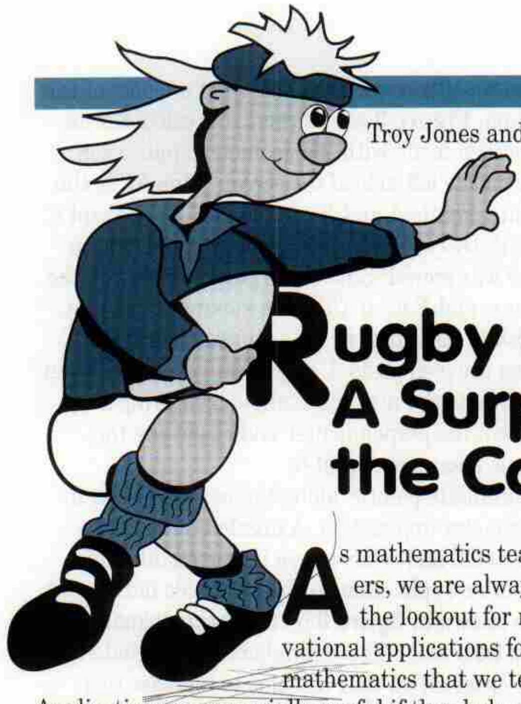

Troy Jones and Steven Jackson

Rugby and Mathematics: A Surprising Link among Geometry, the Conics, and Calculus

As mathematics teachers, we are always on the lookout for motivational applications for the mathematics that we teach. Applications are especially useful if they help students make connections among the various topics of mathematics and can be used by students at several different levels. While thumbing through *The Penguin Dictionary of Curious and Interesting Geometry,* by David Wells (1991), we ran across an entry that led to a very interesting discovery.

German mathematician Johannes Regiomontanus (1436–1476) posed the following question (Wells 1991, pp. 2–3): From what position along a horizontal line can a statue such as the one shown in **figure 1** best be viewed? If the spectator is too close, the statue will appear heavily foreshortened, thus distorting its size. If the spectator is too far away, it will simply be too small to see. An optimal distance for viewing the statue must exist.

A related question might be, From what location can we best view the Statue of Liberty—or any object above eye level? The answer to this question involves finding the location that maximizes the viewing angle and therefore makes the object appear at maximum size.

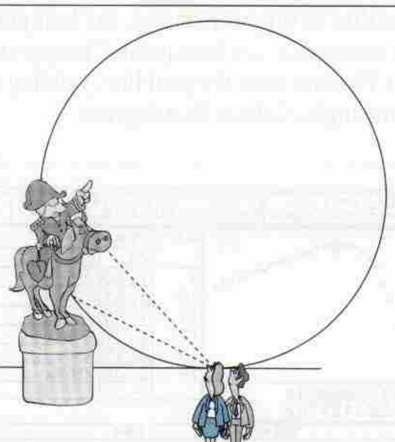

Fig. 1
Determining the ideal position for viewing a statue

Optimization problems have traditionally been reserved until students have acquired calculus skills. Such computer geometry software as The Geometer's Sketchpad and Cabri Geometry give geometry and algebra students the tools to tackle this type of optimization problem in a meaningful way. This early introduction of maximum-minimum situations can also help students experience more success with this important concept in calculus.

THE RUGBY PROBLEM

The Penguin Dictionary of Curious and Interesting Geometry (Wells 1991) mentions the game of rugby as another application of this maximization problem. Although rugby is unfamiliar to most students in the United States, it is popular as an amateur and professional sport in England, France, Ireland, Scotland, Canada, Australia, and New Zealand, among other countries. It was derived around 1823 from intramural soccer games at the Rugby School in England. Although some rugby is played in the United States, almost all U.S. high schools offer rugby's descendant, American football.

An optimal distance for viewing the Statue of Liberty must exist

Rugby and American football are similar in many ways. They use a ball and playing field approximately the same size and shape, and teams in both games attempt to run or pass a ball over the goal line. Players may also kick the ball through the goalposts for points. Unlike in American football, however, rugby players wear little protective equipment; like in soccer, the game continues without the interruption of downs.

A team that succeeds in moving the ball past the goal line must touch the ball down behind the goal

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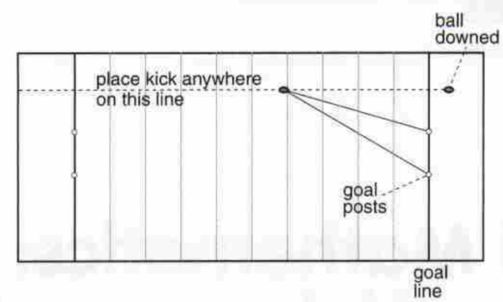

Fig. 2
Kicking a conversion in rugby

A kicker would want to find the location that maximizes the angle between the uprights

line to score a try, which is worth five points. After scoring each try, a player on the scoring team may attempt to convert the try by placekicking the ball through the goalposts. This placekick conversion, which is worth two points, may be kicked from anywhere on a line perpendicular to the goal line and through the spot where the ball was downed past the goal line, as shown in **figure 2.**

To give the kicker an advantage on the conversion, a player crossing the goal line often attempts to run toward the center of the field before downing the ball. If the ball is downed between the uprights, the kicker has only to back up far enough to be able to kick the ball through the uprights. But if the ball is downed outside the uprights, does a best place exist from which to kick the conversion?

The answer, perhaps surprisingly, is yes. Notwithstanding the player's kicking range, a kicker would want to find the location that maximizes the angle between the uprights, thus giving the greatest room for error to the left and right.

This article presents a series of explorations that locate an optimal place for kicking the ball to maximize this angle at the goalposts. These investigations begin with a data-collection activity that uses the interactive geometry software of the TI-92 calculator. The second investigation locates this ideal kicking location through geometric constructions using computer technology and the geometric mean. The third investigation generalizes the set of solutions by using analytic geometry to evaluate the trace of locus points. The final investigation uses calculus to verify this common solution as the ideal position from which to kick the goal.

Through these varied investigations, this single problem can be approached in many of our secondary mathematics courses, including elementary algebra, geometry, intermediate algebra, precalculus, and calculus. **Appendix A** includes a sample student activity that is appropriate for the elementary algebra level.

AN INITIAL EMPIRICAL APPROACH

With the TI-92 calculator, or other interactive geometry software, we can construct a model of this situation. **Figure 3** shows the TI-92 calculator in split-screen mode with the geometry application active on the left side of the screen. Line AB is the goal line. Points A and B are the goalposts. Point C is an arbitrary location along the goal line where the try was scored. Line CD is perpendicular to the goal line, and E is on $\overleftrightarrow{CD}$. The kicking angle that we wish to maximize is α, the angle from E subtending the goal posts. Using the angle tool, we can approximate the maximum angle by moving E up and down the perpendicular and observing the dynamic measurement of α.

We animate point E along the perpendicular in the geometry application. A number of measurements of $\overline{CE}$ and α have been imported into the data-editor application on the right side of the split screen shown in **figure 3** by using the Animate and Collect Data tools. These numbers substantiate the visual observation that α seems to increase to a maximum and then decrease again as E moves away from C.

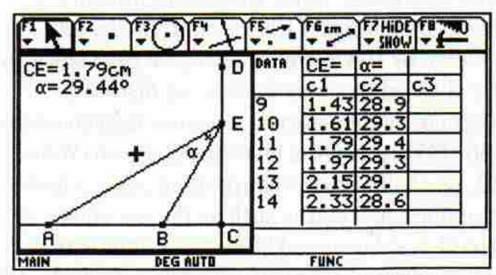

Fig. 3
Interactive geometry and data points in table generated by sliding point E along perpendicular to goal line

Treating these values as ordered pairs in the Data/Matrix editor application, then plotting them, as shown in **figure 4,** gives yet further evidence that a maximum kicking angle exists. This analysis indicates that in this scale model, the best place to kick the conversion is where point E is approximately 1.79 units from the goal line, yielding a maximum angle of about 29.4 degrees.

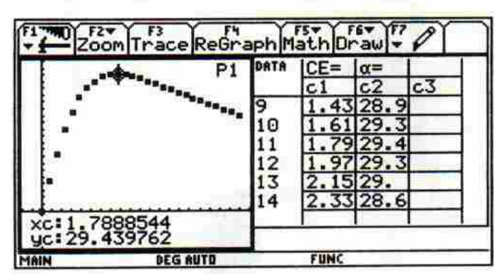

Fig. 4
Data plot and data table showing ideal kicking location

CONSTRUCTING AN OPTIMAL POINT USING GEOMETRY

A geometric solution to this problem also exists. In his book, Wells (1991) mentions that the maximum viewing angle is found when the circle passing through three points—the top of the statue, the bottom of the statue, and the spectator's eye—is also tangent with the horizontal line passing through the spectator's eye. See **figure 1.** A proof of this conjecture is found in **appendix B.**

We can use methods of elementary geometry to construct the point of tangency. We need to use three theorems from geometry, the first of which is the tangent-secant theorem. If a tangent segment of a circle and a secant segment meet at an external point, as shown in **figure 5,** then the length of the tangent segment EC is the geometric mean of the entire length of the secant segment AC and its external part BC. In other words,

(1) $$(EC)^2 = (AC)(BC).$$

The geometric mean EC is the product of two numbers that correspond to the lengths of overlapping segments AC and BC.

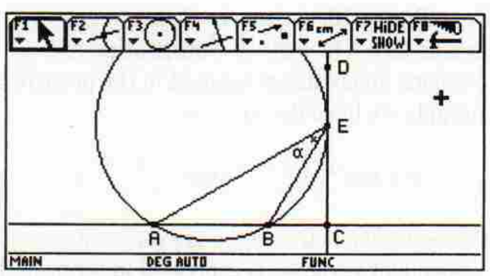

Fig. 5
Tangent-secant theorem applied to rugby conversion kick

To construct this geometric mean, EC, at the optimal kicking location, we first construct the midpoint, M, of $\overline{AC}$, as shown in **figure 6.** We then construct circle M with radius MC and a perpendicular to the goal line through point B. The point of intersection, F, of this perpendicular and circle M creates right triangle AFC because hypotenuse AC is the diameter of circle M. Altitude FB of this right triangle divides the hypotenuse into two segments such that the length of leg FC is the geometric mean of the length of the entire hypotenuse, AC, and the length of the segment of the hypotenuse, BC, that is adjacent to $\overline{FC}$. The overlapping segments of the hypotenuse, $\overline{AC}$, and $\overline{BC}$ are the segments that we need to create the geometric mean in **figure 5.**

The optimal kicking location is at the point at which the circle passing through A and B is tangent to line CD. Both $\overline{CF}$ and $\overline{CE}$ are radii of the same

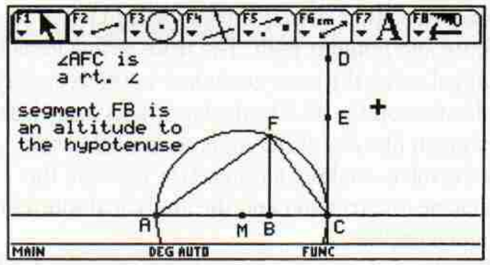

Fig. 6
Geometric mean of right triangle *AFC*

circle; therefore, CE is the geometric mean of AC and BC. See **figure 7.** The kicking location that maximizes the angle α occurs at the intersection E of circle C and $\overleftrightarrow{CD}$.

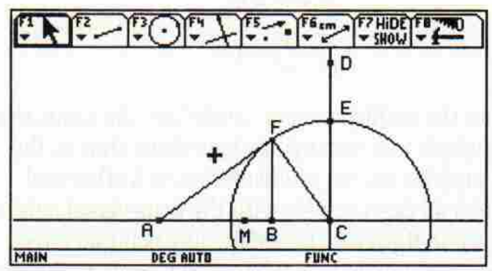

Fig. 7
Construction of ideal kicking location at point *E*

USING ANALYTIC GEOMETRY TO CREATE A LOCUS OF SOLUTIONS

The most surprising discovery in the rugby problem is still to come. We establish a rectangular coordinate system with the x-axis as the goal line and the y-axis as the perpendicular bisector of the goalposts A and B. We label the optimal kicking location point $E(x, y)$, as shown in **figure 8,** and begin to experiment by dragging the downing point $C(x, 0)$ to various locations along the goal line. Since the construction of E depends on this downing point C, the location of E adjusts dynamically as we drag C.

The most surprising discovery is still to come

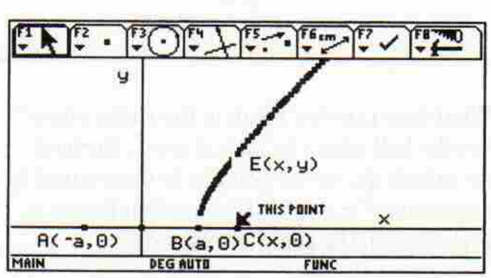

Fig. 8
Trace of locus of ideal kicking locations obtained by dragging point C along the x-axis

By examining the trace of optimal kicking locations, we see that these points appear to follow a definite and familiar path. The trace of this locus is obtained using the trace command in the geometry application on the TI-92 calculator. This locus looks very much like one of the conic sections, but the proof involves making a connection between the geometric construction and the analytical solution obtained earlier.

Equation (1) gives the geometric mean for point E. From **figure 8,** $EC = y$, $AC = x + a$, and $BC = x - a$. Therefore, equation (1) can be written as

$$y^2 = (x + a)(x - a),$$
$$y^2 = x^2 - a^2,$$

which is the equation of a hyperbola. All we need to do is modify it a bit to put it in standard form:

$$y^2 = x^2 - a^2$$
$$x^2 - y^2 = a^2$$
$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$$

Since the coefficients of x^2 and y^2 are the same, the hyperbola is a rectangular hyperbola, that is, the asymptotes are perpendicular to each other and form a 45 degree angle with the transversal axis, as shown in **figure 9.** Therefore, any point on a rectangular hyperbola maximizes the angle to the vertices from a point on a line that is perpendicular to the transversal axis.

The trace of optimal kicking locations follows a definite and familiar path

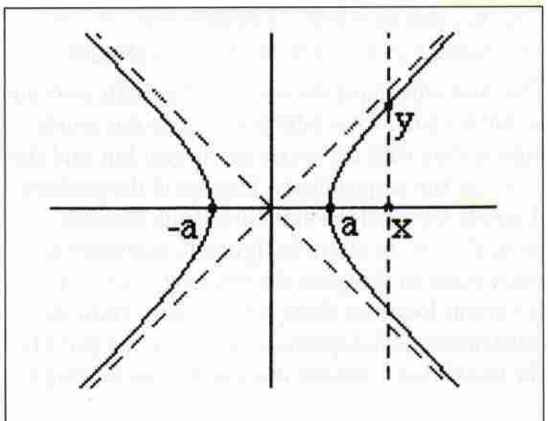

Fig. 9
Rectangular hyperbola

What does this fact mean to the rugby player? Down the ball where he may at any x, the best place to kick the conversion can be determined by the equation $y^2 = x^2 - a^2$. This location lies on a hyperbola with the goalposts as vertices.

PROVING THE RESULT

From an analytical perspective, we can write an algebraic equation relating the kicking angle α as a function of y, the distance from the goal line. With α written as a function of y, we can use calculus methods to find the optimal location from which to kick the conversion. To express α as a function of y, we use the definition of the tangent function for the right triangles ACE and BCE. **Figure 10** relates these angles α and β. From this figure, we note that

$$\tan(\alpha + \beta) = \left(\frac{\text{side opposite}}{\text{side adjacent}}\right).$$

Therefore,

$$\tan(\alpha + \beta) = \left(\frac{x + a}{y}\right)$$

and

$$(\alpha + \beta) = \tan^{-1}\left(\frac{x + a}{y}\right).$$

Similarly, to find the angle β, we note that

$$\tan\beta = \left(\frac{x - a}{y}\right)$$

and

$$\beta = \tan^{-1}\left(\frac{x - a}{y}\right).$$

From **figure 10,** we can see that α can be expressed as $\alpha = (\alpha + \beta) - \beta$. Substituting the expressions involving arc tangent in the preceding paragraph, we have the equation

$$\alpha = \tan^{-1}\left(\frac{x + a}{y}\right) - \tan^{-1}\left(\frac{x - a}{y}\right).$$

In this equation, both x and a are constants, so α, the dependent variable, is expressed in terms of the single independent variable, y. To graph this function on the calculator that uses x as the independent variable, we make the following adjustments: We store an arbitrary value of x, say, 2.03, to the variable m on the home screen; it represents the downing location of the ball from the center of the goalposts. One-half the distance between goalposts is stored to the variable a. We also change the variable y in our equation to the independent variable x

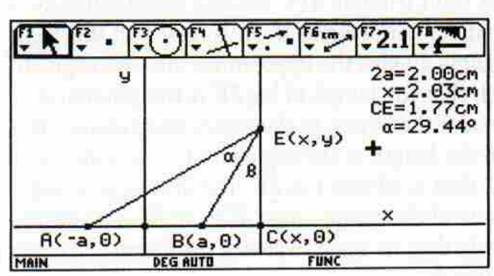

Fig. 10
Kicking location point E and angles to goalposts and goal line

and change the variable α to the dependent variable y. The equation that we enter and graph on the calculator is

$$y = \tan^{-1}\left(\frac{m+a}{x}\right) - \tan^{-1}\left(\frac{m-a}{x}\right).$$

Figure 11 shows a plot of the data, along with a graph of the function. We can find the maximum of the function by using the Maximum command in the Math menu F5.

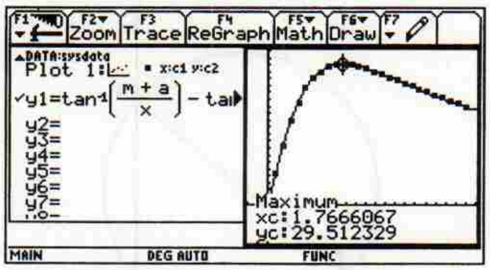

Fig. 11
Scatterplot of data and graph of function showing ideal kicking location

To find this solution using calculus, we assume a fixed value for x, then use the derivative command of the TI-92 calculator's computer algebra system.

$$\alpha = \tan^{-1}\left(\frac{x+a}{y}\right) - \tan^{-1}\left(\frac{x-a}{y}\right)$$

$$\frac{d\alpha}{dy} = \frac{1}{1+\left(\frac{x+a}{y}\right)^2} \cdot \frac{-1(x+a)}{y^2} - \frac{1}{1+\left(\frac{x-a}{y}\right)^2} \cdot \frac{-1(x-a)}{y^2}$$

$$= \frac{-x-a}{y^2+(x+a)^2} + \frac{x-a}{y^2+(x-a)^2}$$

$$= \frac{2ax^2-2ay^2-2a^3}{(y^2+(x+a)^2)(y^2+(x-a)^2)}$$

If α has a maximum, then it will occur when

$$\frac{d\alpha}{dy} = 0,$$

$$2ax^2 - 2ay^2 - 2a^3 = 0,$$

$$2ay^2 = 2ax^2 - 2a^3,$$

$$y^2 = x^2 - a^2,$$

$$y^2 = (x+a)(x-a).$$

Therefore, y is the geometric mean of $(x + a)$ and $(x - a)$, the same result obtained through other methods described in this article.

SUMMARY

Although the practical application of this discovery might be difficult to implement in an actual game of rugby, this problem offers an opportunity to solve a real-world problem while making connections with the mathematics that we teach at the secondary level.

APPENDIX A

SOLVING THE RUGBY PROBLEM FROM AN ELEMENTARY ALGEBRA PERSPECTIVE

The teacher gives each student a copy of the sketch of a rugby field drawn to scale, as shown in **figure 2.** This diagram assumes that the ball has been downed at an arbitrary location, as indicated in **figure 2.** Students, working in pairs, select five different locations along the perpendicular to the goal line and mark these points along the perpendicular. To assure a variety of points, at least two of the students' five points must be beyond the forty-yard marker. After students have marked their kicking locations, they measure the distance from their points to the goal line along the perpendicular. Next, students draw line segments from their kicking location to the two goalposts and measure the resulting kicking angle. Students can record their data in a chart similar to the one shown in **figure 12.**

This problem offers an opportunity to use secondary-level mathematics to solve a real-world problem

Point	Distance from Center of Goalposts	Kicking Angle
1		
2		
3		
4		
5		

Fig. 12
Data-recording chart

After students have collected and recorded their own data, all student pairs share data so that they obtain at least one data point from each ten-yard section of the playing field. Students should then plot these combined data on a coordinate grid that shows the distance from the goal along the horizontal axis and the kicking angle along the vertical axis. Students should be able to see that a maximum angle occurs for a specific downing location. This "ideal kicking location" depends on where the ball is downed. This data plot can also be nicely done using the Stat Editor and Stat Plots functions of most graphing calculators.

APPENDIX B

GEOMETRIC PROOF THAT THE MAXIMUM VIEWING ANGLE OF A STATUE OCCURS AT THE POINT OF TANGENCY TO A CIRCLE PASSING THROUGH THE TOP AND BOTTOM OF THE STATUE AND THE VIEWER'S EYE

In *The Penguin Dictionary of Curious and Interesting Geometry*, Wells (1991) mentions that the maximum angle to view a statue is found where a horizontal line passing through the viewer's eye is tangent to the circle passing through the top and bottom of the statue and the viewer's eye. A proof of this conjecture follows.

PROOF. We consider **figure 13.** We want to prove why H is the optimal viewing location, thus creating the maximum angle from I to points A and B. We know that

$$m\angle AIB = \frac{1}{2}(m\widehat{AB} - m\widehat{JK}).$$

Because the measure of arc AB is constant, the measure of angle AIB is a maximum when the least amount is subtracted from arc AB. This maximum measure occurs at the point of tangency.

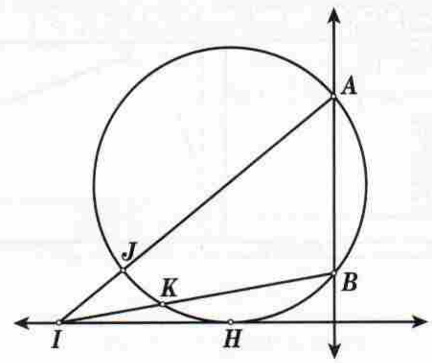

Fig. 13
Why the maximum viewing angle occurs at the point of tangency

REFERENCE

Wells, David. *The Penguin Dictionary of Curious and Interesting Geometry*. New York: Penguin Books, 1991.

The authors wish to express appreciation to the late Arne Engebertson for his encouragement in developing this and other interactive geometry investigations.