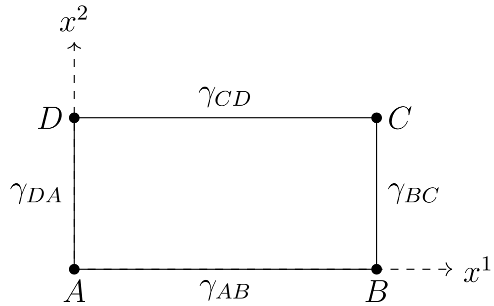
Today we will show that the curvature tensor can be obtained directly from parallel transport. Take two linearly independent vectors $X_p, Y_p \in T_pM$. Choose a coordinate system around $p \in M$, so that $X_p = \frac{\partial}{\partial x^1}$, $Y_p = \frac{\partial}{\partial x^2}$. Let $D_{\epsilon_1,\epsilon_2}$ be the coordinate rectangle with side length ϵ_i in the variable x^i, for $i = 1, 2$. Let $P_{\epsilon_1,\epsilon_2} : E_p \to E_p$ denote parallel translation along the boundary of $D_{\epsilon_1,\epsilon_2}$.

Theorem 31.1. *We have*

$$P_{\epsilon_1,\epsilon_2} s_p = s_p - \epsilon_1\epsilon_2 R_\nabla(X,Y)s_p + o(\epsilon_1^2+\epsilon_2^2)s_p, \tag{31.1}$$

as $\epsilon_1, \epsilon_2 \to 0$.

Proof. Label everything as follows:



Choose a local basis of sections $s_1, \ldots, s_k$ so that $\nabla s_j(A) = 0$, and write a section as $s = \sum_{j=1}^k s^j s_j$ for functions s^j. Parametrize γ_{AB} by $t \mapsto (t, 0, \cdots, 0)$ with $t \in [0, \epsilon_1]$. The ODE for parallel transport along γ_{AB} is

$$\frac{d}{dt}s^j(t,0,\ldots,0) + \Gamma^j_{1l}(t,0,\ldots,0)s^l(t,0,\ldots,0) = 0. \tag{31.2}$$

Using a Taylor exapansion,

$$s^j(t,0,\ldots,0) = s^j_A + \frac{d}{dt}s^j(t,0,\ldots,0)\Big|_{t=0} + \frac{1}{2}\frac{d^2}{dt^2}s^j(t,0,\ldots,0)\Big|_{t=0} + O(t^3), \tag{31.3}$$

as $t \to 0$.

The ODE (31.2) yields

$$\frac{d}{dt}s^j(t,0,\ldots,0)\Big|_{t=0} = -\Gamma^j_{1l}(0,\ldots,0)s^l_A \tag{31.4}$$

and differentiating (31.2) yields

$$\begin{aligned}\frac{d^2}{dt^2}s^j(t,0,\ldots,0)\Big|_{t=0} &= -\frac{d}{dt}\Gamma^j_{1l}(t,0,\ldots,0)\Big|_{t=0}s^l_A - \Gamma^j_{1l}(0,\ldots,0)\frac{d}{dt}s^l(t,0,\ldots,0)\Big|_{t=0}\\ &= -\frac{d}{dt}\Gamma^j_{1l}(t,0,\ldots,0)\Big|_{t=0}s^l_A = -\frac{\partial\Gamma^j_{1l}}{\partial x^1}(0)s^l_A,\end{aligned} \tag{31.5}$$

because the second terms vanishes since we are using an adapted frame.

Putting all this together, we obtain

$$s^j_B = s^j(\epsilon_1,0,\ldots,0) = s^j_A - \frac{1}{2}\epsilon_1^2\frac{\partial\Gamma^j_{1l}}{\partial x^1}(0)s^l_A + O(\epsilon_1^3), \tag{31.6}$$

as $\epsilon_1 \to 0$.

In a similar fashion, we can compute the following parallel transports. Along the curve $\gamma_{BC} : t \mapsto (\epsilon_1, t, 0 \ldots, 0)$,

$$s^j_C = s^j(\epsilon_1, \epsilon_2, \ldots, 0) = s^j_B - \epsilon_1\epsilon_2 \frac{\partial \Gamma^j_{2l}}{\partial x^1}(0) s^l_B - \frac{1}{2}\epsilon_2^2 \frac{\partial \Gamma^j_{2l}}{\partial x^2}(0) s^l_B + l.o.t., \tag{31.7}$$

along the curve $\gamma_{CD} : t \mapsto (\epsilon_1 - t, \epsilon_2, 0, \ldots, 0)$,

$$s^j_D = s^j(0, \epsilon_2, \ldots, 0) = s^j_C + \epsilon_1\epsilon_2 \frac{\partial \Gamma^j_{1l}}{\partial x^2}(0) s^l_C + \frac{1}{2}\epsilon_1^2 \frac{\partial \Gamma^j_{1l}}{\partial x^1}(0) s^l_C + l.o.t., \tag{31.8}$$

and along the curve $\gamma_{DA} : t \mapsto (0, \epsilon_2 - t, 0, \ldots, 0)$,

$$s^j_A = s^j(0, 0, \ldots, 0) = s^j_D + \frac{1}{2}\epsilon_2^2 \frac{\partial \Gamma^j_{2l}}{\partial x^2}(0) s^l_D + O(\epsilon_2^3), \tag{31.9}$$

Adding these four parallel transport equations together yields

$$\begin{aligned} P_{\epsilon_1,\epsilon_2} s_A &= s_A + \epsilon_1\epsilon_2 \Big(\frac{\partial \Gamma^j_{1l}}{\partial x^2}(0) - \frac{\partial \Gamma^j_{2l}}{\partial x^1}(0) \Big) s^l_A s_j(0) + l.o.t. \\ &= s_A - \epsilon_1\epsilon_2 R_\nabla \Big(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \Big) s_A + l.o.t., \end{aligned} \tag{31.10}$$

using the definition of the curvature tensor, and since the frame is adapted at A. □

32 Lecture 32

32.1 Geodesics

Now let use restrict attention to connections in TM; such a connection is called a *linear connection* on M.

Definition 32.1. For a linear connection on M, a curve $\gamma : I \to M$ is a *geodesic* if $\gamma' \equiv \gamma_* \Big(\frac{\partial}{\partial t} \Big)$ is parallel along γ, that is

$$(\gamma^* \nabla)_{\frac{\partial}{\partial t}} \gamma' = 0. \tag{32.1}$$

Choose local coordinate x^i on M and write $x \circ \gamma = (\gamma^1, \ldots, \gamma^n)$. From Proposition 25.4, the condition for a geodesic is locally the second order *nonlinear* ODE

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}(\gamma(t)) \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0. \tag{32.2}$$

The local existence and uniqueness theorem for ODEs says that given $p \in M$ and $X_p \in T_pM$, there is a unique geodesic $\gamma : (-\epsilon, \epsilon) \to M$ satisfying $\gamma(0) = p$, $\gamma'(0) = X_p$.

Example 32.2. Flat metric on a torus T^n, or a cylinder. In Euclidean coordinates on the universal cover, we have $\Gamma^k_{ij} \equiv 0$. In these coordinates, the geodesics are just straight lines.

32.2 The exponential map

If $v \in T_pM$, and there exists a geodesic $\gamma : [0,1] \to M$ satisfying $\gamma(p) = 0$, and $\gamma'(0) = v$, then define $exp_p(v) = \gamma(1)$.

Proposition 32.3. *For any $v \in T_pM$, the curve $\gamma(t) = exp_p(tv)$ is defined for t sufficiently small, and is a geodesic satisfying $\gamma(0) = p$, and $\gamma'(0) = v$.*

Proof. By the ODE existence theorem, there exists a geodesic $\tilde{\gamma} : (-\epsilon, \epsilon) \to M$ such that $\tilde{\gamma}(0) = p$, and $\tilde{\gamma}'(0) = v$. For any $c > 0$, consider $\tilde{\gamma}_c(t) = \tilde{\gamma}(ct)$. Clearly $\tilde{\gamma}_c$ is a geodesic, and

$$\tilde{\gamma}_c'(0) = c \cdot \tilde{\gamma}'(0) = c \cdot v. \tag{32.3}$$

So then $exp_p(c \cdot v) = \tilde{\gamma}_c(1) = \tilde{\gamma}(c)$ is defined for c sufficiently small. □

From this proposition, and the ODE existence theorem, we see that the exponential map is defined in a neighborhood of the origin.

Proposition 32.4. *If γ is a geodesic then the norm of γ' is constant along γ.*

Proof. Note that $\gamma' \in \Gamma(\gamma^* TM)$, and from Proposition 23.3, we have that

$$\frac{d}{dt}(\gamma^* g)(\gamma', \gamma') = 2\gamma^* g\Big((\gamma^*\nabla)_{\frac{d}{dt}}\gamma', \gamma'\Big) = 0 \tag{32.4}$$

□

For any curve $\gamma : I \to M$, and $t_0 \in I$, define the arclength (starting at t_0) by

$$L_{t_0}^t \gamma = \int_{t_0}^t \{\gamma^* g(\gamma', \gamma')\}^{\frac{1}{2}} dt. \tag{32.5}$$

So for any geodesic, the arclenth is a linear function of t. If $\|\gamma'\| = 1$, then we say γ is parametrized by arclength.

Next, if $v \in T_pM$, we have

$$(exp_p)_*|_v : T_v(T_pM) \to T_{exp_p(v)}M. \tag{32.6}$$

Since T_pM is a linear space, we can view this as

$$(exp_p)_*|_v : T_pM \to T_{exp_p(v)}M. \tag{32.7}$$

Lemma 32.5. *The mapping*

$$(exp_p)_*|_0 : T_pM \to T_pM. \tag{32.8}$$

is the identity map.

Proof. Given $v \in T_pM, c(t) = t \cdot v$ is a curve with $c'(0) = v$. From Proposition 32.3, $exp_p(c(t)) = exp_p(t \cdot v)$ is the unique geodesic $\gamma(t)$ with tangent v at $t = 0$. Then

$$(exp_p)_*|_0(v) = \frac{d}{dt} exp_p(c(t))\Big|_{t=0} = \frac{d}{dt}\gamma(t)\Big|_{t=0} = v. \tag{32.9}$$

□

From the inverse function theorem, we have the following corollary.

Corollary 32.6. *The mappping $exp_p : T_pM \to M$ is a local diffeomorphism near p.*

32.3 Gauss Lemma

Assume that g is a Riemannian metric and that ∇ is the Riemannian connection in $TM \to M$.

Lemma 32.7. *The radial geodesics from a point p are orthogonal to the hypersurfaces $S_p(r) = \{exp_p(v) \mid \|v\| = r\}$.*

Proof. Let $v(s)$ be a curve in T_pM with $\|v(s)\| = r_0$, and define $f(r,s) = \exp(rv(s))$. Consider the connection $f^*\nabla$ which is a connection on f^*TM over $[0,r_0] \times (-\epsilon, \epsilon)$. Denote

$$\frac{\partial f}{\partial r} = f_*\Big(\frac{\partial}{\partial r}\Big), \quad \frac{\partial f}{\partial s} = f_*\Big(\frac{\partial}{\partial s}\Big), \tag{32.10}$$

both of which are sections of f^*TM. Using Proposition 23.3 (compatibility of the pullback connection), we have

$$\begin{aligned} \frac{\partial}{\partial r}\Big\{(f^*g)\Big(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}\Big)\Big\} &= (f^*g)\Big((f^*\nabla)_{\frac{\partial}{\partial r}}\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}\Big) + (f^*g)\Big(\frac{\partial f}{\partial r}, (f^*\nabla)_{\frac{\partial}{\partial r}}\frac{\partial f}{\partial s}\Big) \\ &= (f^*g)\Big(\frac{\partial f}{\partial r}, (f^*\nabla)_{\frac{\partial}{\partial r}}\frac{\partial f}{\partial s}\Big), \end{aligned} \tag{32.11}$$

since the radial curves are geodesics. Next, using Proposition 24.1 ("symmetry" of the pullback connection), we have

$$\begin{aligned} \frac{\partial}{\partial r}\Big\{(f^*g)\Big(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}\Big)\Big\} &= (f^*g)\Big(\frac{\partial f}{\partial r}, (f^*\nabla)_{\frac{\partial}{\partial s}}\frac{\partial f}{\partial r}\Big) \\ &= \frac{1}{2}\frac{\partial}{\partial s}\Big\{(f^*g)\Big(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial r}\Big)\Big\}, \end{aligned} \tag{32.12}$$

again using Proposition 23.3. Notice that $\partial f/\partial r$ at the point (r,s) is the tangent vector to the geodesic $\gamma(r)$ from p, with initial tangent vector $v(s)$. Since the norm of a tangent vector to a geodesic is constant in r, we have that

$$(f^*g)\Big(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial r}\Big) = r_0, \tag{32.13}$$

and is therefore independent of s. Consequently, the function

$$(f^*g)\Big(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}\Big) \tag{32.14}$$

must be constant in r. But since $f(0,s) = p$, we have

$$\frac{\partial f}{\partial s}\Big|_{r=0} = 0, \tag{32.15}$$

which finishes the proof. ☐

33 Lecture 33

33.1 Normal Coordinates I

We define Euclidean normal coordinates to be the coordinate system given by the exponential map, together with a Euclidean coordinate system $\{x^i\}$ on T_pM such the the metric $g_{ij}(p) = \delta_{ij}$. We define radial normal coordinates to be

$$\Phi : \mathbb{R}^+ \times S^{n-1} \to M, \tag{33.1}$$

given by

$$(r, \xi) \mapsto \exp(r\xi). \tag{33.2}$$

Proposition 33.1. *In Euclidean normal coordinates,*

$$g = g_{Euc} + O(|x|^2), \text{ as } x \to 0, \tag{33.3}$$

where g_{Euc} is the standard Euclidean metric. In radial normal coordinates, we have

$$\Phi^* g = dr^2 + g_{n-1}, \tag{33.4}$$

where g_{n-1} is a metric on S^{n-1} depending upon r, and satisfying

$$g_{n-1} = r^2 g_{S^{n-1}} + O(r^2), \text{ as } r \to 0, \tag{33.5}$$

where $g_{S^{n-1}}$ is the standard metric on the unit sphere.

Proof. For the first statement, we know that $\exp_*(0) = Id$, so the constant term in the Taylor expansion of g is given by g_{Euc}. Next, we recall that the geodesic equation is

$$\ddot{\gamma}^i + \Gamma^i_{jk}\dot{\gamma}^j\dot{\gamma}^k = 0. \tag{33.6}$$

Since the radial directions are geodesics, we can let $\gamma = rv$, where v is any vector. Evaluating the geodesic equation at the origin, we have

$$\Gamma^i_{jk}(0)v^j v^k = 0, \tag{33.7}$$

for arbitrary v, so $\Gamma^i_{jk}(0) = 0$ (using symmetry). It is then easy to see from the definition of the Christoffel symbols that all first derivatives of the metric then vanish at p.

In normal coordinates, the lines through the origin are geodesics, and therefore have parallel tangent vector field. This implies that the radial component of the metric is dr^2. Then (33.4) follows from the Gauss Lemma. Finally, we see that $g_{Euc} = dr^2 + r^2 g_{S^{n-1}}$, so the second expansion follows from the first.

Remark 33.2. Notice that the term $r^2 g_{S^{n-1}}$ is indeed $O(1)$ as $r \to 0$. Write $h = g_{S^{n-1}}$, and then fixing some coordinate system on S^{n-1}, we compute

$$|r^2 h|^2 = r^4 g^{ip} g^{jq} h_{ij} h_{pq} = h^{ip} h^{jq} h_{ij} h_{pq} = (n-1). \tag{33.8}$$

If that is not convincing, then consider the case of $n = 2$. Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then $r^2 = x^2 + y^2$, and $\theta = \arctan y/x$. It is then easy to compute that

$$dx^2 + dy^2 = dr^2 + r^2 d\theta^2. \tag{33.9}$$

Note that, in a computation analogous to the above, that $|d\theta| = r^{-1}$. That is, $d\theta$ is not of unit norm, but rather $rd\theta$ is.

□

33.2 Geodesics are locally minimizing

Lemma 33.3. *Let $c : [a,b] \to M \setminus p$ be given by a piecewise smooth curve $c(t) = exp_p(u(t)v(t))$, where $0 < u(t) < \epsilon$ and $\|v(t)\| = 1$ Then the length of c*

$$L_a^b c \geq |u(b) - u(a)| \tag{33.10}$$

with equality if and only if u is monotone and v is constant.

Proof. Let $\alpha(r,t) = exp_p(rv(t))$, so that $c(t) = \alpha(u(t),t)$. Then

$$\frac{dc}{dt} = \frac{\partial \alpha}{\partial r} u'(t) + \frac{\partial \alpha}{\partial t}. \tag{33.11}$$

From the Gauss Lemma, the 2 terms on the right hand side are orthogonal. Note also that $\|\frac{\partial\alpha}{\partial r}\| = 1$, by Proposition 32.4, so we have

$$\|\frac{dc}{dt}\|^2 = |u'(t)|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 \geq |u'(t)|^2 \tag{33.12}$$

with equality if and only if $\frac{\partial\alpha}{\partial t} = 0$ which is equivalent to v being constant. Finally,

$$L_a^b c = \int_a^b \|c'(t)\| dt \geq \int_a^b |u'(t)| dt \geq |u(b) - u(a)|. \tag{33.13}$$

□

Corollary 33.4. *If p and q are sufficiently close, then there is a unique minimizing geodesic joining p and q.*

Proof. If the radial geodesic from p to $q = exp_p(v_0)$ does not minimize, then a shorter path would have to go outside of the spherical shell $\|v\| = \|v_0\|$, but by the Lemma, such a path would have longer length. □

Example 33.5. A great circle of S^n is the intersection of S^n with a 2-plane in $\mathbb{R}^{n+1}$. We claim that great circles on S^n are geodesics. To see this, let $p, q \in S^n$ be sufficiently close so that they are joined by a minimizing geodesic γ between. There is a isometric reflection $I : S^n \to S^n$ which fixes the great circle containing p and q. Then $I(\gamma)$ is a curve of the same minimizing length, so by uniqueness $I(\gamma) = \gamma$, which implies that γ must be part of the great circle.

34 Lecture 34

34.1 Distance function and Completeness

Let (M, g) be a connected Riemannian manifold. For $p, q \in M$, define

$$d(p,q) = \inf\{L(\gamma) \mid \gamma \text{ is a piecwise smooth curve from } p \text{ to } q\}. \tag{34.1}$$

Proposition 34.1. *The function $d : M \times M \to \mathbb{R}$ is a metric, that is*

$$d(p,q) \geq 0, \tag{34.2}$$

with equality if and only if $p = q$, and for $p_1, p_2, p_3 \in M$,

$$d(p_1, p_3) \leq d(p_1, p_2) + d(p_2, p_3). \tag{34.3}$$

Furthermore, the topology induced by d is the same as the original topology on M.

Recall that (M, d) is complete in the metric space sense if every Cauchy sequence has a convergent subsequence.

Definition 34.2. A Riemannian manifold (M, g) is geodesically complete if every gedoesic $\gamma : [a, b] \to M$ can be extended to a geodesic defined on all of $\mathbb{R}$.

Theorem 34.3 (Hopf-Rinow)**.** *A Riemannian manifold (M, g) is complete in the metric space sense if and only if it is geodesically complete. In this case, there exists a length minimizing geodesic between any 2 points in M.*

In particular, if M is compact, there exists a length minimizing geodesic between any two points.

34.2 The first variation formula

The length functional is invariant under reparametrizations, which causes problems for variational arguments. To remedy this, consider instead the energy functional, defined for $\gamma : [a, b] \to M$

$$E_a^b(\gamma) = \int_a^b \|\gamma'\|^2 dt = \int_a^b (\gamma^* g)\Big(\gamma_*\Big(\frac{d}{dt}\Big), \gamma_*\Big(\frac{d}{dt}\Big)\Big) dt. \tag{34.4}$$

Note that

$$L_a^b(\gamma) = \int_a^b \|\gamma'\| dt \leq \Big\{ \int_a^b \|\gamma'\|^2 dt \Big\}^{\frac{1}{2}} (b-a)^{\frac{1}{2}}, \tag{34.5}$$

which squares to

$$(L_a^b(\gamma))^2 \leq (b-a) E_a^b(\gamma), \tag{34.6}$$

and equality holds if and only if t is proportional to arclength.

Lemma 34.4. *If γ is a piecewise smooth curve that minimizes length between p and q then γ is a smooth geodesic.*

Proof. If γ minimizes length between p and q, it must also minimize length between any 2 points on γ. Locally, use Lemma 33.3 to prove that γ is a smooth geodesic. $\square$

In particular, minimizing geodesics are critical points for the energy functional. What other paths are critical?

Definition 34.5. A variation of a piecewise smooth curve $\gamma : [a, b] \to M$ is a continuous mapping $\alpha : (-\epsilon, \epsilon) \times [a, b] \to M$ such that $\alpha(0, t) = \gamma(t)$ and there exists a partition $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ so that $\alpha : (-\epsilon, \epsilon) \times (t_i, t_{i+1}) \to M$ is smooth. If $\gamma(a) = p$ and $\gamma(b) = q$, then we say the variation *fixes endpoints* if $\alpha(u, a) = p$ and $\alpha(u, b) = q$ for all $u \in (-\epsilon, \epsilon)$.

Denote

$$\frac{\partial \alpha}{\partial t} = \alpha_*\Big(\frac{\partial}{\partial t}\Big), \quad \frac{\partial \alpha}{\partial u} = \alpha_*\Big(\frac{\partial}{\partial u}\Big), \tag{34.7}$$

both of which are sections of $\alpha^* TM$. We call

$$W_t \equiv \frac{\partial \alpha}{\partial u}\Big|_{u=0} \tag{34.8}$$

the *variation vector field.* Define

$$\Delta_t \gamma' \equiv \gamma'(t+) - \gamma'(t-) \tag{34.9}$$

to be the jump in the velocity vector field at t, which is only possibly nonzero at the points t_i.

Theorem 34.6 (First variation formula)**.** *For any variation α of γ, we have*

$$\frac{1}{2}\frac{d(E(\alpha(u, \cdot))}{du}\Big|_{u=0} = -\sum_t (\gamma^* g)(W_t, \Delta_t \gamma') - \int_a^b (\gamma^* g)\Big(W_t, (\gamma^*\nabla)_{\frac{d}{dt}} \gamma'\Big) dt. \tag{34.10}$$

Proof. Using Proposition 23.3 (compatibility of the pullback connection) and Proposition 24.1 ("symmetry" of the pullback connection), we compute

$$\begin{aligned}
\frac{d(E(\alpha(u, \cdot))}{du} &= \int_a^b \frac{\partial}{\partial u}(\alpha^* g)\Big(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\Big) dt \\
&= 2\int_a^b (\alpha^* g)\Big((\alpha^*\nabla)_{\frac{\partial}{\partial u}} \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\Big) dt \\
&= 2\int_a^b (\alpha^* g)\Big((\alpha^*\nabla)_{\frac{\partial}{\partial t}} \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial t}\Big) dt.
\end{aligned} \tag{34.11}$$

Again using Proposition 23.3, we have

$$\frac{\partial}{\partial t}(\alpha^* g)\Big(\frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial t}\Big) = (\alpha^* g)\Big((\alpha^*\nabla)_{\frac{\partial}{\partial t}} \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial t}\Big) + (\alpha^* g)\Big(\frac{\partial \alpha}{\partial u}, (\alpha^*\nabla)_{\frac{\partial}{\partial t}} \frac{\partial \alpha}{\partial t}\Big). \tag{34.12}$$

Substituting this into the above yields

$$\frac{d(E(\alpha(u,\cdot))}{du} = 2\int_a^b \Big\{\frac{\partial}{\partial t}(\alpha^* g)\Big(\frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial t}\Big) - (\alpha^* g)\Big(\frac{\partial \alpha}{\partial u}, (\alpha^*\nabla)_{\frac{\partial}{\partial t}}\frac{\partial \alpha}{\partial t}\Big)\Big\} dt. \tag{34.13}$$

Evaluating at $u = 0$, and using the fundamental theorem of calculus on the first term yields the first variation formula. □

Corollary 34.7. *If γ is piecewise smooth and critical for E, then γ is smooth and is a geodesic.*

Proof. Let α be a variation which is supported away from the t_i, of the form

$$W(t) = f(t)(\gamma^*\nabla)_{\frac{d}{dt}}\gamma', \tag{34.14}$$

with $f(t) > 0$ away from the t_i. For such a variation, the first term in the first variation formula vanishes, so we have

$$\frac{1}{2}\frac{d(E(\alpha(u,\cdot))}{du}\Big|_{u=0} = -\int_a^b (\gamma^* g)f(t)\Big((\gamma^*\nabla)_{\frac{d}{dt}}\gamma', (\gamma^*\nabla)_{\frac{d}{dt}}\gamma'\Big)dt. \tag{34.15}$$

Since the integrand is non-negative, we conclude it vanishes, so γ satisfies the geodesic equation away from the t_i. Next, pick a variation such that $W(t_i) = \Delta_{t_i}\gamma'$. Then the first variation formula yields

$$\frac{1}{2}\frac{d(E(\alpha(u,\cdot))}{du}\Big|_{u=0} = -\sum_t (\gamma^* g)(\Delta_{t_i}\gamma', \Delta_{t_i}\gamma'), \tag{34.16}$$

which implies that $\Delta_{t_i}\gamma' = 0$. From the ODE existence and uniqueness theorem, γ is smooth at the t_i. □

35 Lecture 35

35.1 The second variation formula

To be completed.

35.2 Jacobi fields

Definition 35.1. A vector field $J \in \Gamma(\gamma^* TM)$ along a geodesic γ, is a Jacobi field if

$$\frac{D^2 J}{dt^2} + R(J,\gamma')\gamma' = 0 \tag{35.1}$$

Definition 35.2. Points p and q are conjugate along a geodesic $\gamma : [a,b] \to M$ if there exists a nonzero Jacobi field J along γ such that $J(a) = 0$ and $J(b) = 0$. The multiplicity of p and q is the dimension of the space of such Jacobi fields.

36 Lecture 36

We view the second variation as a symmetric bilinear form, i.e., for $W_1, W_2 \in T_\gamma\Omega(p,q)$,

$$E_{**}(W_1, W_2) = E_{**}(W_2, W_1) \tag{36.1}$$

Definition 36.1. The index of E_{**}, $Index(E_{**})$ is the maximum dimension of a negative definite subspace of E_{**}. The null space of E_{**} is

$$Null(E_{**}) = \{W \in T_\gamma\Omega \mid E_{**}(W, \tilde{W}) = 0 \ \forall \ \tilde{W} \in T_\gamma\Omega\}. \tag{36.2}$$

The nullity of E_* is the dimension of the nullspace of E_{**}.

Theorem 36.2. *W is in the null space of E_{**} if and only if W is a Jacobi field. The nullity of E_{**} is therefore equal to the multiplicity of p and q.*

Proof. To be completed. □

Example 36.3. If $R \equiv 0$ then there are no conjugate points.

Proposition 36.4. *If γ is a minimizing geodesic between p and q then $E_{**}(W,W) \geq 0$ for all $W \in T_\gamma\Omega$. In other words, the index of E_{**} is zero.*

Proof. To be completed. □

Our goal is to prove a converse of this: if $E_{**}(W,W) > 0$ for all $W \in T_\gamma\Omega$, then γ is minimizing. Before that, we will investigate further the null space of E_{**}.

Lemma 36.5. *If $\gamma : [a,b] \to M$ is a geodesic and if $\alpha(u,t)$ is a variation of γ through geodesics, then $W(t) = \frac{\partial\alpha}{\partial u}|_{u=0}$ is a Jacobi field along γ. Therefore, if α fixes endpoints, then p and q are conjugate.*

Proof. To be completed. □

Example 36.6. Jacobi fields in $\mathbb{R}^n$.

37 Lecture 37

Example 37.1. The space of Jacobi fields on S^n on a great circle between antipodal points vanishing at both endpoints has maximal dimension $n-1$.

Proposition 37.2. *Every Jacobi field along a geodesic $\gamma : [a,b] \to M$ arises from a variation of γ through geodesics. If $J(a) = 0$, then there is a variation which fixes a (but this is not necessarily true also at b if $J(b) = 0$.)*

Proof. To be completed. □

Theorem 37.3. *Let $\gamma : [a,b] \to M$ be a geodesic. If there exists a conjugate point $\gamma(\tau)$ for $\tau \in (a,b)$ then there exists $W \in T_\gamma\Omega$ such that $E_{**}(W,W) < 0$. Consequently, γ cannot be a local minimizer of the energy functional.*

Proof. To be completed. □

38 Lecture 38

Theorem 38.1. *The point $exp_p(v)$ is conjugate to p along the geodesic $\gamma(t) = exp_p(tv)$ if and only if v is a critical point of exp_p.*

Proof. To be completed. □

Corollary 38.2. *If $p \in M$ then for almost every $q \in M$, p is not conjugate to q along any geodesic.*

Proof. Sard's Theorem. □

Remark 38.3. Denote by $Conj(p)$ the set of points which are conjugate to q along some geodesic. Then $\mathcal{H}^{n-2}(Conj(p)) < \infty$, [**?**].

Lemma 38.4. *For $v, w \in T_pM$, let $\gamma(t)$ be the geodesic $exp_p(tv)$. Let $J(t)$ be the Jacobi field along γ with $J(0) = 0$ and $\frac{DJ}{dt}|_{t=0} = w$. Then $(exp_p)_{*,v}(w) = J(1)$, where we identify $T_v(T_pM) \cong T_pM$.*

Proof. To be completed. □

Lemma 38.5. *Suppose $g(R(A,B)B,A) \leq 0$ for all $A, B \in T_pM$ and for all $p \in M$. Then no 2 points in M are conjugate along any geodesic.*

Let $\Pi \subset T_pM$ be a 2-plane, and let $X_p, Y_p \in T_pM$ span Π. Then

$$K(\Pi) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}. \tag{38.1}$$

is called the *sectional curvature* of the 2-plane Π.

Lemma 38.6. *$K(\Pi)$ is independent of the particular chosen basis for Π*

Proof. To be completed. □

39 Lecture 39

Theorem 39.1 (Cartan-Hadamard)**.** *Let M be complete and simply connected with nonpositive sectional curvature. Then M is diffeomorphic to $\mathbb{R}^n$. Furthermore, any 2 points in M are joined by a unique geodesic.*

Proof. To be completed. □

Corollary 39.2. *If M is complete and nonpositive sectional curvature then $\pi_i(M) = 0$ for $i > 1$.*

Proof. Homotopy sequence of a fibration. □

Example 39.3. S^n does not admit a metric with nonpositive sectional curvature. Nor does $S^p \times S^q$ with $p > 1$. Nor does $M_1 \times M_2$ with M_1 simply connected.

Corollary 39.4. *If M is complete and nonpositive sectional curvature then $\pi_1(M)$ contains no element of finite order other than the identity.*

Proof. Needs some theory of cohomology of finite groups. □

40 Lecture 40

40.1 Curvature in the tangent bundle

The curvature tensor is defined by

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{40.1}$$

for vector fields X, Y, and Z. We define

$$Rm(X,Y,Z,W) \equiv -\langle \mathcal{R}(X,Y)Z, W \rangle. \tag{40.2}$$

We will refer to $\mathcal{R}$ as the curvature tensor of type $(1,3)$ and to Rm as the curvature tensor of type $(0,4)$.

The algebraic symmetries are:

$$\mathcal{R}(X,Y)Z = -\mathcal{R}(Y,X)Z \tag{40.3}$$
$$0 = \mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y \tag{40.4}$$
$$Rm(X,Y,Z,W) = -Rm(X,Y,W,Z) \tag{40.5}$$
$$Rm(X,Y,W,Z) = Rm(W,Z,X,Y). \tag{40.6}$$

In a coordinate system we define quantities ${R_{ijk}}^l$ by

$$\mathcal{R}(\partial_i, \partial_j)\partial_k = {R_{ijk}}^l \partial_l, \tag{40.7}$$

or equivalently,

$$\mathcal{R} = {R_{ijk}}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l. \tag{40.8}$$

Define quantities R_{ijkl} by

$$R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l), \tag{40.9}$$

or equivalently,

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l. \tag{40.10}$$

Then

$$R_{ijkl} = -\langle \mathcal{R}(\partial_i, \partial_j)\partial_k, \partial_l \rangle = -\langle {R_{ijk}}^m \partial_m, \partial_l \rangle = -{R_{ijk}}^m g_{ml}. \tag{40.11}$$

Equivalently,

$$R_{ijlk} = {R_{ijk}}^m g_{ml}, \tag{40.12}$$

that is, we lower the upper index to the *third* position.

Remark 40.1. Some authors choose to lower this index to a different position. One has to be very careful with this, or you might end up proving that S^n has negative curvature!

In coordinates, the algebraic symmetries of the curvature tensor are

$$R_{ijk}{}^{l} = -R_{jik}{}^{l} \tag{40.13}$$
$$0 = R_{ijk}{}^{l} + R_{jki}{}^{l} + R_{kij}{}^{l} \tag{40.14}$$
$$R_{ijkl} = -R_{ijlk} \tag{40.15}$$
$$R_{ijkl} = R_{klij}. \tag{40.16}$$

Of course, we can write the first 2 symmetries as a $(0, 4)$ tensor,

$$R_{ijkl} = -R_{jikl} \tag{40.17}$$
$$0 = R_{ijkl} + R_{jkil} + R_{kijl}. \tag{40.18}$$

Note that using (40.16), the algebraic Bianchi identity (40.18) may be written as

$$0 = R_{ijkl} + R_{iklj} + R_{iljk}. \tag{40.19}$$

We next compute the curvature tensor in coordinates.

$$\begin{aligned}
\mathcal{R}(\partial_i, \partial_j)\partial_k &= R_{ijk}{}^{l}\partial_l \\
&= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k \\
&= \nabla_{\partial_i}(\Gamma^l_{jk}\partial_l) - \nabla_{\partial_j}(\Gamma^l_{ik}\partial_l) \\
&= \partial_i(\Gamma^l_{jk})\partial_l + \Gamma^l_{jk}\Gamma^m_{il}\partial_m - \partial_j(\Gamma^l_{ik})\partial_l - \Gamma^l_{ik}\Gamma^m_{jl}\partial_m \\
&= \Big(\partial_i(\Gamma^l_{jk}) + \Gamma^m_{jk}\Gamma^l_{im} - \partial_j(\Gamma^l_{ik}) - \Gamma^m_{ik}\Gamma^l_{jm}\Big)\partial_l,
\end{aligned} \tag{40.20}$$

which is the formula

$$\boxed{R_{ijk}{}^{l} = \partial_i(\Gamma^l_{jk}) - \partial_j(\Gamma^l_{ik}) + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}} \tag{40.21}$$

Fix a point p. Exponential coordinates around p form a normal coordinate system at p. That is $g_{ij}(p) = \delta_{ij}$, and $\partial_k g_{ij}(p) = 0$, which is equivalent to $\Gamma^k_{ij}(p) = 0$. The Christoffel symbols are

$$\Gamma^l_{jk} = \frac{1}{2}g^{lm}\Big(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk}\Big). \tag{40.22}$$

In normal coordinates at the point p,

$$\partial_i\Gamma^l_{jk} = \frac{1}{2}\delta^{lm}\Big(\partial_i\partial_k g_{jm} + \partial_i\partial_j g_{km} - \partial_i\partial_m g_{jk}\Big). \tag{40.23}$$

We then have at p

$$R_{ijk}{}^{l} = \frac{1}{2}\delta^{lm}\Big(\partial_i\partial_k g_{jm} - \partial_i\partial_m g_{jk} - \partial_j\partial_k g_{im} + \partial_j\partial_m g_{ik}\Big). \tag{40.24}$$

Lowering an index, we have at p

$$\begin{aligned}
R_{ijkl} &= -\frac{1}{2}\Big(\partial_i\partial_k g_{jl} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il} + \partial_j\partial_l g_{ik}\Big) \\
&= -\frac{1}{2}\Big(\partial^2 \owedge g\Big).
\end{aligned} \tag{40.25}$$

The $\owedge$ symbol is the Kulkarni-Nomizu product, which takes 2 symmetric $(0,2)$ tensors and gives a $(0,4)$ tensor with the same algebraic symmetries of the curvature tensor, and is defined by

$$\begin{aligned} A \owedge B(X,Y,Z,W) =& A(X,Z)B(Y,W) - A(Y,Z)B(X,W) \\ & - A(X,W)B(Y,Z) + A(Y,W)B(X,Z). \end{aligned}$$

To remember: the first term is $A(X,Z)B(Y,W)$, skew symmetrize in X and Y to get the second term. Then skew-symmetrize both of these in Z and W.

40.2 Sectional curvature, Ricci tensor, and scalar curvature

Let $\Pi \subset T_pM$ be a 2-plane, and let $X_p, Y_p \in T_pM$ span Π. Then

$$K(\Pi) = \frac{Rm(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{40.26}$$

is independent of the particular chosen basis for Π, and is called the *sectional curvature* of the 2-plane Π. The sectional curvatures in fact determine the full curvature tensor:

Proposition 40.2. *Let Rm and Rm' be two $(0,4)$-curvature tensors which satisfy $K(\Pi) = K'(\Pi)$ for all 2-planes Π, then $Rm = Rm'$.*

Proof. To be completed. □

From this proposition, if $K(\Pi) = k_0$ is constant for all 2-planes Π, then we must have

$$Rm(X,Y,Z,W) = k_0\Big(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\Big), \tag{40.27}$$

That is

$$Rm = \frac{k_0}{2} g \owedge g. \tag{40.28}$$

In coordinates, this is

$$R_{ijkl} = k_0(g_{ik}g_{jl} - g_{jk}g_{il}). \tag{40.29}$$

We define the *Ricci tensor* as the $(0,2)$-tensor

$$Ric(X,Y) = tr(U \to \mathcal{R}(U,X)Y). \tag{40.30}$$

We clearly have

$$Ric(X,Y) = R(Y,X), \tag{40.31}$$

so $Ric \in \Gamma(S^2(T^*M))$. We let R_{ij} denote the components of the Ricci tensor,

$$Ric = R_{ij} dx^i \otimes dx^i, \tag{40.32}$$

where $R_{ij} = R_{ji}$. From the definition,

$$R_{ij} = R_{lij}{}^{l} = g^{lm} R_{limj}. \tag{40.33}$$

Notice for a space of constant curvature, we have

$$R_{jl} = g^{ik} R_{ijkl} = k_0 g^{ik}(g_{ik} g_{jl} - g_{jk} g_{il}) = (n-1) k_0 g_{jl}, \tag{40.34}$$

or invariantly

$$Ric = (n-1) k_0 g. \tag{40.35}$$

The *Ricci endomorphism* is defined by

$$Rc(X) \equiv \sharp\Big(Ric(X, \cdot) \Big). \tag{40.36}$$

The *scalar curvature* is defined as the trace of the Ricci endomorphism

$$R \equiv tr(X \to Rc(X)). \tag{40.37}$$

In coordinates,

$$R = g^{pq} R_{pq} = g^{pq} g^{lm} R_{lpmq}. \tag{40.38}$$

Note for a space of constant curvature k_0,

$$R = n(n-1) k_0. \tag{40.39}$$

41 Lecture 41

41.1 Spaces of constant curvature

Recall the Jacobi equation:

$$\frac{D^2}{dt^2} J + R(J, \dot{\gamma})\dot{\gamma} = 0. \tag{41.1}$$

Obviously, $(at + b)\dot{\gamma}$ is a Jacobi field for any constants a and b.

Proposition 41.1. *Let (M, g) have constant curvature k_0, and γ be a unit speed geodesic. Then the Jacobi Fields along γ which vanish at $t = 0$ and which are orthogonal to $\dot{\gamma}$ are given by $f(t)E$ where E is a parallel normal field, and f is given by*

$$f = \begin{cases} Ct & k_0 = 0 \\ C \sin(\sqrt{k_0} \cdot t) & k_0 > 0 \\ C \sinh(\sqrt{-k_0} \cdot t) & k_0 < 0 \end{cases}. \tag{41.2}$$

Proof. Let E be a parallel normal vector field along γ, and consider $f(t)E$. Since g has constant curvature k_0, from (40.27) above, we have

$$R(E,\dot\gamma)\dot\gamma = -k_0(\langle E,\dot\gamma\rangle\dot\gamma - \langle\dot\gamma,\dot\gamma\rangle E) = k_0 E, \tag{41.3}$$

since by assumption E is orthogonal to $\dot\gamma$, and γ is a unit speed geodesic. Plugging this into the Jacobi equation,

$$(\ddot f + k_0 f)E = 0, \tag{41.4}$$

which has the stated solutions. □

Corollary 41.2. *If g has constant curvature k_0, then in radial normal coordinates the metric has the form*

$$g = \begin{cases} dr^2 + r^2 g_{S^{n-1}} & k_0 = 0 \\ dr^2 + \frac{1}{k_0}\sin^2(\sqrt{k_0}\cdot r) g_{S^{n-1}} & k_0 > 0 \\ dr^2 + \frac{1}{|k_0|}\sinh^2(\sqrt{|k_0|}\cdot r) g_{S^{n-1}} & k_0 < 0 \end{cases}. \tag{41.5}$$

Proof. Pulling the metric back to T_pM using the exponential map, we have a metric on T_pM for which lines through the origin are geodesics. Consider the map $\gamma(s,t) = t\xi(s)$, where $\xi(s)$ is any curve. For s fixed, this is a geodesic, so is a 1-parameter variation of geodesics. Call $\xi(0) = \alpha$ and $\xi'(0) = \beta$. From above, we see that

$$\frac{\partial}{\partial s}\gamma\Big|_{s=0} = t\beta \tag{41.6}$$

is a Jacobi field along the geodesic $t \mapsto t\alpha$. From Proposition 33.1, we already know that the metric in radial normal coordinates has the form (33.4). So assume that β is orthogonal to α in the Euclidean metric, and that $|\alpha| = 1$. We claim that the Jacobi Field $t\beta$ is orthogonal to α along this geodesic. To see this we compute

$$\frac{d^2}{dt^2}(g(t\beta,\alpha)) = g\Big(\frac{D^2}{dt^2}(t\beta),\alpha\Big) \text{ (since } \alpha \text{ is parallel)} \tag{41.7}$$

$$= g(-R(t\beta,\alpha)\alpha,\alpha) = 0, \tag{41.8}$$

from the skew-symmetry of the curvature tensor. This obviously implies that $g(\beta,\alpha)$ is constant in t, and must vanish identically since it vanishes at the origin. From Proposition 41.1 we conclude that

$$t\beta = \begin{cases} CtE & k_0 = 0 \\ C\sin(\sqrt{k_0}\cdot t)E & k_0 > 0 \\ C\sinh(\sqrt{-k_0}\cdot t)E & k_0 < 0 \end{cases}, \tag{41.9}$$

where E is parallel.

If $k_0 = 0$, this says that β is a parallel normal field. In particular, $|\beta|$ is independent of the radius, and $|\beta|(r\alpha) = |\beta|(0)$. So the metric in normal cordinates is the Euclidean metric everywhere, which has the stated form in radial coordinates.

If $k_0 > 0$, then

$$\frac{t}{\sin(\sqrt{k_0} \cdot t)} \beta \tag{41.10}$$

is parallel, which implies that

$$|\beta|(r\alpha) = \frac{\sin(\sqrt{k_0} \cdot r)}{\sqrt{k_0} \cdot r} |\beta|(0). \tag{41.11}$$

In radial coordinates, the metric on the sphere of radius r pulls pack to $r^2 g_{S^{n-1}}$, so the r cancels out and we arrive at (41.5). A similar argument holds in the $k_0 < 0$ case. □

This implies that any two space forms of the same constant curvature are locally isometric, but not necessarily globally! The above coordinate system can fail for two reasons. First, one can hit the cut locus, in which case the coordinate system is not injective. Second, the expression for the metric can become degenerate, this is called a conjugate point. Discuss the cut locus in a few examples, such as tori, spheres, projective spaces, lens spaces.

In general, we have the following:

Theorem 41.3. *If (M, g) is simply-connected and constant sectional curvature $K = 0, 1, -1$ then (M, g) is isometric to Euclidean space, S^n with the round metric, or hyperbolic space H^n.*

42 Lecture 42

42.1 Theorem of Bonnet-Myers

Theorem 42.1. *If (M, g) is complete and Ric $\geq \frac{n-1}{a^2} g$ for a constant $a > 0$, then the diameter $diam_g \leq \pi a$ and $\pi_1(M)$ is finite.*

Proof. Let $\gamma : [0, 1] \to M$ be a geodesic of length L. Plug in $\sin(\pi t) P_i$, where P_i is parallel orthonormal frame field into the second variation formula, and then take a sum. to see that the index will be positive if $L > \pi a$. Then γ could not be minimizing by Theorem 37.3. □

42.2 Taylor expansion of a metric in normal coordinates

Theorem 42.2. *In normal coordinates, a metric g admits the expansion*

$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{kijl} x^k x^l + O(|x|^3) \tag{42.1}$$

as $x \to 0$, where all coefficients are evaluated at 0.

Proof. To compute this, we argue as in the proof of Corollary 41.2. Choose β orthogonal to α in the Euclidean metric, and assume that $|\alpha| = 1$. Then $J = t\beta$ is a Jacobi field along the geodesic $t \mapsto t\alpha$. We want to expand the function $f(t) \equiv g(t\beta, t\beta)(t\alpha)$ as a function of t. Obviously, $f(0) = 0$, and

$$\partial_t f = \partial_t(g(J, J)) = 2g(D_t J, J). \tag{42.2}$$

Evaluating at 0, $f'(0) = 0$, since $J(0) = 0$. Next,

$$\partial_t^2 g(J, J) = 2g(D_t^2 J, J) + 2g(D_t J, D_t J). \tag{42.3}$$

Evaluating (42.3) at 0, since $J(0) = 0$, and $D_t J = \beta$, we have

$$f''(0) = 2g_0(\beta, \beta), \tag{42.4}$$

where g_0 denotes the Euclidean metric at the origin.

To simplify notation, we will let R_α denote the endomorphism $J \mapsto R(\alpha, J)\alpha$, so we can write

$$\partial_t^2 g(J, J) = 2g(R_\alpha(J), J) + 2g(D_t J, D_t J). \tag{42.5}$$

Note that R_α is self-adjoint, i.e.,

$$g(R_\alpha(X), Y) = g(R(\alpha, X)\alpha, Y) = g(R(\alpha, Y)\alpha, X) = g(X, R_\alpha Y), \tag{42.6}$$

from the symmetry of the curvature tensor (40.6).

Differentiating (42.3),

$$\partial_t^3(g(J, J)) = 2g(D_t^3 J, J) + 6g(D_t^2 J, D_t J). \tag{42.7}$$

Evaluating at 0, since $J(0) = 0$, and $D_t^2 J = R_\alpha(J)$, we have

$$f'''(0) = 0. \tag{42.8}$$

Differentiating (42.7),

$$\partial_t^4(g(J, J)) = 2g(D_t^4 J, J) + 8g(D_t^3 J, D_t J) + 6g(D_t^2 J, D_t^2 J). \tag{42.9}$$

Note that

$$D_t^3 J = D_t(D_t^2 J) = D_t(R_\alpha(J)) = (D_t R_\alpha)(J) + R_\alpha(D_t J). \tag{42.10}$$

Evaluating (42.9) at $t = 0$, we obtain

$$f^{(iv)}(0) = 8g_0(R_\alpha(\beta), \beta). \tag{42.11}$$

Performing a Taylor expansion around $t = 0$, we have shown that

$$g(\beta, \beta)(t\alpha) = g_0(\beta, \beta) + \frac{t^2}{3} g_0(R_\alpha(\beta), \beta) + O(t^3) \tag{42.12}$$

as $t \to 0$. We let $\alpha = (x^i/t)\partial_i$, and $\beta = \beta^j\partial_j$. The first term on the right hand side of (42.12) is simply

$$g_0(\beta,\beta) = \delta_{ij}\beta^i\beta^j. \tag{42.13}$$

The second term on the right hand side of (42.12) is

$$\begin{aligned}
\frac{t^2}{3}g_0(R_\alpha(\beta),\beta) &= \frac{t^2}{3}g_0(R(\alpha,\beta)\alpha,\beta) && (42.14)\\
&= \frac{1}{3}g_0(R(x^k\partial_k,\beta^i\partial_i)x^l\partial_l,\beta^j\partial_j) && (42.15)\\
&= \frac{1}{3}x^kx^lR_{kil}{}^m\delta_{mj}\beta^i\beta^j && (42.16)\\
&= \frac{1}{3}R_{kijl}x^kx^l\beta^i\beta^j. && (42.17)
\end{aligned}$$

□

43 Lecture 43

43.1 Covariant derivatives of tensor fields

Let E and E' be vector bundles over M, with covariant derivative operators ∇, and ∇', respectively. The covariant derivative operators in $E \otimes E'$ and $Hom(E,E')$ are

$$\nabla_X(s\otimes s') = (\nabla_X s)\otimes s' + s\otimes(\nabla'_X s') \tag{43.1}$$

$$(\nabla_X L)(s) = \nabla'_X(L(s)) - L(\nabla_X s), \tag{43.2}$$

for $s \in \Gamma(E), s' \in \Gamma(E')$, and $L \in \Gamma(Hom(E,E'))$. Note also that the covariant derivative operator in $\Lambda(E)$ is given by

$$\nabla_X(s_1\wedge\cdots\wedge s_r) = \sum_{i=1}^r s_1\wedge\cdots\wedge(\nabla_X s_i)\wedge\cdots\wedge s_r, \tag{43.3}$$

for $s_i \in \Gamma(E)$.

These rules imply that if T is an (r,s) tensor, then the covariant derivative ∇T is an $(r,s+1)$ tensor given by

$$\nabla T(X,Y_1,\dots,Y_s) = \nabla_X(T(Y_1,\dots Y_s)) - \sum_{i=1}^s T(Y_1,\dots,\nabla_X Y_i,\dots,Y_s). \tag{43.4}$$

We next consider the above definitions in components for (r,s)-tensors. For the case of a vector field $X \in \Gamma(TM)$, ∇X is a $(1,1)$ tensor field. By the definition of a connection, we have

$$\nabla_m X \equiv \nabla_{\partial_m}X = \nabla_{\partial_m}(X^j\partial_j) = (\partial_m X^j)\partial_j + X^j\Gamma^l_{mj}\partial_l = (\nabla_m X^i + X^l\Gamma^i_{ml})\partial_i. \tag{43.5}$$

In other words,

$$\nabla X = \nabla_m X^i (dx^m \otimes \partial_i), \tag{43.6}$$

where

$$\nabla_m X^i = \partial_m X^i + X^l \Gamma^i_{ml}. \tag{43.7}$$

However, for a 1-form ω, (43.2) implies that

$$\nabla \omega = (\nabla_m \omega_i) dx^m \otimes dx^i, \tag{43.8}$$

with

$$\nabla_m \omega_i = \partial_m \omega_i - \omega_l \Gamma^l_{im}. \tag{43.9}$$

The definition (43.1) then implies that for a general (r,s)-tensor field S,

$$\begin{aligned}\nabla_m S^{i_1 \dots i_r}_{j_1 \dots j_s} \equiv \partial_m S^{i_1 \dots i_r}_{j_1 \dots j_s} + S^{l i_2 \dots i_r}_{j_1 \dots j_s} \Gamma^{i_1}_{ml} + \dots + S^{i_1 \dots i_{r-1} l}_{j_1 \dots j_s} \Gamma^{i_r}_{ml} \\ - S^{i_1 \dots i_r}_{l j_2 \dots j_s} \Gamma^l_{m j_1} - \dots - S^{i_1 \dots i_r}_{j_1 \dots j_{s-1} l} \Gamma^l_{m j_s}.\end{aligned} \tag{43.10}$$

Remark 43.1. Some authors instead write covariant derivatives with a semi-colon

$$\nabla_m S^{i_1 \dots i_r}_{j_1 \dots j_s} = S^{i_1 \dots i_r}_{j_1 \dots j_s ; m}. \tag{43.11}$$

However, the ∇ notation fits nicer with our conventions, since the *first* index is the direction of covariant differentiation.

Notice the following calculation,

$$(\nabla g)(X, Y, Z) = X g(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0, \tag{43.12}$$

so the metric is parallel. Note that in coordinates, this says that

$$0 = \nabla_m g_{ij} = \partial_m g_{ij} - \Gamma^p_{mi} g_{pj} - \Gamma^p_{mj} g_{ip}, \tag{43.13}$$

which yield the formula

$$\partial_k g_{ij} = \Gamma^p_{ki} g_{pj} + \Gamma^p_{kj} g_{ip}. \tag{43.14}$$

This is sometimes written as

$$\partial_k g_{ij} = [ki, j] + [kj, i], \tag{43.15}$$

where $[ij; k]$ are called the Christoffel symbols of the first kind defined by

$$[ij, k] \equiv \frac{1}{2}\Big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \Big). \tag{43.16}$$

Next, let $I : TM \to TM$ denote the identity map, which is naturally a $(1,1)$ tensor. We have

$$(\nabla I)(X, Y) = \nabla_X (I(Y)) - I(\nabla_X Y) = \nabla_X Y - \nabla_X Y = 0, \tag{43.17}$$

so the identity map is also parallel.
Note that the following statements are equivalent

- $\flat \in Hom(TM, T^*M)$ is parallel
- $\flat$ commutes with covariant differentiation.
- $\nabla_m(g_{ij}X^j) = g_{ij}\nabla_m X^j$.

Similarly, the induced metric on T^*M is parallel, and the following are equivalent.

- $\sharp \in Hom(T^*M, TM)$ is parallel
- $\sharp$ commutes with covariant differentiation.
- $\nabla_m(g^{ij}\omega_j) = g^{ij}\nabla_m\omega_j$.

Finally, note that the following are equivalent

- Tr mapping from (p,q)-tensors to $(p-1, q-1)$ tensors is parallel.
- Tr commutes with covariant differentiation.
- $\nabla_m\left(\delta^{j_1}_{i_1} X^{i_1 i_2 \dots}_{j_1 j_2 \dots}\right) = \delta^{j_1}_{i_1}\nabla_m X^{i_1 i_2 \dots}_{j_1 j_2 \dots}$.

43.2 Double covariant derivatives

For an (r,s) tensor field T, we will write the double covariant derivative as

$$\nabla^2 T = \nabla\nabla T, \tag{43.18}$$

which is an $(r, s+2)$ tensor.

Proposition 43.2. *If T is an (r,s)-tensor field, then the double covariant derivative satisfies*

$$\nabla^2 T(X, Y, Z_1, \dots, Z_s) = \nabla_X(\nabla_Y T)(Z_1, \dots, Z_s) - (\nabla_{\nabla_X Y} T)(Z_1, \dots Z_s). \tag{43.19}$$

Proof. The left hand side of (43.19) is

$$\begin{aligned} \nabla^2 T(X, Y, Z_1, \dots, Z_s) &= \nabla(\nabla T)(X, Y, Z_1, \dots, Z_s) \\ &= \nabla_X(\nabla T(Y, Z_1, \dots, Z_s)) - \nabla T(\nabla_X Y, Z_1, \dots, Z_s) \\ &\quad - \sum_{i=1}^{s} \nabla T(Y, \dots, \nabla_X Z_i, \dots Z_s). \end{aligned} \tag{43.20}$$

The right hand side of (43.19) is

$$\begin{aligned} &\nabla_X(\nabla_Y T)(Z_1, \dots, Z_s) - (\nabla_{\nabla_X Y} T)(Z_1, \dots Z_s) \\ &= \nabla_X(\nabla_Y T(Z_1, \dots, Z_s)) - \sum_{i=1}^{s}(\nabla_Y T)(Z_1, \dots, \nabla_X Z_i, \dots, Z_s) \\ &\quad - \nabla T(\nabla_X Y, Z_1, \dots, Z_s). \end{aligned} \tag{43.21}$$

The first term on the right hand side of (43.21) is the same as first term on the right hand side of (43.20). The second term on the right hand side of (43.21) is the same as third term on the right hand side of (43.20). Finally, the last term on the right hand side of (43.21) is the same as the second term on the right hand side of (43.20). $\square$

Remark 43.3. When we write

$$\nabla_i \nabla_j T^{j_1 \dots j_r}_{i_i \dots i_s} \tag{43.22}$$

we mean the components of the double covariant derivative of T as a $(r, s+2)$ tensor. This does NOT mean to take one covariant derivative ∇T, plug in ∂_j to get an (r, s) tensor, and then take a covariant derivative in the ∂_i direction; this would yield only the first term on the right hand side of (43.19).

For illustration, let's compute an example in coordinates. If $\omega \in \Omega^1(M)$, then

$$\begin{aligned} \nabla_i \nabla_j \omega_k &= \partial_i(\nabla_j \omega_k) - \Gamma^p_{ij} \nabla_p \omega_k - \Gamma^p_{ik} \nabla_j \omega_p \\ &= \partial_i(\partial_j \omega_k - \Gamma^l_{jk}\omega_l) - \Gamma^p_{ij}(\partial_p \omega_k - \Gamma^l_{pk}\omega_l) - \Gamma^p_{ik}(\partial_j \omega_p - \Gamma^l_{jp}\omega_k). \end{aligned} \tag{43.23}$$

Expanding everything out, we can write this formally as

$$\nabla^2 \omega = \partial^2 \omega_k + \Gamma * \partial \omega + (\partial \Gamma + \Gamma * \Gamma) * \omega, \tag{43.24}$$

where $*$ denotes various tensor contractions. Notice that the coefficient of ω on the right looks similar to the curvature tensor in coordinates (40.21). This is closely related to Weitzenböck formulas which we will discuss later.

44 Lecture 44

44.1 Commuting covariant derivatives

Let $X, Y, Z \in \Gamma(TM)$, and compute using Proposition 43.2

$$\begin{aligned} \nabla^2 Z(X,Y) - \nabla^2 Z(Y,X) &= \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z - \nabla_Y(\nabla_X Z) - \nabla_{\nabla_Y X} Z \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{\nabla_X Y - \nabla_Y X} Z \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \\ &= \mathcal{R}(X,Y)Z, \end{aligned} \tag{44.1}$$

which is just the definition of the curvature tensor. In coordinates,

$$\boxed{\nabla_i \nabla_j Z^k = \nabla_j \nabla_i Z^k + R_{ijm}{}^k Z^m.} \tag{44.2}$$

We extend this to $(p, 0)$-tensor fields:

$$\begin{aligned} &\nabla^2(Z_1 \otimes \cdots \otimes Z_p)(X,Y) - \nabla^2(Z_1 \otimes \cdots \otimes Z_p)(Y,X) \\ &= \nabla_X(\nabla_Y(Z_1 \otimes \cdots \otimes Z_p)) - \nabla_{\nabla_X Y}(Z_1 \otimes \cdots \otimes Z_p) \\ &\quad - \nabla_Y(\nabla_X(Z_1 \otimes \cdots \otimes Z_p)) - \nabla_{\nabla_Y X}(Z_1 \otimes \ldots \otimes Z_p \\ &= \nabla_X\Big(\sum_{i=1}^p Z_1 \otimes \cdots \nabla_Y Z_i \otimes \cdots \otimes Z_p\Big) - \sum_{i=1}^p Z_1 \otimes \cdots \nabla_{\nabla_X Y} Z_i \otimes \cdots \otimes Z_p \\ &- \nabla_Y\Big(\sum_{i=1}^p Z_1 \otimes \cdots \nabla_X Z_i \otimes \cdots \otimes Z_p\Big) + \sum_{i=1}^p Z_1 \otimes \cdots \nabla_{\nabla_Y X} Z_i \otimes \cdots \otimes Z_p. \end{aligned} \tag{44.3}$$

With a slight abuse of notation, this may be rewritten as

$$
\begin{aligned}
&\nabla^2(Z_1 \otimes \cdots \otimes Z_p)(X,Y) - \nabla^2(Z_1 \otimes \cdots \otimes Z_p)(Y,X) \\
&= \sum_{j=1}^{p} \sum_{i=1, i\neq j}^{p} Z_1 \otimes \nabla_X Z_j \otimes \cdots \nabla_Y Z_i \otimes \cdots \otimes Z_p \\
&\qquad - \sum_{j=1}^{p} \sum_{i=1, i\neq j}^{p} Z_1 \otimes \nabla_Y Z_j \otimes \cdots \nabla_X Z_i \otimes \cdots \otimes Z_p \\
&\qquad\qquad + \sum_{i=1}^{p} Z_1 \otimes \cdots \otimes (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z_i \otimes \cdots \otimes Z_p \\
&= \sum_{i=1}^{p} Z_1 \otimes \cdots \otimes \mathcal{R}(X,Y) Z_i \otimes \cdots \otimes Z_p.
\end{aligned}
\tag{44.4}
$$

In coordinates, this is

$$
\boxed{\nabla_i \nabla_j Z^{i_1 \dots i_p} = \nabla_j \nabla_i Z^{i_i \dots i_p} + \sum_{k=1}^{p} R_{ijm}{}^{i_k} Z^{i_1 \dots i_{k-1} m i_{k+1} \dots i_p}.}
\tag{44.5}
$$

Proposition 44.1. *For a* 1*-form* ω*, we have*

$$
\nabla^2\omega(X,Y,Z) - \nabla^2\omega(Y,X,Z) = \omega(\mathcal{R}(Y,X)Z).
\tag{44.6}
$$

Proof. Using Proposition 43.2, we compute

$$
\begin{aligned}
&\nabla^2\omega(X,Y,Z) - \nabla^2\omega(Y,X,Z) \\
&= \nabla_X(\nabla_Y \omega)(Z) - (\nabla_{\nabla_X Y}\omega)(Z) - \nabla_Y(\nabla_X \omega)(Z) - (\nabla_{\nabla_Y X}\omega)(Z) \\
&= X(\nabla_Y \omega(Z)) - \nabla_Y \omega(\nabla_X Z) - \nabla_X Y(\omega(Z)) + \omega(\nabla_{\nabla_X Y} Z) \\
&\quad - Y(\nabla_X \omega(Z)) + \nabla_X \omega(\nabla_Y Z) + \nabla_Y X(\omega(Z)) - \omega(\nabla_{\nabla_Y X} Z) \\
&= X(\nabla_Y \omega(Z)) - Y(\omega(\nabla_X Z)) + \omega(\nabla_Y \nabla_X Z) - \nabla_X Y(\omega(Z)) + \omega(\nabla_{\nabla_X Y} Z) \\
&\quad - Y(\nabla_X \omega(Z)) + X(\omega(\nabla_Y Z)) - \omega(\nabla_X \nabla_Y Z) + \nabla_Y X(\omega(Z)) - \omega(\nabla_{\nabla_Y X} Z) \\
&= \omega\Big(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z\Big) + X(\nabla_Y \omega(Z)) - Y(\omega(\nabla_X Z)) - \nabla_X Y(\omega(Z)) \\
&\quad - Y(\nabla_X \omega(Z)) + X(\omega(\nabla_Y Z)) + \nabla_Y X(\omega(Z)).
\end{aligned}
\tag{44.7}
$$

The last six terms are

$$
\begin{aligned}
&X(\nabla_Y \omega(Z)) - Y(\omega(\nabla_X Z)) - \nabla_X Y(\omega(Z)) \\
&- Y(\nabla_X \omega(Z)) + X(\omega(\nabla_Y Z)) + \nabla_Y X(\omega(Z)) \\
&= X\Big(Y(\omega(Z)) - \omega(\nabla_Y Z)\Big) - Y(\omega(\nabla_X Z)) - [X,Y](\omega(Z)) \\
&- Y\Big(X(\omega(Z)) - \omega(\nabla_X Z)\Big) + X(\omega(\nabla_Y Z)) \\
&= 0.
\end{aligned}
\tag{44.8}
$$

□

Remark 44.2. It would have been a lot easier to assume we were in normal coordinates, and ignore terms involving first covariant derivatives of the vector fields, but we did the above for illustration.

In coordinates, this formula becomes

$$\boxed{\nabla_i \nabla_j \omega_k = \nabla_j \nabla_i \omega_k - R_{ijk}{}^{p} \omega_p.} \tag{44.9}$$

As above, we can extend this to $(0, s)$ tensors using the tensor product, in an almost identical calculation to the $(r, 0)$ tensor case. Finally, putting everything together, the analogous formula in coordinates for a general (r, s)-tensor T is

$$\boxed{\nabla_i \nabla_j T^{i_1 \dots i_r}_{j_1 \dots j_s} = \nabla_j \nabla_i T^{i_1 \dots i_r}_{j_1 \dots j_s} + \sum_{k=1}^{r} R_{ijm}{}^{i_k} T^{i_1 \dots i_{k-1} m i_{k+1} \dots i_r}_{j_1 \dots j_s} - \sum_{k=1}^{s} R_{ijj_k}{}^{m} T^{i_1 \dots i_r}_{j_1 \dots j_{k-1} m j_{k+1} \dots j_s}.} \tag{44.10}$$

44.2 Gradient, Hessian, and Laplacian

As an example of the above, we consider the Hessian of a function. For $f \in C^1(M, \mathbb{R})$, the *gradient* is defined as

$$\nabla f = \sharp(df), \tag{44.11}$$

which is a vector field. This is standard notation, although in our notation above, $\nabla f = df$, where this ∇ denotes the covariant derivative. The *Hessian* is the $(0, 2)$-tensor defined by the double covariant derivative of a function, which by Proposition 43.2 is given by

$$\nabla^2 f(X, Y) = \nabla_X (\nabla_Y f) - \nabla_{\nabla_X Y} f = X(Yf) - (\nabla_X Y) f. \tag{44.12}$$

In components, this formula is

$$\nabla^2 f(\partial_i, \partial_j) = \nabla_i \nabla_j f = \partial_i \partial_j f - \Gamma^k_{ij} (\partial_k f). \tag{44.13}$$

The symmetry of the Hessian

$$\nabla^2 f(X, Y) = \nabla^2 f(Y, X), \tag{44.14}$$

then follows easily from the symmetry of the Riemannian connection. Notice that no curvature terms appear in this formula, which happens only in this special case.

The Laplacian of a function is the trace of the Hessian when considered as an endomorphism,

$$\Delta f = tr\big(X \mapsto \sharp(\nabla^2 f(X, \cdot))\big), \tag{44.15}$$

so in coordinates is given by

$$\Delta f = g^{ij} \nabla_i \nabla_j f. \tag{44.16}$$

This turns out to be equal to

$$\Delta f = \frac{1}{\sqrt{\det(g)}} \partial_i \big(g^{ij} \partial_j f \sqrt{\det(g)} \big). \tag{44.17}$$

In a local orthonormal frame $\{e_i\}, i = 1 \dots n$, the formula for the Hessian looks like

$$\begin{aligned} (\nabla^2 f)(e_i, e_j) &= \nabla_{e_i}(\nabla_{e_j} f) - \nabla_{\nabla_{e_i} e_j} f \\ &= e_i(e_j f) - (\nabla_{e_i} e_j) f, \end{aligned} \tag{44.18}$$

and the Laplacian is given by the expression

$$\Delta f = \sum_{i=1}^{n} \nabla^2 f(e_i, e_i) = \sum_{i=1}^{n} e_i(e_i f) - \sum_{i=1}^{n} (\nabla_{e_i} e_i) f. \tag{44.19}$$

44.3 Differential Bianchi Identity

Higher covariant derivatives of the curvature tensor must satisfy certain identities, the first of which is the following, which is known as the differential Bianchi identity.

Proposition 44.3. *The covariant derivative of the curvature tensor ∇Rm satisfies the relation*

$$\nabla Rm(X, Y, Z, V, W) + \nabla Rm(Y, Z, X, V, W) + \nabla Rm(Z, X, Y, V, W) = 0. \tag{44.20}$$

Proof. Since the equation is tensorial, we can compute in a normal coordinate system near a point p, letting the vector fields be the coordinate partials. This means we can ignore terms involving only first covariant derivatives of the vector fields. Also, Lie brackets can be ignored since they vanish identically in a neighborhood of p. We compute

$$\begin{aligned} &\nabla Rm(X, Y, Z, V, W) + \nabla Rm(Y, Z, X, V, W) + \nabla Rm(Z, X, Y, V, W) \\ &= X(Rm(Y, Z, V, W)) + Y(Rm(Z, X, V, W)) + Z(Rm(X, Y, V, W)) \\ &= -X \langle \mathcal{R}(Y, Z)V, W \rangle - Y \langle \mathcal{R}(Z, X)V, W \rangle - Z \langle \mathcal{R}(X, Y)V, W \rangle \\ &= -\langle \nabla_X \mathcal{R}(Y, Z)V, W \rangle - \langle \nabla_Y \mathcal{R}(Z, X)V, W \rangle - \langle \nabla_Z \mathcal{R}(X, Y)V, W \rangle \\ &= -\langle \nabla_X \nabla_Y \nabla_Z V - \nabla_X \nabla_Z \nabla_Y V, W \rangle - \langle \nabla_Y \nabla_Z \nabla_X V - \nabla_Y \nabla_X \nabla_Z V, W \rangle \\ &\qquad - \langle \nabla_Z \nabla_X \nabla_Y V - \nabla_Z \nabla_Y \nabla_X V, W \rangle \\ &= -\langle \nabla_X \nabla_Y \nabla_Z V - \nabla_Y \nabla_X \nabla_Z V, W \rangle - \langle \nabla_Y \nabla_Z \nabla_X V - \nabla_Z \nabla_Y \nabla_X V, W \rangle \\ &\qquad - \langle \nabla_Z \nabla_X \nabla_Y V - \nabla_X \nabla_Z \nabla_Y V, W \rangle \\ &= Rm(X, Y, \nabla_Z V, W) + Rm(Y, Z, \nabla_X V, W) + Rm(Z, X, \nabla_Y V, W) \equiv 0. \end{aligned} \tag{44.21}$$

□

In coordinates, this is equivalent to

$$\boxed{\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0.} \tag{44.22}$$

Let us raise an index,

$$\nabla_i R_{jkm}{}^l + \nabla_j R_{kim}{}^l + \nabla_k R_{ijm}{}^l = 0. \tag{44.23}$$

Contract on the indices i and l,

$$0 = \nabla_l R_{jkm}{}^l + \nabla_j R_{klm}{}^l + \nabla_k R_{ljm}{}^l = \nabla_l R_{jkm}{}^l - \nabla_j R_{km} + \nabla_k R_{jm}. \tag{44.24}$$

This yields the Bianchi identity

$$\boxed{\nabla_l R_{jkm}{}^l = \nabla_j R_{km} - \nabla_k R_{jm}.} \tag{44.25}$$

In invariant notation, this is sometimes written as

$$\delta \mathcal{R} = d^{\nabla} Ric, \tag{44.26}$$

where $d^{\nabla} : S^2(T^*M) \to \Lambda^2(T^*M) \otimes T^*M$, is defined by

$$d^{\nabla} h(X, Y, Z) = \nabla h(X, Y, Z) - \nabla h(Y, Z, X), \tag{44.27}$$

and δ is the *divergence operator*.

Next, trace (44.25) on the indices k and m,

$$g^{km} \nabla_l R_{jkm}{}^l = g^{km} \nabla_j R_{km} - g^{km} \nabla_k R_{jm}. \tag{44.28}$$

Since the metric is parallel, we can move the g^{km} terms inside,

$$\nabla_l g^{km} R_{jkm}{}^l = \nabla_j g^{km} R_{km} - \nabla_k g^{km} R_{jm}. \tag{44.29}$$

The left hand side is

$$\begin{aligned} \nabla_l g^{km} R_{jkm}{}^l &= \nabla_l g^{km} g^{lp} R_{jkpm} \\ &= \nabla_l g^{lp} g^{km} R_{jkpm} \\ &= \nabla_l g^{lp} R_{jp} = \nabla_l R^l_j. \end{aligned} \tag{44.30}$$

So we have the Bianchi identity

$$\boxed{2 \nabla_l R^l_j = \nabla_j R.} \tag{44.31}$$

Invariantly, this can be written

$$\delta Rc = \frac{1}{2} dR. \tag{44.32}$$

Corollary 44.4. *Let (M, g) be a connected Riemannian manifold. If $n > 2$, and there exists a function $f \in C^{\infty}(M)$ satisfying $Ric = fg$, then $Ric = (n-1)k_0 g$, where k_0 is a constant.*

Proof. Taking a trace, we find that $R = nf$. Using (44.31), we have

$$2 \nabla_l R^l_j = 2 \nabla_l \Big(\frac{R}{n} \delta^l_j \Big) = \frac{2}{n} \nabla_l R = \nabla_l R. \tag{44.33}$$

Since $n > 2$, we must have $dR = 0$, which implies that R, and therefore f, is constant. $\square$

A metric satisfying $Ric = \Lambda g$ for a constant Λ is called an *Einstein metric*.

45 Lecture 45

45.1 The divergence of a tensor

If T is an (r, s)-tensor, we define the *divergence* of T, div T to be the $(r, s-1)$ tensor

$$(\text{div } T)(Y_1, \dots, Y_{s-1}) = tr\Big(X \to \sharp(\nabla T)(X, \cdot, Y_1, \dots, Y_{s-1})\Big), \tag{45.1}$$

that is, we trace the covariant derivative on the *first* two covariant indices. In coordinates, this is

$$(\text{div } T)^{i_1 \dots i_r}_{j_1 \dots j_{s-1}} = g^{ij} \nabla_i T^{i_1 \dots i_r}_{j j_1 \dots j_{s-1}}. \tag{45.2}$$

Using an local orthonormal frame $\{e_i\}, i = 1 \dots n$, the divergence can also be written as

$$(\text{div } T)(Y_1, \dots Y_{s-1}) = \sum_{i=1}^{n} (\nabla_{e_i} T)(e_i, Y_1, \dots, Y_{s-1}). \tag{45.3}$$

If X is a vector field, define

$$(\text{div } X) = tr(\nabla X), \tag{45.4}$$

which is in coordinates

$$\text{div } X = \delta^i_j \nabla_i X^j = \nabla_j X^j. \tag{45.5}$$

For vector fields and 1-forms, these two are of course closely related:

Proposition 45.1. *For a vector field X,*

$$\text{div } X = \text{div } (\flat X). \tag{45.6}$$

Proof. We compute

$$\text{div } X = \delta^i_j \nabla_i X^j = \delta^i_j \nabla_i g^{jl} X_l = \delta^i_j g^{jl} \nabla_i X_l = g^{il} \nabla_i X_l = \text{div } (\flat X). \tag{45.7}$$

□

In a local orthonormal frame $\{e_i\}, i = 1 \dots n$, the divergence of a 1-form is given by

$$\begin{aligned} \text{div } \omega &= \sum_{i=1}^{n} (\nabla_{e_i} \omega)(e_i) \\ &= \sum_{i=1}^{n} e_i(\omega(e_i)) - \omega\Big(\sum_{i=1}^{n} \nabla_{e_i} e_i\Big), \end{aligned} \tag{45.8}$$

whereas the divergence of a vector field is given by

$$\text{div } X = \sum_{i=1}^{n} \langle \nabla_{e_i} X, e_i \rangle. \tag{45.9}$$

45.2 Volume element and Hodge star

If M is oriented, we define the Riemannian volume element dV to be the oriented unit norm element of $\Lambda^n(T^*M_x)$. Equivalently, if $\omega^1, \ldots, \omega^n$ is a positively oriented ONB of T^*M_x, then

$$dV = \omega^1 \wedge \cdots \wedge \omega^n. \tag{45.10}$$

In coordinates,

$$dV = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n. \tag{45.11}$$

Recall the Hodge star operator $* : \Lambda^p \to \Lambda^{n-p}$ defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dV_x, \tag{45.12}$$

where $\alpha, \beta \in \Lambda^p$.

Remark 45.2. *The inner product in* (45.12) *is the inner product on p-forms, not the tensor inner product.*

The folowing proposition summarizes the main properties of the Hodge star operator that we will require.

Proposition 45.3. *The Hodge star operator satisfies the following.*

1. *The Hodge star is an isometry from Λ^p to Λ^{n-p}.*

2. $*(\omega^1 \wedge \cdots \wedge \omega^p) = \omega^{p+1} \wedge \cdots \wedge \omega^n$ *if $\omega^1, \ldots, \omega^n$ is a positively oriented ONB of T^*M_x. In particular, $*1 = dV$, and $*dV = 1$.*

3. *On Λ^p, $*^2 = (-1)^{p(n-p)}$.*

4. *For $\alpha, \beta \in \Lambda^p$,*

$$\langle \alpha, \beta \rangle = *(\alpha \wedge *\beta) = *(\beta \wedge *\alpha). \tag{45.13}$$

5. *If $\{e_i\}$ and $\{\omega^i\}$ are dual ONB of T_xM, and T_x^*M, respectively, and $\alpha \in \Lambda^p$, then*

$$*(\omega^j \wedge \alpha) = (-1)^p i_{e_j}(*\alpha), \tag{45.14}$$

where $i_X : \Lambda^p \to \Lambda^{p-1}$ is interior multiplication *defined by*

$$i_X\alpha(X_1, \ldots, X_{p-1}) = \alpha(X, X_1, \ldots, X_{p-1}). \tag{45.15}$$

6. *For $\alpha \in \Omega^p(M)$, in a coordinate system,*

$$(*\alpha)_{i_1 \ldots i_{n-p}} = \frac{1}{p!}\alpha^{j_1 \ldots j_p}\sqrt{\det(g)}\epsilon_{j_1 \ldots j_p i_1 \ldots i_{n-p}}, \tag{45.16}$$

where the ϵ symbol is equal to 1 *if $(j_1, \ldots, j_p, i_1, \ldots, i_{n-p})$ is an even permutation of $(1, \ldots, n)$, equal to -1 if it is an odd permutation, and zero otherwise.*

Proof. The proof is left to the reader. □

Remark 45.4. *Note that interior multiplication is not canonically defined – it depends upon our identification of p-forms with alternating tensors of type* $(0,p)$.

Remark 45.5. *In general, locally there will be two different Hodge star operators, depending upon the two different choices of local orientation. Each will extend to a global Hodge star operator if and only if M is orientable. However, one can still construct global operators using the Hodge star, even if M is non-orientable, an example of which will be the Laplacian.*

45.3 Exterior derivative and covariant differentiation

We next give a formula relating the exterior derivative and covariant differentiation.

Proposition 45.6. *The exterior derivative* $d : \Omega^p \to \Omega^{p+1}$ *is given by*

$$d\omega(X_0, \dots, X_p) = \sum_{j=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \dots, \hat{X}_j, \dots, X_p), \tag{45.17}$$

(recall the notation means that the $\hat{X}_j$ term is omitted). If $\{e_i\}$ and $\{\omega^i\}$ are dual ONB of T_xM, and T_x^*M, then this may be written

$$d\omega = \sum_{i=1}^{n} \omega^i \wedge \nabla_{e_i}\omega. \tag{45.18}$$

In coordinates, this is

$$(d\omega)_{i_0 \dots i_p} = \sum_{j=0}^{p} (-1)^j \nabla_{i_j} \omega_{i_0 \dots \hat{i_j} \dots i_p}. \tag{45.19}$$

Proof. Recall the formula for the exterior derivative [War83, Proposition 2.25],

$$\begin{aligned} d\omega(X_0, \dots, X_p) = &\sum_{j=0}^{p} (-1)^j X_j \Big(\omega(X_0, \dots, \hat{X}_j, \dots, X_p) \Big) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p). \end{aligned} \tag{45.20}$$

Since both sides of the equation (45.17) are tensors, we may assume that $[X_i, X_j]_x = 0$, at a fixed point x. Since the connection is Riemannian, we also have $\nabla_{X_i} X_j(x) = 0$. We then compute at the point x.

$$\begin{aligned} d\omega(X_0, \dots, X_p)(x) &= \sum_{j=0}^{p} (-1)^j X_j \Big(\omega(X_0, \dots, \hat{X}_j, \dots, X_p) \Big)(x) \\ &= \sum_{j=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \dots, \hat{X}_j, \dots, X_p)(x), \end{aligned} \tag{45.21}$$

using the definition of the covariant derivative. This proves the first formula (45.17). The formula (45.19) is just (45.17) in a coordinate system.

For (45.18), note that

$$\nabla_{X_j}\omega = \nabla_{(X_j)^i e_i}\omega = \sum_{i=1}^{n} \omega^i(X_j) \cdot (\nabla_{e_i}\omega), \tag{45.22}$$

so we have

$$\begin{aligned} d\omega(X_0, \ldots, X_p) &= \sum_{j=0}^{p}(-1)^j \sum_{i=1}^{n} \omega^i(X_j) \cdot (\nabla_{e_i}\omega)(X_0, \ldots, \hat{X}_j, \ldots, X_p) \\ &= \sum_i (\omega^i \wedge \nabla_{e_i}\omega)(X_0, \ldots, X_p), \end{aligned} \tag{45.23}$$

where we used (1.21) to obtain the last equality. □

46 Lecture 46

46.1 The divergence theorem for a Riemannian manifold

We begin with a useful formula for the divergence of a vector field.

Proposition 46.1. *For a vector field X,*

$$*(\text{div } X) = (\text{div } X)dV = d(i_X dV) = \mathcal{L}_X(dV). \tag{46.1}$$

In a coordinate system, we have

$$\text{div } X = \frac{1}{\sqrt{\det(g)}}\partial_i\Big(X^i\sqrt{\det(g)}\Big). \tag{46.2}$$

Proof. Fix a point $x \in M$, and let $\{e_i\}$ be an orthonormal basis of T_xM. In a small neighborhood of x, parallel translate this frame along radial geodesics. For such a frame, we have $\nabla_{e_i}e_j(x) = 0$. Such a frame is called an *adapted* moving frame field at x. Let $\{\omega^i\}$ denote the dual frame field. We have

$$\begin{aligned} \mathcal{L}_X(dV) &= (di_X + i_X d)dV = d(i_X dV) \\ &= \sum_i \omega^i \wedge \nabla_{e_i}\Big(i_X(\omega^1 \wedge \cdots \wedge \omega^n)\Big) \\ &= \sum_i \omega^i \wedge \nabla_{e_i}\Big((-1)^{j-1}\sum_{j=1}^{n} \omega^j(X)\omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^n\Big) \\ &= \sum_{i,j}(-1)^{j-1}e_i\big(\omega^j(X)\big)\omega^i \wedge \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^n \\ &= \sum_i \omega^i(\nabla_{e_i}X)dV \\ &= (\text{div } X)dV = *(\text{div } X). \end{aligned} \tag{46.3}$$

Applying $*$ to this formula, we have

$$
\begin{aligned}
\text{div } X &= *d(i_X dV) \\
&= *d(i_X \sqrt{\det(g)} dx^1 \wedge \dots \wedge dx^n) \\
&= *d\Big(\sum_{j=1}^{n} (-1)^{j-1} X^j \sqrt{\det(g)} dx^1 \wedge \dots \hat{dx^j} \dots \wedge dx^n \Big) \\
&= *\Big(\partial_i (X^i \sqrt{\det(g)}) dx^1 \wedge \dots \wedge dx^n \Big) \\
&= *\Big(\partial_i (X^i \sqrt{\det(g)}) \frac{1}{\sqrt{\det(g)}} dV \Big) \\
&= \frac{1}{\sqrt{\det(g)}} \partial_i \Big(X^i \sqrt{\det(g)} \Big).
\end{aligned}
\tag{46.4}
$$

□

Corollary 46.2. *Let (M, g) be compact, orientable and with boundary ∂M. If X is a vector field of class C^1, and f is a function of class C^1, then*

$$
\int_M (\text{div } X) f dV = -\int_M df(X) dV + \int_{\partial M} \langle X, \hat{n} \rangle f dS, \tag{46.5}
$$

where $\hat{n}$ is the outer unit normal. If ω is a one-form of class C^1, then

$$
\int_M (\text{div } \omega) f dV = -\int_M \langle \omega, df \rangle dV + \int_{\partial M} \omega(\hat{n}) f dS. \tag{46.6}
$$

If u and v are functions of class C^2, then

$$
\int_M (\Delta u) v dV = -\int_M \langle \nabla u, \nabla v \rangle dV + \int_{\partial M} \langle \nabla u, \hat{n} \rangle v dS, \tag{46.7}
$$

and

$$
\int_M (\Delta u) v dV - \int_M u (\Delta v) dV = \int_{\partial M} \langle \nabla u, \hat{n} \rangle v dS - \int_{\partial M} v \langle \nabla u, \hat{n} \rangle dS. \tag{46.8}
$$

Consequently, if M is compact without boundary, then Δ is a self-adjoint operator.

Proof. We compute

$$
d(f i_X dV) = df \wedge (i_X dV) + f d(i_X dV). \tag{46.9}
$$

Using Stokes' Theorem and Proposition 46.1,

$$
\int_M f (\text{div } X) dV + \int_M df \wedge (i_X dV) = \int_{\partial M} f i_X dV. \tag{46.10}
$$

A computation like above shows that

$$
df \wedge (i_X dV) = df(X) dV. \tag{46.11}
$$

Next, on ∂M, decompose $X = X_T + X_N$ into its tangential and normal components. Then

$$\begin{aligned} i_X dV &= dV(X_T + X_N, \dots) \\ &= dV(\langle X, \hat{n}\rangle \hat{n}, \dots) \\ &= \langle X, \hat{n}\rangle dS, \end{aligned} \tag{46.12}$$

since the volume element on the boundary is $dS = i_{\hat{n}} dV$. The proof for 1-forms is the dual argument. Green's first formula (46.7) follows using $\Delta u = \mathrm{div}(\nabla u)$, and Green's second formula (46.8) follows from (46.7). $\square$

We point out the following. The formula (46.5), gives a nice way to derive the coordinate formula for the divergence as follows. Fix a coordinate system, and assume that X and f have compact support in these coordinates. Then

$$\begin{aligned} \int_M f(\mathrm{div}\ X) dV &= -\int_M df(X) dV \\ &= -\int_M \partial_i f dx^i (X^j \partial_j) \sqrt{\det(g)} dx \\ &= -\int_{\mathbb{R}^n} \partial_i f X^i \sqrt{\det(g)} dx \\ &= \int_{\mathbb{R}^n} f \partial_i \Big(X^i \sqrt{\det(g)} \Big) dx \\ &= \int_M f \frac{1}{\sqrt{\det(g)}} \partial_i \Big(X^i \sqrt{\det(g)} \Big) dV. \end{aligned} \tag{46.13}$$

Since this is true for any f, we must have

$$\mathrm{div}\ X = \frac{1}{\sqrt{\det(g)}} \partial_i \Big(X^i \sqrt{\det(g)} \Big). \tag{46.14}$$

This formula yields a slightly non-obvious formula for the contraction of the Christoffel symbols on the upper and one lower index.

Corollary 46.3. *The Christoffel symbols satisfy*

$$\sum_{i=1}^n \Gamma^i_{ij} = \frac{1}{2} g^{pq} \partial_j g_{pq} = \frac{1}{\sqrt{\det(g)}} \partial_j \sqrt{\det(g)} = \frac{1}{2} \partial_j \log \det(g). \tag{46.15}$$

Proof. The first equality follows easily from the coordinate formula for the Christoffel symbols (23.6). Next, on one hand, we have the formula

$$\mathrm{div}\ X = \nabla_i X^i = \partial_i X^i + \Gamma^i_{ip} X^p. \tag{46.16}$$

On the other hand we have

$$\mathrm{div}\ X = \partial_i X^i + \frac{1}{\sqrt{\det(g)}} \Big(\partial_p \sqrt{\det(g)} \Big) X^p. \tag{46.17}$$

Since this is true for an arbitrary vector field X, the coefficient of X^p must be the same. $\square$

Exercise 46.4. *Prove the middle equality in* (46.15) *directly. (Hint: use Jacobi's formula for the derivative of a determinant.)*

This also yields a useful formula for the Laplacian of a function.

Corollary 46.5. *In a coordinate system, the Laplacian of a function is given by*

$$\Delta f = \frac{1}{\sqrt{\det(g)}} \partial_i \Big(g^{ij} \partial_j f \sqrt{\det(g)} \Big). \tag{46.18}$$

Proof. Since $\Delta f = \mathrm{div}(\nabla f)$, just let $X^i = g^{ij}\partial_i f$ in (46.14). $\square$

46.2 Integration and adjoints

We begin with an integration-by-parts formula for (r,s)-tensor fields.

Proposition 46.6. *Let (M,g) be compact and without boundary, T be an (r,s)-tensor field, and S be a $(r,s+1)$ tensor field. Then*

$$\int_M \langle \nabla T, S \rangle dV = - \int_M \langle T, \mathrm{div}\ S \rangle dV. \tag{46.19}$$

Proof. Let us view the inner product $\langle T, S\rangle$ as a 1-form ω. In coordinates

$$\omega = \langle T, S \rangle = T^{j_1 \dots j_s}_{i_1 \dots i_r} S^{i_1 \dots i_r}_{j j_1 \dots j_s} dx^j. \tag{46.20}$$

Note the indices on T are reversed, since we are taking an inner product. Taking the divergence, since g is parallel we compute

$$\begin{aligned} \mathrm{div}\ (\langle T, S \rangle) &= \nabla^j (T^{j_1 \dots j_s}_{i_1 \dots i_r} S^{i_1 \dots i_r}_{j j_1 \dots j_s}) \\ &= \nabla^j (T^{j_1 \dots j_s}_{i_1 \dots i_r}) S^{i_1 \dots i_r}_{j j_1 \dots j_s} + T^{j_1 \dots j_s}_{i_1 \dots i_r} \nabla^j S^{i_1 \dots i_r}_{j j_1 \dots j_s} \\ &= \langle \nabla T, S \rangle + \langle T, \mathrm{div}\ S \rangle. \end{aligned} \tag{46.21}$$

The result then follows from Proposition 45.1 and Corollary 46.2. $\square$

Remark 46.7. *Some authors define $\nabla^* = -div$. Then*

$$\int_M \langle \nabla T, S \rangle dV = \int_M \langle T, \nabla^* S \rangle dV, \tag{46.22}$$

so that ∇^* is the formal L^2-adjoint of ∇, for example [Pet06].

Recall the adjoint of d, $\delta : \Omega^p \to \Omega^{p-1}$, is defined by

$$\delta \omega = (-1)^{n(p+1)+1} * d * \omega. \tag{46.23}$$

Proposition 46.8. *For (M,g) compact without boundary, the operator δ is the L^2 adjoint of d,*

$$\int_M \langle \delta \alpha, \beta \rangle dV = \int_M \langle \alpha, d\beta \rangle dV, \tag{46.24}$$

where $\alpha \in \Omega^p(M)$, and $\beta \in \Omega^{p-1}(M)$.

Proof. We compute

$$
\begin{aligned}
\int_M \langle \alpha, d\beta \rangle dV &= \int_M d\beta \wedge *\alpha \\
&= \int_M \Big(d(\beta \wedge *\alpha) + (-1)^p \beta \wedge d * \alpha \Big) \\
&= \int_M (-1)^{p+(n-p+1)(p-1)} \beta \wedge * * d * \alpha \\
&= \int_M \langle \beta, (-1)^{n(p+1)+1} * d * \alpha \rangle dV \\
&= \int_M \langle \beta, \delta\alpha \rangle dV.
\end{aligned}
\tag{46.25}
$$

□

We note the following. If $\alpha \in \Omega^p(T^*M)$, then we can define the divergence operator $\mathrm{div} : \Omega^p(M) \to \Omega^{p-1}(M)$ as follows.

$$
\mathrm{div}\ \alpha = \sum_{j=1}^n i_{e_j} \nabla_{e_j} \alpha. \tag{46.26}
$$

This is a well-defined global operator $\mathrm{div} : \Omega^p(M) \to \Omega^{p-1}(M)$, and agrees with our previous definition of div under our identification of p-forms with alternating tensors. To see this, fix a point $x \in M$, and let $\{e_i\}$ and $\{\omega^i\}$ denote an adapted orthonormal frame field at x. Recall that p-form is written as

$$
\alpha = \frac{1}{p!} \sum_{1 \leq i_1, i_2, \ldots, i_p \leq n} \alpha_{i_1 \ldots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}. \tag{46.27}
$$

So then (46.26), evaluated at x, is

$$
\begin{aligned}
\mathrm{div}\ \alpha &= \frac{1}{p!} \sum_{1 \leq i_1, i_2, \ldots, i_p \leq n} \sum_{j=1}^n e_j(\alpha_{i_1 \ldots i_p}) i_{e_j}(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) \\
&= \frac{1}{p!} \sum_{1 \leq i_1, i_2, \ldots, i_p \leq n} \sum_{j=1}^n e_j(\alpha_{i_1 \ldots i_p}) \sum_{k=1}^p (-1)^{k-1} \delta_j^{i_k} (\omega^{i_1} \wedge \cdots \wedge \widehat{\omega^{i_k}} \wedge \cdots \wedge \omega^{i_p}) \\
&= \frac{1}{p!} \sum_{1 \leq i_1, i_2, \ldots, i_p \leq n} \sum_{k=1}^p e_{i_k}(\alpha_{i_1 \ldots i_p}) (-1)^{k-1} (\omega^{i_1} \wedge \cdots \wedge \widehat{\omega^{i_k}} \wedge \cdots \wedge \omega^{i_p}) \\
&= \frac{1}{(p-1)!} \sum_{1 \leq i_1, i_2, \ldots, i_{p-1} \leq n} \sum_{k=1}^n e_k(\alpha_{k i_1 \ldots i_{p-1}}) (\omega^{i_1} \wedge \cdots \wedge \omega^{i_{p-1}}).
\end{aligned}
\tag{46.28}
$$

So the components of div α at x are

$$
(\mathrm{div}\ \alpha)_{i_1 \ldots i_{p-1}} = \sum_{k=1}^n e_k(\alpha_{k i_1 \ldots i_{p-1}}). \tag{46.29}
$$

On the other hand, the alternating $(0,p)$-tensor corresponding to α is

$$\tilde{\alpha} = \sum_{1 \leq i_1, i_2, \dots, i_p \leq n} \alpha_{i_1 \dots i_p} \omega^{i_1} \otimes \cdots \otimes \omega^{i_p}, \tag{46.30}$$

and the definition of div $\tilde{\alpha}$ from (45.2), evaluated at x, is

$$(\mathrm{div}\ \tilde{\alpha})_{i_1 \dots i_{p-1}} = \sum_{j=1}^{n} \nabla_{e_j} \alpha_{j i_1 \dots i_{p-1}} = \sum_{j=1}^{n} e_j(\alpha_{j i_1 \dots i_{p-1}}). \tag{46.31}$$

Consequently, our definitions agree. The next proposition says that our divergence operator agrees with the Hodge δ operator, up to a sign, a fact which is not at all obvious.

Proposition 46.9. *On Ω^p, $\delta = -\mathrm{div}$.*

Proof. Let $\omega \in \Omega^p$. Choose locally defined dual ONB $\{e_i\}$ and $\{\omega^i\}$. We compute

$$\begin{aligned}
(\mathrm{div}\ \omega) &= \sum_j i_{e_j} \nabla_{e_j} \omega \\
&= \sum_j (-1)^{p(n-p)} \Big(i_{e_j} \big(* * (\nabla_{e_j} \omega) \big) \Big) \\
&= (-1)^{p(n-p)} \sum_j (-1)^{n-p} * (\omega^j \wedge * \nabla_{e_j} \omega) \\
&= (-1)^{(p+1)(n-p)} \sum_j * \big(\omega^j \wedge \nabla_{e_j} (*\omega) \big) \\
&= (-1)^{n(p+1)} (* d * \omega).
\end{aligned} \tag{46.32}$$

□

An alternative proof of the proposition is as follows. Assume that $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^p(M)$ are supported in a coordinate system. Then using Proposition 46.8, formula (45.19), and Proposition 46.6, we have

$$\begin{aligned}
\int_M \langle \alpha, \delta\beta \rangle dV &= \int_M \langle d\alpha, \beta \rangle dV \\
&= \frac{1}{p!} \int_M (d\alpha)_{i_0 \dots i_{p-1}} \beta^{i_0 \dots i_{p-1}} dV \\
&= \frac{1}{p!} \int_M \sum_{j=0}^{p-1} (-1)^j \nabla_{i_j} \alpha_{i_0 \dots \hat{i_j} \dots i_{p-1}} \beta^{i_0 \dots i_{p-1}} dV \\
&= \frac{1}{(p-1)!} \int_M \nabla_{i_0} \alpha_{i_1 \dots i_{p-1}} \beta^{i_0 \dots i_{p-1}} dV \\
&= \frac{1}{(p-1)!} \int_M \langle \nabla\alpha, \beta \rangle_{ten} dV \\
&= \frac{1}{(p-1)!} \int_M \langle \alpha, -\mathrm{div}\ \beta \rangle_{ten} dV \\
&= \int_M \langle \alpha, -\mathrm{div}\ \beta \rangle dV.
\end{aligned} \tag{46.33}$$

where $\langle \cdot, \cdot \rangle_{ten}$ denotes the tensor inner product. Thus both δ and $-\text{div}$ are L^2 adjoints of d. The result then follows from uniqueness of the L^2 adjoint.

Exercise 46.10. *Try and prove Proposition 46.9 directly in coordinates, using the coordinate formulas* (45.2), (45.16), *and* (45.19).

47 Lecture 47

47.1 The Hodge Laplacian and the rough Laplacian

For T an (r, s)-tensor, the rough Laplacian is an (r, s) tensor given by

$$\Delta T = \text{div } \nabla T, \tag{47.1}$$

and is given in coordinates by

$$(\Delta T)^{i_1 \dots i_r}_{j_1 \dots j_s} = g^{ij} \nabla_i \nabla_j T^{i_1 \dots i_r}_{j_1 \dots j_s}. \tag{47.2}$$

If $\omega \in \Omega^p(M)$, the rough Laplacian is defined by

$$\Delta \omega = \sum_{j=1}^{n} \nabla_{e_j} \nabla_{e_j} \omega, \tag{47.3}$$

and this agrees with the rough Laplacian above under our identification of p-forms with alternating tensors.

For $\omega \in \Omega^p$ we define the *Hodge laplacian* $\Delta_H : \Omega^p \to \Omega^p$ by

$$\Delta_H \omega = (d\delta + \delta d)\omega. \tag{47.4}$$

We say a p-form is *harmonic* if it is in the kernel of Δ_H.

Proposition 47.1. *If M is compact without boundary, then for T and S both (r, s)-tensors,*

$$\int_M \langle \Delta T, S \rangle dV = -\int_M \langle \nabla T, \nabla S \rangle dV = \int_M \langle T, \Delta S \rangle dV. \tag{47.5}$$

For $\alpha, \beta \in \Omega^p$,

$$\int_M \langle \Delta_H \alpha, \beta \rangle dV = \int_M \langle d\alpha, d\beta \rangle dV + \int_M \langle \delta\alpha, \delta\beta \rangle dV = \int_M \langle \alpha, \Delta_H \beta \rangle dV. \tag{47.6}$$

Consequently, a p-form is harmonic ($\Delta_H \alpha = 0$) if and only if it is both closed and co-closed ($d\alpha = 0$ and $\delta\alpha = 0$).

Proof. Formula (47.5) is an application of (47.1) and Proposition 46.6. For the second, from Proposition 46.8,

$$\begin{aligned}\int_M \langle \Delta_H \alpha, \beta\rangle dV &= \int_M \langle (d\delta + \delta d)\alpha, \beta\rangle dV \\ &= \int_M \langle d\delta\alpha, \beta\rangle dV + \int_M \langle \delta d\alpha, \beta\rangle dV \\ &= \int_M \langle \delta\alpha, \delta\beta\rangle dV + \int_M \langle d\alpha, d\beta\rangle dV \\ &= \int_M \langle \alpha, d\delta\beta\rangle dV + \int_M \langle \alpha, \delta d\beta\rangle dV \\ &= \int_M \langle \alpha, \Delta_H \beta\rangle dV.\end{aligned} \tag{47.7}$$

The last statement follows easily by letting $\alpha = \beta$ in (47.6). □

Note that Δ maps alternating $(0, p)$ tensors to alternating $(0, p)$ tensors, therefore it induces a map $\Delta : \Omega^p \to \Omega^p$ (note that on [Poo81, page 159] it is stated that the rough Laplacian of an r-form is in general not an r-form, but this seems to be incorrect). On p-forms, Δ and Δ_H are two self-adjoint linear second order differential operators. How are they related? Next, we will look at the simplest case of 1-forms.

47.2 1-forms

Consider the case of 1-forms.

Proposition 47.2. *Let $\omega \in \Omega^1(M)$.*

$$\Delta\omega = -\Delta_H(\omega) + Ric(\sharp\omega, \cdot), \tag{47.8}$$

and

$$g(\Delta_H \omega, \omega) = \frac{1}{2}\Delta_H |\omega|^2 + |\nabla\omega|^2 + Ric(\sharp\omega, \sharp\omega). \tag{47.9}$$

Proof. We compute

$$\begin{aligned}(\delta d\omega)_j &= -g^{pq}\nabla_p (d\omega)_{qj} \\ &= -g^{pq}\nabla_p(\nabla_q \omega_j - \nabla_j \omega_q) \\ &= -g^{pq}\nabla_p\nabla_q\omega_j + g^{pq}\nabla_p\nabla_j\omega_q \\ &= -\Delta\omega_j + g^{pq}\nabla_p\nabla_j\omega_q.\end{aligned} \tag{47.10}$$

Next,

$$\begin{aligned}(d\delta\omega)_j &= \big(d\big(-g^{pq}\nabla_p\omega_q\big)\big)_j \\ &= -\nabla_j(g^{pq}\nabla_p\omega_q) \\ &= -g^{pq}\nabla_j\nabla_p\omega_q.\end{aligned} \tag{47.11}$$

Adding these together,

$$
\begin{aligned}
(\Delta_H \omega)_j &= -\Delta \omega_j + g^{pq}(\nabla_p \nabla_j - \nabla_j \nabla_p)\omega_q \\
&= -\Delta \omega_j + g^{pq}(-R_{pjq}{}^{i}\omega_i) \\
&= -\Delta \omega_j - g^{pq}(R_{pjiq}\omega^i) \\
&= -\Delta \omega_j + R_{ij}\omega^i,
\end{aligned}
\tag{47.12}
$$

recalling that our convention is to lower the upper index of the $(1,3)$ curvature tensor to the third position. This proves (47.8).

Next, we claim that

$$
-g(\Delta\omega, \omega) = \frac{1}{2}\Delta_H |\omega|^2 + |\nabla \omega|^2. \tag{47.13}
$$

To see this, compute

$$
\begin{aligned}
\Delta_H |\omega|^2 &= -g^{kl}\nabla_k \nabla_l (g^{ij}\omega_i \omega_j) \\
&= -g^{kl} g^{ij} \nabla_k \big((\nabla_l \omega_i)\omega_j + \omega_i \nabla_l \omega_j \big) \\
&= -g^{kl} g^{ij} \big((\nabla_k \nabla_l \omega_i)\omega_j + \nabla_l \omega_i \nabla_k \omega_j + \nabla_k \omega_i \nabla_l \omega_j + \omega_i \nabla_k \nabla_l \omega_j \big) \\
&= -2g(\Delta\omega, \omega) - 2|\nabla\omega|^2.
\end{aligned}
\tag{47.14}
$$

Pairing (47.8) with ω and using (47.13) proves (47.9). □

Theorem 47.3 (Bochner)**.** *If (M, g) is compact and has positive semi-definite Ricci curvature, then any harmonic 1-form is parallel. In this case $b_1(M) \leq n$. If, in addition, Ric is positive definite at some point, then any harmonic 1-form is trivial. In this case $b_1(M) = 0$.*

Proof. If ω satisfies $\Delta_H \omega = 0$, then integrating (47.9) and using the divergence theorem yields

$$
0 = \int_M |\nabla \omega|^2 dV + \int_M Ric(\sharp\omega, \sharp\omega) dV. \tag{47.15}
$$

This clearly implies that $\nabla \omega \equiv 0$, thus ω is parallel, so is determined everywhere by its value at any point. If in addition Ric is strictly positive somewhere, ω must vanish identically. The conclusion on the first Betti number follows from the Hodge Theorem. □

47.3 Killing vector fields

Proposition 47.4. *For a vector field X,*

$$
g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = \mathcal{L}_X g(Y, Z), \tag{47.16}
$$

and

$$
\mathrm{tr}(\mathcal{L}_X g) = 2 \ \mathrm{div}(X). \tag{47.17}
$$

In coordinates

$$
\big(\mathcal{L}_X g\big)_{ij} = g_{jk} \nabla_i X^k + g_{ik} \nabla_j X^k. \tag{47.18}
$$

Proof. Recalling the formula for the Lie derivative of a $(0,2)$ tensor,

$$\begin{aligned}
\mathcal{L}_X g(Y,Z) &= X(g(Y,Z)) - g([X,Y],Z) - g(Y,[X,Z]) \\
&= X(g(Y,Z)) - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) \\
&= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) + X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\
&= g(\nabla_Y X, Z) + g(Y, \nabla_Z X).
\end{aligned} \tag{47.19}$$

noting that the 3 latter terms vanish since g is parallel, which proves (47.16). ☐

A vector field X is a *Killing field* if the 1-parameter group of local diffeomorphisms generated by X consists of local isometries of g.

Proposition 47.5. *A vector field is a Killing field if and only if $\mathcal{L}_X g = 0$.*

Proof. Let ϕ_t denote the 1-parameter group of X,

$$\begin{aligned}
\frac{d}{ds}(\phi_s^* g)\Big|_t &= \frac{d}{ds}(\phi_{s+t}^* g)\Big|_0 \\
&= \phi_t^* \frac{d}{ds}(\phi_s^* g)\Big|_0 \\
&= \phi_t^* \mathcal{L}_X g.
\end{aligned} \tag{47.20}$$

It follows that $\phi_t^* g = g$ for every t if and only if $\mathcal{L}_X g = 0$. ☐

Note that, in particular, a Killing field is divergence free. We next have a formula due to Bochner

Proposition 47.6. *Let $\omega \in \Omega^1(M)$.*

$$\mathrm{div}(\mathcal{L}_X g) = \flat(\Delta X) + d(\mathrm{div}\ X) + Ric(X, \cdot), \tag{47.21}$$

and

$$\mathrm{div}(\mathcal{L}_X g)(X) = \frac{1}{2}\Delta |X|^2 - |\nabla X|^2 + X(\mathrm{div}\ X) + Ric(X,X). \tag{47.22}$$

Proof. We compute

$$\begin{aligned}
(\mathrm{div}(\mathcal{L}_X g))_j &= g^{pq}\nabla_p (\mathcal{L}_X g)_{qj} \\
&= g^{pq}\nabla_p \big(g_{jk}\nabla_q X^k + g_{qk}\nabla_j X^k\big) \\
&= g^{pq} g_{jk} \nabla_p \nabla_q X^k + g^{pq} g_{qk} \nabla_p \nabla_j X^k \\
&= g_{jk}(\Delta X)^l + \delta_k^p (\nabla_j \nabla_p X^k + R_{pjl}{}^k X^l) \\
&= g_{jk}(\Delta X)^l + \nabla_j (\mathrm{div}\ X) + R_{jl} X^l,
\end{aligned} \tag{47.23}$$

which proves (47.21).

Next, we claim that

$$g(\Delta X, X) = \frac{1}{2}\Delta|X|^2 - |\nabla X|^2. \tag{47.24}$$

To see this, compute

$$\begin{aligned} \Delta|X|^2 &= g^{kl}\nabla_k\nabla_l(g_{ij}X^iX^j) \\ &= g^{kl}g_{ij}\nabla_k\big((\nabla_l X^i)X^j + X^i\nabla_l X^j\big) \\ &= g^{kl}g^{ij}\big((\nabla_k\nabla_l X^i)X^j + \nabla_l X^i\nabla_k X^j + \nabla_k X^i\nabla_l X^j + X^i\nabla_k\nabla_l X^j\big) \\ &= 2g(\Delta X, X) + 2|\nabla X|^2. \end{aligned} \tag{47.25}$$

Pairing (47.21) with X and using (47.24) proves (47.22).

□

We next have

Corollary 47.7. *Let (M, g) be compact, and X be a Killing field. If Ric is negative semidefinite, then X is parallel and $Ric(X, X) = 0$. If, in addition, the Ricci tensor is negative definite at some point, then $X \equiv 0$.*

Proof. If X is Killing, then by (47.17), div $X = 0$. Integrating (47.22) over M yields

$$\int_M Ric(X, X)dV - \int_M |\nabla X|^2 dV = 0. \tag{47.26}$$

If Ric is negative semidefinite, then clearly $\nabla X = 0$, so X is parallel. It also folows from this that if Ric is negative definite somewhere, then $X \equiv 0$. □

Corollary 47.8. *Let (M, g) be compact, and let $Iso(M, g)$ denote the isometry group of (M, g). If (M, g) has negative semi-definite Ricci tensor, then $dim(Iso(M, g)) \leq n$. If, in addition, the Ricci tensor is negative definite at some point, then $Iso(M, g)$ is finite.*

Proof. We recall that the isometry group of a compact Riemannian manifold is a compact Lie group with Lie algebra the space of Killing vector fields with the Lie bracket. If the isometry group is not finite, then there exists a non-trivial 1-parameter group $\{\phi_t\}$ of isometries. By Proposition 47.5, this generates a non-trivial Killing vector field. From Corollary 47.7, X is parallel and $Ric(X, X) = 0$. Since X is parallel, it is determined by its value at a single point, so the dimension of the space of Killing vector fields is less than n, which implies that $dim(\mathrm{Iso}(M, g)) \leq n$. If Ric is negative definite at some point x, then $X \equiv 0$ so there are no non-trivial Killing fields. Consequently, there are no nontrival 1-parameter groups of isometries, so $\mathrm{Iso}(M, g)$ must be finite since it is a compact Lie group. □

Note that an n-dimensional flat torus $S^1 \times \cdots \times S^1$ attains equality in the above inequality. Note also that by Gauss-Bonnet, any metric on a surface of genus $g \geq 2$ must have a point of negative curvature, so any non-positively curved metric on a surface of genus $g \geq 2$ must have finite isometry group.

48 Lecture 48

We next generalize this to p-forms.

Definition 48.1. For $\omega \in \Omega^p$, we define a $(0,p)$-tensor field ρ_ω by

$$\rho_\omega(X_1,\dots,X_p) = \sum_{i=1}^n \sum_{j=1}^p \Big(\mathcal{R}_{\Lambda^p}(e_i,X_j)\omega\Big)(X_1,\dots,X_{j-1},e_i,X_{j+1},\dots,X_p), \tag{48.1}$$

where $\{e_i\}$ is an ONB at $x \in M$.

Remark 48.2. Recall what this means. The Riemannian connection induces a metric connection in the bundle $\Lambda^p(T^*M)$. The curvature of this connection therefore satisfies

$$\mathcal{R}_{\Lambda^p} \in \Gamma\Big(\Lambda^2(T^*M)\otimes \mathfrak{so}(\Lambda^p(T^*M))\Big). \tag{48.2}$$

We leave it to the reader to show that (48.1) is well-defined.

The relation between the Laplacians is given by

Theorem 48.3. *Let $\omega \in \Omega^p$. Then*

$$\Delta_H \omega = -\Delta\omega + \rho_\omega. \tag{48.3}$$

We also have the formula

$$\langle \Delta_H\omega,\omega\rangle = \frac{1}{2}\Delta_H|\omega|^2 + |\nabla\omega|^2 + \langle\rho_\omega,\omega\rangle. \tag{48.4}$$

Proof. Take $\omega \in \Omega^p$, and vector fields $X, Y_1,\dots,Y_p$. We compute

$$\begin{aligned}(\nabla\omega - d\omega)(X,Y_1,\dots,Y_p) &= (\nabla_X\omega)(Y_1,\dots,Y_p) - d\omega(X,Y_1,\dots,Y_p) &(48.5)\\ &= \sum_{j=1}^p (\nabla_{Y_j}\omega)(Y_1,\dots,Y_{j-1},X,Y_{j+1},\dots,Y_p), &(48.6)\end{aligned}$$

using Proposition 45.6. Fix a point $x \in M$. Assume that $(\nabla Y_j)_x = 0$, by parallel translating the values of Y_j at x. Also take e_i to be an adapted moving frame at p. Using Proposition 46.9, we compute at x

$$\begin{aligned}(\text{div }\nabla\omega + \delta d\mu)(Y_1,\dots,Y_p) &= \text{div }(\nabla\omega - d\omega)(Y_1,\dots,Y_r)\\ &= \sum_{i=1}^n \Big(\nabla_{e_i}(\nabla\omega - d\omega)\Big)(e_i,Y_1,\dots,Y_p)\\ &= \sum_{i=1}^n \Big(e_i(\nabla\omega - d\omega)\Big)(e_i,Y_1,\dots,Y_p)\\ &= \sum_{i=1}^n\sum_{j=1}^p e_i\Big((\nabla_{Y_j}\omega)(Y_1,\dots,Y_{j-1},e_i,Y_{j+1},\dots,Y_p)\\ &= \sum_{i=1}^n\sum_{j=1}^p (\nabla_{e_i}\nabla_{Y_j}\omega)(Y_1,\dots,Y_{j-1},e_i,Y_{j+1},\dots,Y_p)\end{aligned} \tag{48.7}$$

We also have

$$
\begin{aligned}
d\delta\omega(Y_1,\dots,Y_p) &= \sum_{j=1}^{p}(-1)^{j+1}(\nabla_{Y_j}\delta\omega)(Y_1,\dots,\hat{Y}_j,\dots,Y_p)\\
&= \sum_{j=1}^{p}(-1)^{j}Y_j\left(\sum_{i=1}^{n}(\nabla_{e_i}\omega)(e_i,Y_1,\dots,\hat{Y}_j,\dots,Y_p)\right)\\
&= -\sum_{i=1}^{n}\sum_{j=1}^{p}(\nabla_{Y_j}\nabla_{e_i}\omega)(Y_1,\dots,Y_{j-1},e_i,Y_{j+1},\dots,Y_p).
\end{aligned}
\tag{48.8}
$$

The commutator $[e_i,Y_j](x)=0$, since $\nabla_{e_i}Y_j(x)=0$, and $\nabla_{Y_j}e_i(x)=0$, by our choice. Consequently,

$$
(\Delta_H\omega+\Delta\omega)(Y_1,\dots,Y_p) = (\Delta_H\omega+\mathrm{div}\ \nabla\omega)(Y_1,\dots,Y_p) = \rho_\omega(Y_1,\dots,Y_p). \tag{48.9}
$$

This proves (48.3). For (48.4), we compute at x

$$
\begin{aligned}
\mathrm{div}\ \nabla\omega(Y_1,\dots,Y_p) &= \sum_i \nabla_{e_i}(\nabla\omega)(e_1,Y_1,\dots,Y_p)\\
&= \sum_i e_i(\nabla_{e_i}\omega)(Y_1,\dots,Y_p)\\
&= \sum_i(\nabla_{e_i}\nabla_{e_i}\omega)(Y_1,\dots,Y_p).
\end{aligned}
\tag{48.10}
$$

Next, again at x,

$$
\begin{aligned}
\langle -\mathrm{div}\nabla\omega,\omega\rangle &= -\sum_i\langle\nabla_{e_i}\nabla_{e_i}\omega,\omega\rangle\\
&= -\sum_i e_i\left(\langle\nabla_{e_i}\omega,\omega\rangle - \langle\nabla_{e_i}\omega,\nabla_{e_i}\omega\rangle\right)\\
&= -\frac{1}{2}\sum_i(e_ie_i|\omega|^2)+|\nabla\omega|^2\\
&= \frac{1}{2}\Delta_H|\omega|^2+|\nabla\omega|^2.
\end{aligned}
\tag{48.11}
$$

□

Remark 48.4. The rough Laplacian is therefore “roughly” the Hodge Laplacian, up to curvature terms. Note also in (48.4), the norms are tensor norms, since the right hand side has the term $|\nabla\omega|^2$ and $\nabla\omega$ is not a differential form. We are using (1.17) to identify forms and alternating tensors.

49 Lecture 49

49.1 Manifolds with positive curvature operator

We begin with a general property of curvature in exterior bundles.

Proposition 49.1. *Let ∇ be a connection in a vector bundle $\pi : E \to M$. As before, extend ∇ to a connection in $\Lambda^p(E)$ by defining it on decomposable elements*

$$\nabla_X(s_1 \wedge \cdots \wedge s_p) = \sum_{i=1}^{p} s_1 \wedge \cdots \wedge \nabla_X s_i \wedge \cdots \wedge s_p. \tag{49.1}$$

For vector fields X, Y, $\mathcal{R}_{\Lambda^p(E)}(X,Y) \in End(\Lambda^p(E))$ acts as a derivation

$$\mathcal{R}_{\Lambda^p(E)}(X,Y)(s_1 \wedge \cdots \wedge s_p) = \sum_{i=1}^{p} s_1 \wedge \cdots \wedge \mathcal{R}_\nabla(X,Y)(s_i) \wedge \cdots \wedge s_p. \tag{49.2}$$

Proof. We prove for $p = 2$, the case of general p is left to the reader. Since this is a tensor equation, we may assume that $[X,Y] = 0$. We compute

$$\begin{aligned}
&\mathcal{R}_{\Lambda^2(E)}(X,Y)(s_1 \wedge s_2) = \nabla_X \nabla_Y (s_1 \wedge s_2) - \nabla_Y \nabla_X (s_1 \wedge s_2)\\
&= \nabla_X \Big((\nabla_Y s_1) \wedge s_2 + s_1 \wedge (\nabla_Y s_2) \Big) - \nabla_Y \Big((\nabla_X s_1) \wedge s_2 + s_1 \wedge (\nabla_X s_2) \Big)\\
&= (\nabla_X \nabla_Y) s_1 \wedge s_2 + \nabla_Y s_1 \wedge \nabla_X s_2 + \nabla_X s_1 \wedge \nabla_Y s_2 + s_1 \wedge (\nabla_X \nabla_Y) s_2\\
&- (\nabla_Y \nabla_X) s_1 \wedge s_2 - \nabla_X s_1 \wedge \nabla_Y s_2 - \nabla_Y s_1 \wedge \nabla_X s_2 - s_1 \wedge (\nabla_Y \nabla_X) s_2\\
&= \Big(\mathcal{R}_\nabla(X,Y)(s_1) \Big) \wedge s_2 + s_1 \wedge \Big(\mathcal{R}_\nabla(X,Y)(s_2) \Big).
\end{aligned} \tag{49.3}$$

□

We apply this to the bundle $E = \Lambda^p(T^*M)$. Recall for a 1-form ω,

$$\nabla_i \nabla_j \omega_l = \nabla_j \nabla_i \omega_l - R_{ijl}{}^k \omega_k. \tag{49.4}$$

In other words,

$$(\mathcal{R}(\partial_i, \partial_j)\omega)_l = -R_{ijl}{}^k \omega_k, \tag{49.5}$$

where the left hand side means the curvature of the connection in T^*M, but the right hand side is the Riemannian curvature tensor. For a p-form $\omega \in \Omega^p$, with components $\omega_{i_1 \dots i_p}$, Proposition 49.1 says that

$$\Big(\mathcal{R}_{\Lambda^p}(e_\alpha, e_\beta)\omega \Big)_{i_1 \dots i_p} = -\sum_{k=1}^{p} R_{\alpha\beta i_k}{}^l \omega_{i_1 \dots i_{k-1} l i_{k+1} \dots i_p}, \tag{49.6}$$

where the left hand side now means the curvature of the connection in $\Lambda^p(T^*M)$.

Next, we look at ρ_ω in coordinates. It is written

$$(\rho_\omega)_{i_i \dots i_p} = g^{\alpha l} \sum_{j=1}^{p} \big(\mathcal{R}_{\Lambda^p}(\partial_\alpha, \partial_{i_j})\omega \big)_{i_1 \dots i_{j-1} l i_{j+1} \dots i_p}. \tag{49.7}$$

Using (49.6), we may write ρ_ω as

$$\begin{aligned}(\rho_\omega)_{i_i\dots i_p} = -g^{\alpha l}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p} R_{\alpha i_j i_k}{}^{m}\omega_{i_1\dots i_{j-1}l i_{j+1}\dots i_{k-1}m i_{k+1}\dots i_p}\\ - g^{\alpha l}\sum_{j=1}^{p} R_{\alpha i_j l}{}^{m}\omega_{i_1\dots i_{j-1}m i_{j+1}\dots i_p}\end{aligned} \tag{49.8}$$

Let us rewrite the above formula in an orthonormal basis,

$$\begin{aligned}(\rho_\omega)_{i_i\dots i_p} = -\sum_{l,m=1}^{n}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p} R_{l i_j m i_k}\omega_{i_1\dots i_{j-1}l i_{j+1}\dots i_{k-1}m i_{k+1}\dots i_p}\\ + \sum_{m=1}^{n}\sum_{j=1}^{p} R_{i_j m}\omega_{i_1\dots i_{j-1}m i_{j+1}\dots i_p}.\end{aligned} \tag{49.9}$$

Using the algebraic Bianchi identity (40.18), this is

$$R_{l i_j m i_k} + R_{l m i_k i_j} + R_{l i_k i_j m} = 0, \tag{49.10}$$

which yields

$$R_{l i_j m i_k} - R_{m i_j l i_k} = R_{l m i_j i_k}. \tag{49.11}$$

Substituting into (49.9) and using skew-symmetry,

$$\begin{aligned}(\rho_\omega)_{i_i\dots i_p} &= -\frac{1}{2}\sum_{l,m=1}^{n}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p} (R_{l i_j m i_k} - R_{m i_j l i_k})\omega_{i_1\dots i_{j-1}l i_{j+1}\dots i_{k-1}m i_{k+1}\dots i_p}\\ &\quad + \sum_{m=1}^{m}\sum_{j=1}^{p} R_{i_j m}\omega_{i_1\dots i_{j-1}m i_{j+1}\dots i_p}\\ &= -\frac{1}{2}\sum_{l,m=1}^{n}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p} R_{l m i_j i_k}\omega_{i_1\dots i_{j-1}l i_{j+1}\dots i_{k-1}m i_{k+1}\dots i_p}\\ &\quad + \sum_{m=1}^{m}\sum_{j=1}^{p} R_{i_j m}\omega_{i_1\dots i_{j-1}m i_{j+1}\dots i_p}.\end{aligned} \tag{49.12}$$

Theorem 49.2. *If (M^n, g) is closed and has non-negative curvature operator, then any harmonic form is parallel. In this case, $b_1(M) \leq \binom{n}{k}$. If in addition, the curvature operator is positive definite at some point, then any harmonic p-form is trivial for $p = 1 \dots n-1$. In this case, $b_p(M) = 0$ for $p = 1 \dots n-1$.*

Proof. Let ω be a harmonic p-form. Integrating the Weitzenböck formula (48.4), we obtain

$$0 = \int_M |\nabla\omega|^2 dV + \int_M \langle\rho_\omega, \omega\rangle dV. \tag{49.13}$$

It turns out the the second term is positive if the manifold has positive curvature operator [Poo81, Chapter 4], [Pet06, Chapter 7]. Thus $|\nabla\omega| = 0$ everywhere, so ω is parallel. A parallel form is determined by its value at a single point, so using the Hodge Theorem, we obtain the first Betti number estimate. If the curvature operator is positive definite at some point, then we see that ω must vanish at that point, which implies the second Betti number estimate. Note this only works for $p = 1 \dots n-1$, since ρ_ω is zero in these cases. □

This says that all of the real cohomology of a manifold with positive curvature operator vanishes except for H^n and H^0. We say that M is a rational homology sphere (which necessarily has $\chi(M) = 2$). If M is simply-connected and has positive curvature operator, then is M diffeomorphic to a sphere? In dimension 3 this was answered affirmatively by Hamilton in [**?**]. Hamilton also proved the answer is yes in dimension 4 [**?**]. Very recently, Böhm and Wilking have shown that the answer is yes in all dimensions [**?**]. The technique is using the Ricci flow, which we will discuss shortly.

We also mention that recently, Brendle and Schoen have shown that manifolds with 1/4-pinched curvature are diffeomorphic to space forms, again using the Ricci flow. If time permits, we will also discuss this later [**?**].

Remark 49.3. On 2-forms, the Weitzenböck formula is

$$(\Delta_H\omega)_{ij} = -(\Delta w)_{ij} - \sum_{l,m} R_{lmij}\omega_{lm} + \sum_m R_{im}\omega_{mj} + \sum_m R_{jm}\omega_{im}. \tag{49.14}$$

Through a careful analysis of the curvature terms, M. Berger was able to prove a vanishing theorem for $H^2(M,\mathbb{R})$ provided that the sectional curvature is pinched between 1 and $2(n-1)/(8n-5)$ [**?**].

50 Lecture 50

References

[Bes87] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987.

[Lee97] John M. Lee, *Riemannian manifolds*, Graduate Texts in Mathematics, vol. 176, Springer-Verlag, New York, 1997, An introduction to curvature.

[Pet06] Peter Petersen, *Riemannian geometry*, second ed., Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006.

[Poo81] Walter A. Poor, *Differential geometric structures*, McGraw-Hill Book Co., New York, 1981.

[Spi79] Michael Spivak, *A comprehensive introduction to differential geometry. Vol. I*, second ed., Publish or Perish, Inc., Wilmington, Del., 1979. MR 532830

[War83] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York, 1983, Corrected reprint of the 1971 edition.

Department of Mathematics, University of California, Irvine, CA 92697
E-mail Address: jviaclov@uci.edu